THE COLLECTED

MATHEMATICAL WORKS

OF

GEORGE WILLIAM HILL

VOLUME FOUR

Ἄστρων κάτοιδα νυκτέρων ὁμήγυριν.—Æschylus.

THE COLLECTED

MATHEMATICAL WORKS

OF

GEORGE WILLIAM HILL

VOLUME FOUR

PUBLISHED BY THE CARNEGIE INSTITUTION OF WASHINGTON.
APRIL, 1907.

CARNEGIE INSTITUTION OF WASHINGTON

Publication No. 9 (Volume Four)

The Lord Baltimore Press
BALTIMORE, MD., U. S. A.

CONTENTS.

THE

COLLECTED MATHEMATICAL WORKS

OF

G. W. HILL

VOLUME IV

MEMOIR No. 51.

The Secular Variation of the Motion of the Moon's Perigee.

(Astronomical Journal, Vol. X, pp. 73–74, 1890.)

In Nos. 220 and 221 of this Journal appears an article by Mr. Stockwell, in which, in opposition to all previous investigators, an acceleration is found for the motion of this element of the lunar orbit. The introductory history of this matter given by the author is incomplete, as he passes over without notice the most recent, and perhaps the only correct, determination of the coefficient, viz.: that by Delaunay (*Comptes Rendus*, Tome LXXIV, pp. 152, 153). Delaunay finds that, assuming $-1270''i^2$ as the value of $n\int(e'^2-e'^2_0)dt$, the coefficients of i^2 in the motions of the perigee and node are severally $-39''.986$ and $+6''.778$.

Before astronomers accept the new values advanced by Mr. Stockwell, in order to judge intelligently about the matter, they doubtless would like to know how it happens that the author differs so greatly from his predecessors. No information in the article itself is given on this point; it is strange that the author should deem this a matter of no importance. The explanation is not far to seek; the quantities e and ω, which appear in the author's equation (1), are treated as though they suffered no periodic perturbations. Yet the equation is not true unless these two symbols denote severally the eccentricity and longitude of the perigee in the instantaneous orbit which the moon is, at the moment, describing. All the previous investigators, while they do not employ equation (1), have employed methods in which this principle is tacitly admitted. It should be pointed out that

equation (1) is not in the shape astronomers generally give it. It appears that the lunar elements have been partially eliminated from it by the introduction of the differential coefficients

$$\frac{dv}{dt} \text{ and } \frac{dr}{dt}$$

But the method followed by the author requires that the elimination should be complete; no instantaneous element of the moon should appear in the equation. Very simple equations exist suitable for this purpose; such, for instance, as

$$\mu e \cos(v-\omega) = r^3\frac{dv^2}{dt^2} - \mu$$

$$\mu e \sin(v-\omega) = r^2\frac{dr}{dt}\frac{dv}{dt}$$

The equation would then take the form

$$\frac{d\omega}{dt} = \frac{r^2\frac{dv}{dt}\left[r^3\frac{dv^2}{dt^2}-\mu\right]\cdot\left(\frac{dR}{dr}\right) - \frac{dr}{dt}\left[r^3\frac{dv^2}{dt^2}+\mu\right]\cdot\left(\frac{dR}{dv}\right)}{\left[r^3\frac{dv^2}{dt^2}-\mu\right]^2 + r^4\frac{dr^2}{dt^2}\frac{dv^2}{dt^2}}$$

in which the right member is expressed purely as a function of the four quantities

$$r,\ v,\ \frac{dr}{dt} \text{ and } \frac{dv}{dt}$$

It may be thought the equation in this shape is too cumbrous for use; and so it is. For this reason it has not been employed by astronomers. The method followed by Delaunay, for instance, involves far less labor. This method furnishes the equations

$$\frac{dg}{dt} = -\frac{dR}{dG}, \quad \frac{dh}{dt} = -\frac{dR}{dH}$$

But it is more convenient to express R in terms of a, e and γ, than in terms of L, G and H; hence we write

$$\frac{d\omega}{dt} = -\left(\frac{da}{dG}+\frac{da}{dH}\right)\frac{dR}{da} - \left(\frac{de}{dG}+\frac{de}{dH}\right)\frac{dR}{de} - \left(\frac{d\gamma}{dG}+\frac{d\gamma}{dH}\right)\frac{dR}{d\gamma}$$

This equation holds true through all the transformations Delaunay makes for the purpose of removing from R the various periodic terms it contains by reason of the action of the sun. In these transformations, however, e' the solar eccentricity is treated as a constant. It is, however,

variable, and it is permitted to conceive it as developed in ascending powers of t. Hence, wherever, in the old development of R, we wrote e', we ought now to write $e'_0 + ft$, e_0 and f being regarded as constants. The new portion of R is then

$$\frac{dR}{de'}ft$$

for, limiting the determination of the secular variation to the term proportional to t^2, it is not necessary to go beyond the first power of t. And, in this new portion of R, we ought to make all the transformations of variables which have as object the removal of the periodic terms of the earlier portion of R. Again, it is plain that the e', which appears in the still-remaining non-periodic term of the earlier portion of R, should be regarded as constant; that is, e'_0 should be substituted for it. Also, the making the transformations in

$$\frac{dR}{de'}ft$$

is the same thing as substituting in it, for the three coordinates of the moon their values as affected by solar action. This being done, and the resulting function limited to its non-periodic term, it is plain the term we are in search of will be given by the integration of the equation

$$\frac{d\omega}{dt} = -\left\{\left(\frac{da}{dG}+\frac{da}{dH}\right)\frac{d^2R}{da\,de'}+\left(\frac{de}{dG}+\frac{de}{dH}\right)\frac{d^2R}{de\,de'}+\left(\frac{d\gamma}{dG}+\frac{d\gamma}{dH}\right)\frac{d^2R}{d\gamma\,de'}\right\}ft$$

In this care must be taken to have a, e and γ of the same signification in all the factors. When we neglect the inclination of the lunar orbit the last term may be neglected.

The advantage of the employment of such a formula as this over the one employed by Mr. Stockwell consists in the circumstance that since the first factors are non-periodic, the second may be limited to their non-periodic terms. The expression is correct, however far we may wish to push the approximation in reference to powers of the solar disturbing force.

It would be impossible to give here a redetermination of the coefficient in dispute to the degree of approximation adopted by Delaunay, on account of its length. Nevertheless, some approximate statements may be of interest. In another place I have found (*Astronomical Papers of the American Ephemeris*, Vol. III, p. 388),

$$an\left(\frac{da}{dG}+\frac{da}{dH}\right) = -\,0.003395, \qquad a^2ne\left(\frac{de}{dG}+\frac{de}{dH}\right) = -\,1.049$$

When we neglect the inclination of the lunar orbit and the solar parallax, with Delaunay's notation, we have

$$R = \tfrac{1}{4} n'^2 \frac{a'^3}{r'^3} r^2 + \tfrac{3}{4} n'^2 \frac{a'^3}{r'^3} r^2 \cos 2(h + \nu - h' - \nu')$$

Thence we derive

$$\frac{dR}{de'} = -3R\frac{d.\log r'}{de'} - \frac{dR}{dv}\frac{dv'}{de'}$$

Here it will be sufficiently accurate to put

$$-3\frac{d.\log r'}{de'} = -\tfrac{3}{2}e' + 3\cos l\,, \qquad \frac{dv'}{de'} = 2\sin l'$$

Then, if we suppose the disturbing function and its derivative contains severally the terms

$$R = a^2n^2(A_0 + A_1 e' \cos l'), \qquad \frac{dR}{dv} = Ba^2n^2e' \sin l'$$

where A_0, A_1 and B are functions of a and e only, we shall have

$$\frac{dR}{de'} = a^2n^2\left\{\tfrac{3}{2}(A_1 - A_0) - B\right\}e'$$

From Pontécoulant's *Théorie Analytique du Système du Monde*, Tome IV, pp. 100, 106, 185, 253 and 256, we get

$$\begin{aligned} A_0 &= \tfrac{1}{4} m^2 - \tfrac{179}{96} m^4 + (\tfrac{3}{8} m^2 + \tfrac{225}{32} m^3 + \tfrac{16751}{512} m^4)\, e^2 \\ A_1 &= \tfrac{3}{4} m^2 - \tfrac{163}{16} m^4 + (\tfrac{9}{8} m^2 + \tfrac{975}{32} m^3 + \tfrac{105357}{512} m^4)\, e^2 \\ B &= (\tfrac{75}{8} m^3 + \tfrac{825}{32} m^4)\, e^2 \end{aligned}$$

Thus the function $\dfrac{dR}{de'}$ would contain the terms

$$\{\tfrac{3}{4} m^2 - \tfrac{799}{64} m^4 + (\tfrac{9}{8} m^2 + \tfrac{825}{32} m^3 + \tfrac{119709}{512} m^4)\, e^2\}a^2n^2e'$$

Pontécoulant's and Delaunay's e are not quite identical; but neglecting the difference, substituting this expression in our formula, and adopting $-1270''i^2$ as the value of $n\int(e'^2 - e_0'^2)dt$, we get as the coefficient $-33''$ instead of Delaunay's $-40''$. The difference is caused, in the main, by our neglect of the terms of higher orders.

In conclusion, it should be stated that it is not true that when we have

derived a formula such as

$$\omega = \omega_0 + (A + Be'^2)\,t$$

on the supposition that the solar eccentricity is constant, we shall have, as the proper correction to this when the eccentricity is variable, the formula

$$\delta\omega = B\int(e'^2 - e'^{\,2})\,dt$$

Delaunay has given the value of B (*Comptes Rendus*, Tome LXXIV, pp. 15–17), and a difference will be observed between the coefficients.

MEMOIR No. 52.

Additional Terms in the Great Inequalities of Jupiter and Saturn.

(Astronomical Journal, Vol. XI, pp. 49–51, 1891.)

The discussion of the observations of Jupiter and Saturn, now in progress in the office of the *American Ephemeris and Nautical Almanac*, has reached such a stage that we can exhibit the results in reference to the correction of the mean longitude of Jupiter as it is given in my new theory of Jupiter and Saturn, (*Astronomical Papers of the American Ephemeris*, Vol. IV, p. 558). Dividing the material into eleven groups roughly corresponding to as many revolutions of the planet, values of δL were obtained, which are shown in the following table:

Interval.	Mean Year.	δL	Weight.
1750-1765	1758	$-0''.50+0.137\mu$	4.2
1766-1777	1772	$-0.71-0.037\mu$	1.4
1778-1789	1783	$-0.52-0.063\mu$	0.9
1790-1801	1796	$-1.28-0.111\mu$	2.8
1802-1813	1808	$-1.33+0.133\mu$	5.5
1814-1825	1820	$-1.30+0.040\mu$	6.1
1826-1837	1832	$+0.01-0.097\mu$	11.7
1838-1849	1844	$-0.04-0.098\mu$	16.7
1850-1861	1856	$+0.02-0.068\mu$	16.6
1862-1873	1868	$+0.28+0.179\mu$	16.5
1874-1887	1881	$+0.16+0.139\mu$	19.0

In the values of δL the modifications caused by a change in the mass of Saturn are shown by the terms involving μ, an indeterminate so chosen that $\mu = 1''$ corresponds to an augmentation of Bessel's mass $\frac{1}{3501.6}$ by a thousandth part. The column of weights is added that some idea of the degree of precision of the several values of δL may be obtained.

These values can be very well represented by a linear function of the time, with the exception of the three belonging to the interval 1790–1825, which have abnormally large negative values. And the latter cannot be brought into harmony with the rest by assigning to μ any possible value; in fact, the solution of the equations gives for μ an insignificant quantity.

The disagreement might be attributed to personal equations in the observers; but this seems not likely, as the three values of 1796, 1808 and 1820 are closely coincident, and yet are founded on material coming, in the first from Greenwich and Palermo, in the second from Greenwich, Palermo and Paris, and in the third from Greenwich, Paris and Königsberg.

Again we may suppose that there is some inequality of long period in the mean longitude of Jupiter not taken into consideration in the theory. If this arises from the mutual action of Jupiter and Saturn, the observations of the latter should more clearly exhibit the effects of this perturbation. However, there does not seem to be any indication of an inequality in Saturn two and a half times as great as that which would seem to have place in Jupiter.

In order that nothing might be wanting to the clearing up of this difficulty, I determined to compute the terms in the great inequalities which depend on three times the principal argument, that is to say, on the argument $15g' - 6g$. These terms have hitherto been neglected, and from induction one would suppose their coefficients were at the limit of smallness, permitting their being passed over, at least if regard is had only to the representation of the observations. The period of these terms is about 310 years, and their coefficients are seemingly in the neighborhood of $0''.1$.

As we have here to deal with very small quantities, we may confine our attention to what are presumably the largest components of these coefficients. Thus we assume that these terms arise solely from the variations $n\delta z$ and $n'\delta z'$ of the mean longitudes of the planets, and that all consideration of the variables ν and ν' may be omitted.

The developments given in the New Theory (*Astr. Papers*, Vol. IV, p. 50) do not reach as far as the argument $15g' - 6g$, but they may be easily extended to this point by the mode of induction explained at pp. 45–46. By considering the terms involving the six arguments from $15g' - 8\varepsilon$ to $15g' - 13\varepsilon$, we conclude that the function $\frac{a'}{\Delta}$ contains the terms,

$$+ 0.0000000038 \cos(15g' - 6\varepsilon) + 0.0000000132 \sin(15g' - 6\varepsilon)$$
$$+ 0.0000000734 \cos(15g' - 7\varepsilon) - 0.0000000374 \sin(15g' - 7\varepsilon)$$

and, in consequence, the terms

$$- 0.0000000114 \cos(15g' - 6g) + 0.0000000136 \sin(15g' - 6g).$$

By means of this expression, we can complete the terms of T and T' dependent on $5g' - 2g$ and its multiples given at pp. 75–91, so that, writing

V for $5g' - 2g$ they stand as follows:

$$\begin{aligned} T = &- 0''.07676841 \sin V - 0''.18132165 \cos V \\ &+ 0''.0007626 \sin 2V - 0''.0007509 \cos 2V \\ &+ 0''.0000066 \sin 3V + 0''.0000079 \cos 3V, \\ T' = &+ 1''.1766033 \sin V + 2''.7790743 \cos V \\ &- 0''.0116886 \sin 2V + 0''.0115086 \cos 2V \\ &- 0''.000101 \sin 3V - 0''.000120 \cos 3V. \end{aligned}$$

It is evident that, as far as these terms are concerned, with sufficient approximation, we have the equation

$$T + 0.06524557\, T' = 0.$$

Hence, after the inequalities of Saturn have been obtained, it will be only necessary to multiply them by the factor -0.4024 to have those of Jupiter. Dealing therefore with Saturn alone, we have to consider that V in the expression for T' receives the increment $\delta V = 5n'\delta z' - 2n\delta z$, and thus becomes

$$T' + \frac{dT'}{dV}\delta V + \frac{1}{2}\frac{d^2T'}{dV^2}(\delta V)^2.$$

From the expressions of $n\delta z$ and $n'\delta z'$ (*Astr. Papers*, Vol. IV, pp. 403, 449) we get

$$\delta V = 5n'\delta z' - 2n\delta z = - 6595'' \sin V - 15593'' \cos V - 107''.5 \sin 2V + 113''.2 \cos 2V,$$

as also

$$\tfrac{1}{2}(\delta V)^2 = + 242''.0 \cos 2V + 249''.2 \sin 2V.$$

Substituting these values and confining our attention to the terms involving $3V$, we find that T' becomes

$$T' = - 0''.002377 \sin 3V - 0''.000925 \cos 3V.$$

Integrating this twice we get

$$n'\delta z' = + 0''.286 \sin (3V + 21^\circ.3).$$

And, multiplying this by the factor -0.4024,

$$n\delta z = + 0''.115 \sin (3V + 201^\circ.3).$$

It has been assumed here that V will be more correctly denoted by $5g' - 2g - 82''t$ than by $5g' - 2g$, consequently the integrating factor has been taken at 10.58.

The evections associated with these long-period inequalities are not much beneath them in magnitude, and we propose to compute them in the same approximate way.

By induction from the terms involving the six arguments from $7g' - 6g$ to $12g' - 6g$ in the functions $a' \dfrac{d\Omega'}{dg'}$ and $a' r' \dfrac{d\Omega'}{dr'}$ (*Astr. Papers*, Vol. IV, p. 71) it is found that the latter contain severally the terms

$$a' \frac{d\Omega'}{dg'} = + 0''.000140 \sin(14g' - 6g) + 0''.000176 \cos(14g' - 6g),$$
$$a' r' \frac{d\Omega'}{dr'} = + 0''.000112 \cos(14g' - 6g) - 0''.000071 \sin(14g' - 6g).$$

By means of these additional terms we complete the expression of T', p. 83, so that the terms which involve V now become

$$\begin{aligned} T' = &+ 2''.50567 \sin(-\gamma' + V) - 20''.38129 \cos(-\gamma' + V) \\ &+ 0''.10783 \sin(-\gamma' + 2V) - 0''.02865 \cos(-\gamma' + 2V) \\ &+ 0''.00039 \sin(-\gamma' + 3V) + 0''.00042 \cos(-\gamma' + 3V) \end{aligned}$$

Exactly as before, we must now suppose that in T', V receives the increment δV, and thus that T' becomes

$$T' + \frac{dT'}{dV}\delta V + \tfrac{1}{2}\frac{d^2T'}{dV^2}(\delta V)^2.$$

Making the substitution and preserving only the terms involving $3V$, we get

$$T' = + 0''.01055 \sin(-\gamma' + 3V) + 0''.01576 \cos(-\gamma' + 3V).$$

Integrating this once, making $\gamma' = g'$, and then integrating again, the effect on the coefficients is the same as multiplying them by 11.11, and we obtain

$$n'\delta z' = + 0''.212 \sin(14g' - 6g + 55^\circ.6).$$

Proceeding in like manner for Jupiter we obtain

$$a \frac{d\Omega}{dg} = + 0''.000012 \sin(15g' - 7g) - 0''.000012 \cos(15g' - 7g),$$
$$ar \frac{d\Omega}{dr} = + 0''.000011 \cos(15g' - 7g) + 0''.000029 \sin(15g' - 7g).$$

By means of these expressions, we are enabled to add to T terms involving $3V$, so that it now becomes

$$\begin{aligned} T = &- 2''.02853 \sin(-\gamma + V) + 0''.15263 \cos(-\gamma + V) \\ &- 0''.00484 \sin(-\gamma + 2V) - 0''.00979 \cos(-\gamma + 2V) \\ &+ 0''.00003 \sin(-\gamma + 3V) - 0''.00005 \cos(-\gamma + 3V). \end{aligned}$$

Supposing, in this expression, that V receives the increment δV, it is found that T contains the terms

$$T = + 0''.00103 \sin(-\gamma + 3V) - 0''.00131 \cos(-\gamma + 3V).$$

We then obtain $n\delta z$ by multiplying the coefficients by 25.88 and substituting g for γ. Thus

$$n\delta z = +\ 0''.043 \sin (15g' - 7g + 308^\circ).$$

From induction it is supposed that these two evections will be obtained with greater precision if, in place of $14g' - 6g$ and $15g' - 7g$, we put $14g' - 6g - 180''t$ and $15g' - 7g - 120''t$.

Gathering together our results, the mean longitude of Jupiter ought to be increased by the terms

$$\begin{aligned} n\delta z = &+ 0''.115 \sin (15g' - 6g - 246''t + 201^\circ.3) \\ &+ 0''.043 \sin (15g' - 7g - 120''t + 308^\circ), \end{aligned}$$

and the mean longitude of Saturn by the terms

$$\begin{aligned} n'\delta z' = &+ 0''.286 \sin (15g' - 6g - 246''t + 21^\circ.3) \\ &+ 0''.212 \sin (14g' - 6g - 180''t + 55.6). \end{aligned}$$

It will be seen that the long-period inequality for Jupiter is not of a magnitude sufficient to remove the difficulty stated at the beginning of this article. However, it appears worth while to have computed these inequalities, if the only result is that the doubt is removed.

MEMOIR No. 53.

On the Connection of Precession and Nutation with the Figure of the Earth.

(Astronomical Journal, Vol. XIII, pp. 1–6, 1893.)

Some difficulties have been encountered in reconciling the various values derived for the compression of the earth. On reading the discussions which have been published on this subject, it is suggested that they might possibly be removed if the formulas, on which the derivation depends, were made more nearly rigorous. In the present article I propose to supply some omissions in the theory of precession and nutation, one of the sources whence has been derived a value for the compression. The expressions hitherto given for the constant of precession and the coefficient of the principal term of nutation have been obtained by the substitution of elliptical values for the lunar coordinates. The only deviation from this mode of proceeding I have been able to find is in Prof. Harkness's *The Solar Parallax and its Related Constants*, where, in the first of his equations (158) for Serret's $\frac{1}{a}$ he substitutes $\frac{1+x'}{a}$, x' standing for

$$\left(\tfrac{1}{6}+\tfrac{1}{4}e'^2\right)m^2-\tfrac{179}{288}m^4-\tfrac{97}{48}m^5$$

in Delaunay's notation. But the part of the constant of precession produced by lunar action and the coefficient of the principal term of nutation are augmented by about a 340th part through the solar perturbations of the lunar coordinates. Prof. Harkness's innovation brings us much nearer the truth, but is not quite rigorous. When the object is simply to show how these phenomena result from the forces in action, and an appeal is finally to be made to observation for the values of the constants involved, no great objection can be made to the old method. But when we wish to derive from these observed values the mass of the moon and the ratio of the moments of inertia of the earth it is important that the factors involved should be determined with some approach to rigor.

The differential equations for precession and nutation, first stated by Poisson, are

$$\frac{d\omega}{dt}=\frac{1}{Cn\sin\omega}\frac{dV}{d\psi},\qquad \frac{d\psi}{dt}=-\frac{1}{Cn\sin\omega}\frac{dV}{d\omega},$$

$$V=-\tfrac{3}{2}(C-A)\left[\frac{m}{r^3}\sin^2\delta+\frac{m'}{r'^3}\sin^2\delta'\right].$$

Here ω denotes the obliquity of the equator to a fixed ecliptic, ψ the amount of retrograde motion of its node on this plane, C is the moment of inertia of the earth about its axis of rotation, A half the sum of the moments of inertia about the two principal axes lying in the plane of the equator, m and m' are the masses severally of the moon and sun, r and r' the distances of their centers from the center of the earth, and δ and δ' their declinations.

When we treat precession alone, it suffices to substitute for the terms $\frac{m}{r^3}\sin^2\delta + \frac{m'}{r'^3}\sin^2\delta'$ their non-periodic portions. In the case of the second of these, neglecting all periodic perturbations, we can assume that the sun moves about the earth in an ellipse whose elements are slowly changing. From the theory of elliptic motion we know that $\frac{a^3}{r^3}\cos 2f$ and $\frac{a^3}{r^3}\sin 2f$ have no non-periodic terms; also the addition of a function independent of ω and ψ to V does not impair its use for our purposes. Thus it is plain we may substitute for $\sin^2\delta'$

$$-\tfrac{1}{2}\cos^2\omega' = -\tfrac{1}{2}[\cos i\cos\omega - \sin i\sin\omega\cos(\psi+\theta)]^2,$$

where ω' denotes the obliquity of the actual equator to the actual ecliptic, and i the inclination, and θ the longitude of the node of the latter on the fixed ecliptic. And, for $\frac{a'^3}{r'^3}$, may be substituted its non-periodic term $(1-e'^2)^{-\frac{3}{2}}$, e' denoting the eccentricity of the earth's orbit.

If, with Delaunay, we denote the longitude and latitude of the moon by V and U, we have

$$\sin\delta = \cos\omega'\sin U + \sin\omega'\cos U\sin(V+\psi)$$

$$\frac{m}{r^3}\sin^2\delta = \frac{m}{r^3}\cos^2\omega'\sin^2 U + \frac{m}{r^3}\sin\omega'\cos\omega'\sin 2U\sin(V+\psi)$$
$$+ \quad \frac{m}{r^3}\sin^2\omega'\cos^2 U - \tfrac{1}{2}\frac{m}{r^3}\sin^2\omega'\cos^2 U\cos 2(V+\psi).$$

As it is evident that the two terms of the latter equation, which involve V, are wholly periodic, they may be rejected; and from the expression we may subtract $\frac{m}{r^3}\sin^2 U$, which does not contain ω or ψ. Thus, for precession, we may substitute for $\frac{m}{r^3}\sin^2\delta$,

$$\tfrac{1}{2}\frac{m}{r^3}[1-3\sin^2 U]\sin^2\omega'.$$

Let us denote the non-periodic term of $\frac{a^3}{r^3}[1 - 3\sin^2 U]$ by N. The value of this, corresponding to elliptic expressions for the lunar coordinates, is easily found. For the non-periodic term of $\frac{a^3}{r^3}$ is $(1 - e^2)^{-\frac{3}{2}}$, and $\sin U$ is equal to the sine of the inclination multiplied by the sine of the true argument of latitude. Thus the elliptic value of N, in Delaunay's symbols, is

$$(1 - e^2)^{-\frac{3}{2}}(1 - 6\gamma^2 + 6\gamma^4)$$

To obtain the part of N which arises from solar perturbation, we consider first the function $\frac{a^3}{r^3}$. Let $\delta\mathrm{r}$ denote the perturbation of $\frac{a}{r}$, so that

$$\frac{a^3}{r^3} = \left(\frac{a}{r_0} + \delta\mathrm{r}\right)^3 = \frac{a^3}{r_0^3} + 3\frac{a^2}{r_0^2}\delta\mathrm{r} + 3\frac{a}{r_0}\delta\mathrm{r}^2 + \delta\mathrm{r}^3$$

In this formula it is sufficient to put

$$3\frac{a^2}{r_0^2} = 3 + \tfrac{3}{2}e^2 + \tfrac{9}{8}e^4 + [6e + \tfrac{9}{4}e^3]\cos l + \tfrac{15}{2}e^2\cos 2l,$$

$$3\frac{a}{r_0} = 3 + [3e - \tfrac{3}{8}e^3]\cos l + 3e^2\cos 2l.$$

We propose to compute N to terms of the seventh order inclusive, and shall make use of Delaunay's expressions for the lunar coordinates. However, in the case of $\frac{a}{r}$, the expression goes only to terms of the fifth order, and we need the non-periodic term of this coordinate to terms of the seventh order. Fortunately Adams has published the expression for this term to this degree of approximation (*Monthly Notices*, Vol. XXXVIII, p. 472). We also need the coefficient of $\cos l$ in the same coordinate to terms of the sixth order. To get this we resort to Pontécoulant's expression. His e and γ, however, differ from Delaunay's quantities denoted by the same symbols, and a comparison of their coefficients of $\sin l$ and $\sin F$, severally in the expressions for the moon's longitude and latitude, shows that, in order to obtain Delaunay's form for the coefficients, we ought to substitute for Pontécoulant's e the expression

$$[1 - \tfrac{3}{4}m^2 + \tfrac{75}{128}m^3 + \tfrac{6947}{512}m^4 + \tfrac{1606525}{24576}m^5]e + [\tfrac{263}{32}m^2 + \tfrac{51345}{1024}m^3]e^3 + [\tfrac{9}{2}m^2 + \tfrac{6125}{128}m^3]ee'^2$$
$$+ [\tfrac{51}{16}m^2 - \tfrac{87}{8}m^3]\gamma^2 e,$$

and, for his γ, the expression

$$[2 - \tfrac{33}{64}m^3 - \tfrac{241}{256}m^4 + \tfrac{82495}{12288}m^5]\gamma - [1 - \tfrac{55}{32}m^2 + \tfrac{339}{128}m^3]\gamma^3 - \tfrac{1}{4}\gamma^5 - \tfrac{737}{256}m^2e^2\gamma$$
$$- \tfrac{27}{4}m^2e'^2\gamma + e^2\gamma^3 + \tfrac{7}{32}e^4\gamma$$

Pontécoulant's expression for the coefficient of $\cos l$ in $\frac{a}{r}$ is (*Théorie Analytique*, Tom. IV, pp. 138, 276, 332),

$$\begin{aligned}[1 - \tfrac{1}{8}e^2 + \tfrac{1}{192}e^4 + \tfrac{1}{6}m^2 - \tfrac{645}{128}m^3 - \tfrac{152129}{4608}m^4 - \tfrac{3474907}{24576}m^5 - (\tfrac{797}{96}m^2 + \tfrac{28265}{512}m^3)\,e^2 \\ - (\tfrac{43}{8}m^2 + \tfrac{8215}{128}m^3)\,e'^2 - (\tfrac{83}{128}m^2 - \tfrac{1707}{256}m^3)\,\gamma^2]\,e.\end{aligned}$$

However, it must be noted that in the final result at p. 568 he has 315 instead of the 1707 at the end of this formula. I adopt the earlier stated number. Substituting for Pontécoulant's e and γ their values just given, we get as the coefficient of $\cos l$ in terms of Delaunay's constants,

$$\begin{aligned}[1 - \tfrac{7}{12}m^2 - \tfrac{285}{64}m^3 - \tfrac{45091}{2304}m^4 - \tfrac{341017}{4096}m^5]\,e + [-\tfrac{1}{8} + \tfrac{19}{96}m^2 + \tfrac{2295}{512}m^3]\,e^3 \\ + [-\tfrac{7}{8}m^2 - \tfrac{1045}{64}m^3]\,ee'^2 + [\tfrac{19}{32}m^2 + \tfrac{1011}{64}m^3]\,\gamma^2 e + \tfrac{1}{192}e^5\end{aligned}$$

Delaunay's value, as far as it goes, agrees with this, except that he has the terms $\frac{5}{2}\gamma^4 e - \frac{5}{4}\gamma^2 e^3$, overlooked by Pontécoulant.

By squaring and cubing δr as given by Delaunay, and preserving only the terms which are useful to us, we get

$$\begin{aligned}\delta r^2 = {} & \tfrac{19}{36}m^4 + \tfrac{19}{6}m^5 + \tfrac{10441}{864}m^6 + \tfrac{15793}{432}m^7 + [\tfrac{225}{128}m^2 + \tfrac{3765}{256}m^3 + \tfrac{2915423}{36864}m^4 + \tfrac{540285}{1536}m^5]\,e^2 \\ & + [\tfrac{119}{24}m^4 + \tfrac{171}{4}m^5]\,e'^2 + [-2m^4 - \tfrac{67}{6}m^5]\,\gamma^2 + [\tfrac{225}{32}m^2 + \tfrac{12705}{256}m^3]\,e^4 \\ & + [-\tfrac{225}{32}m^2 - \tfrac{9555}{128}m^3]\,e^2\gamma^2 + [\tfrac{1207}{128}m^2 + \tfrac{68285}{512}m^3]\,e^2e'^2 + [\tfrac{25}{8} - \tfrac{675}{32}m]\,e^2\gamma^4 \\ & + [\tfrac{225}{512}m^2 + \tfrac{1215}{256}m^3]\,\frac{a^2}{a'^2} + [\tfrac{25}{32} - \tfrac{225}{32}m]\,e'^2\,\frac{a^2}{a'^2} \\ & + [\tfrac{15}{8}m^3 + \tfrac{3931}{288}m^4 + \tfrac{82115}{512}m^5 + (\tfrac{225}{32}m^2 + \tfrac{1635}{32}m^3)\,e^2 + \tfrac{55}{8}m^3e'^2 - \tfrac{15}{2}m^3\gamma^2]\,e\cos l \\ & + \tfrac{495}{128}m^3e^2\cos 2l \\ \delta r^3 = {} & \tfrac{55}{216}m^6 + \tfrac{19}{12}m^7 + [\tfrac{225}{256}m^4 + \tfrac{2925}{512}m^5]\,e^2\end{aligned}$$

By the substitution of these values in the preceding equation we ascertain the non-periodic term of $\frac{a^3}{r^3}$; to which we annex the few periodic terms which are necessary for the complete determination of N:

$$\begin{aligned}\frac{a^3}{r^3} = {} & (1 - e^2)^{-\frac{3}{2}} + \tfrac{1}{2}m^2 - \tfrac{9}{32}m^4 + \tfrac{55}{16}m^5 + \tfrac{2159}{96}m^6 + \tfrac{1997}{24}m^7 \\ & + [\tfrac{483}{128}m^2 + \tfrac{8595}{256}m^3 + \tfrac{814965}{4096}m^4 + \tfrac{3695593}{4096}m^5]\,e^2 - [6m^4 + \tfrac{67}{2}m^5]\,\gamma^2 \\ & + [\tfrac{3}{4}m^2 + \tfrac{153}{64}m^4 + \tfrac{1485}{32}m^5]\,e'^2 + [\tfrac{1845}{64}m^2 + \tfrac{51615}{256}m^3]\,e^4 \\ & - [\tfrac{213}{16}m^2 + \tfrac{21015}{128}m^3]\,e^2\gamma^2 + [-3m^2 + \tfrac{27}{8}m^3]\,\gamma^4 + \tfrac{15}{16}m^2e'^4 \\ & + [\tfrac{3333}{128}m^2 + \tfrac{185055}{512}m^3]\,e^2e'^2 - \tfrac{15}{4}e^4\gamma^2 + [\tfrac{135}{8} - \tfrac{2025}{32}m]\,e^2\gamma^4 \\ & + [\tfrac{963}{512}m^2 + \tfrac{4095}{256}m^3]\,\frac{a^2}{a'^2} + [\tfrac{75}{32} - \tfrac{675}{32}m]\,e'^2\,\frac{a^2}{a'^2} \\ & + 3e\cos l + \tfrac{9}{2}e^2\cos 2l + [6m^2 - 9m^3 - \tfrac{45}{2}e^2]\,\gamma^2\cos 2F + [-\tfrac{15}{2} + \tfrac{405}{16}m]\,\gamma^2 e\cos(2F - l) \\ & + \tfrac{147}{8}m\gamma^2 e\cos(2F + l) + [3m^2 + \tfrac{19}{2}m^3 + \tfrac{137}{6}m^4 \\ & + (\tfrac{135}{8}m + \tfrac{1941}{32}m^2)\,e^2 - \tfrac{15}{2}m^2e'^2 - 6m^2\gamma^2]\cos 2D \\ & + [\tfrac{45}{8}m + \tfrac{657}{32}m^2]\,e\cos(2D - l) + \tfrac{147}{16}m^2 e\cos(2D + l) + \tfrac{21}{2}m^2e'\cos(2D - l') \\ & - \tfrac{3}{2}m^2e'\cos(2D + l') - 9m^2\gamma^2\cos(2D - 2F)\end{aligned}$$

For computing the periodic development of $\sin^2 U$, it is sufficient to take

$$\sin^2 U = U^2 - \tfrac{1}{3} U^4 + \tfrac{2}{45} U^6$$

And let us suppose that $U = U_0 + U_1$, U_0 denoting the elliptic portion, and U_1 the perturbational portion. Then, employing the symbol δ to denote the perturbational part.

$$\begin{aligned}\delta(\sin^2 U) &= (2U_0 - \tfrac{4}{3}U_0^3 + \tfrac{4}{15}U_0^5)\,U_1 + (1 - 2U_0^2)\,U_1^2 - \tfrac{4}{3}U_0U_1^3 \\ &= \sin 2U_0 \cdot U_1 + \cos 2U_0 \cdot U_1^2 - \tfrac{4}{3}U_0U_1^3\end{aligned}$$

But, on making $U_0 = 2\gamma \sin F$ and $U_1 = \tfrac{3}{4}\gamma m \sin(2D - F)$, it is perceived that the last term of the second member contains no constant part; hence we neglect it; and for $\cos 2U_0$ it suffices to substitute its constant term $1 - 4\gamma^2$. Thus

$$\delta(\sin^2 U) = \sin 2U_0 \cdot U_1 + (1 - 4\gamma^2)\,U^2$$

For $\sin 2U_0$ it is sufficient to write the expression

$$\sin 2U_0 = [4\gamma - 4\gamma e^2 - 8\gamma^3] \sin F + 4\gamma e \sin(F + l) - 4\gamma e \sin(F - l) - \tfrac{1}{2}\gamma e^2 \sin(F - 2l)$$

We obtain the needed expression for U_1 from Delaunay by subtracting his value of U_0 (vol. I, p. 58), from his value of U (vol. II, p. 862). It is necessary to take all the inequalities of the second and third orders with one of the fourth, but their coefficients must be carried to terms severally of the fifth and fourth orders inclusive. Thus,

$$\begin{aligned}U_1 = {} & [\tfrac{3}{4}m + \tfrac{9}{32}m^2]\,\gamma e' \sin(F - l') - [\tfrac{3}{4}m - \tfrac{69}{32}m^2]\,\gamma e' \sin(F + l') \\ & - [\tfrac{1}{2}m^2 - \tfrac{21}{8}m^3]\,\gamma e \sin(F + l) \\ & + [\tfrac{189}{32}m^2 + \tfrac{375}{32}m^3 - \tfrac{135}{32}me^2 + \tfrac{135}{8}m\gamma^2]\,\gamma e \sin(F - l) \\ & + [-\tfrac{5}{4} + \tfrac{135}{32}m]\,\gamma e^2 \sin(F - 2l) \\ & + [\tfrac{11}{8}m^2 + \tfrac{59}{12}m^2 + \tfrac{135}{16}me^2 - \tfrac{3}{8}m\gamma^2]\,\gamma \sin(2D + F) \\ & + [\tfrac{15}{4}m + \tfrac{241}{16}m^2]\,\gamma e \sin(2D + F - l) \\ & + [\tfrac{3}{4}m + \tfrac{25}{16}m^2 + \tfrac{2957}{768}m^3 + \tfrac{84703}{9216}m^4 \\ & \qquad + (\tfrac{27}{16}m + \tfrac{423}{64}m^2)\,e^2 - (\tfrac{15}{8}m + \tfrac{199}{16}m^2)\,e'^2 \\ & \qquad + (\tfrac{9}{8}m - \tfrac{175}{32}m^2)\,\gamma^2]\,\gamma \sin(2D - F) \\ & + [\tfrac{7}{4}m + \tfrac{255}{32}m^2]\,\gamma e' \sin(2D - F - l') \\ & \qquad - [\tfrac{3}{4}m + \tfrac{115}{32}m^2]\,\gamma e' \sin(2D - F + l') \\ & \qquad + [\tfrac{3}{4}m + \tfrac{23}{16}m^2]\,\gamma e \sin(2D - F + l) \\ & \qquad + [3m + \tfrac{105}{8}m^2]\,\gamma e \sin(2D - F - l)\end{aligned}$$

For the elliptic value of $\sin^2 U_0$ it suffices to take

$$\sin^2 U_0 = 2\gamma^2 - 2\gamma^4 - 2\gamma^2 \cos 2F + 4\gamma^2 e \cos(F - l) - 4\gamma^2 e \cos(F + l)$$

By performing the multiplications we get

$$
\begin{aligned}
\sin 2U_0 \cdot U_1 = & -\left[\tfrac{205}{16} m^2 + \tfrac{459}{16} m^3\right] e^2\gamma^2 + \left[\tfrac{5}{16} + \tfrac{945}{128} m\right] e^4\gamma^2 - \tfrac{135}{4} me^2\gamma^4 \\
& + \left[\tfrac{173}{16} m^2 + \tfrac{291}{16} m^3 + \left(\tfrac{5}{2} - \tfrac{135}{8} m\right) e^2 + \tfrac{135}{4} m\gamma^2\right] \gamma^2 e \cos l \\
& + \left[-\tfrac{5}{2} + \tfrac{135}{16} m\right] \gamma^2 e^2 \cos 2l \\
& + \left[-\tfrac{3}{2} m - \tfrac{3}{8} m^2 + \tfrac{273}{128} m^3 + 3me^2 + \tfrac{15}{4} me'^2\right] \gamma^2 \cos 2D \\
& - \tfrac{7}{2} m\gamma^2 e' \cos (2D - l') + \tfrac{3}{2} m\gamma^2 e' \cos (2D + l') \\
& - 3m\gamma^2 e \cos (2D + l) + \left[3m + \tfrac{39}{4} m^2\right] \gamma^2 e \cos (2D - l) \\
& + \tfrac{3}{2} m\gamma^2 \cos (2D - 2F) \\
U_1^2 = & \left[\tfrac{9}{32} m^2 + \tfrac{75}{64} m^3 + \tfrac{5175}{1024} m^4 + \tfrac{60425}{3072} m^5\right] \gamma^2 \\
& \qquad + \left[\tfrac{837}{64} m^2 + \tfrac{14865}{128} m^3\right] e^2\gamma^2 \\
& + \left[\tfrac{31}{32} m^2 + \tfrac{795}{128} m^3\right] e'^2\gamma^2 + \left[\tfrac{27}{32} m^2 - \tfrac{183}{64} m^3\right] \gamma^4 \\
& \qquad + \left[\tfrac{25}{32} - \tfrac{675}{128} m\right] e^4\gamma^2 \\
& + \left[\tfrac{45}{16} m^2 + \tfrac{351}{16} m^3\right] \gamma^2 e \cos l
\end{aligned}
$$

The sum of the three parts, of which it is composed, gives

$$
\begin{aligned}
\sin^2 U = & \left[2 + \tfrac{9}{32} m^2 + \tfrac{75}{64} m^3 + \tfrac{5175}{1024} m^4 + \tfrac{60425}{3072} m^5\right] \gamma^2 \\
& \qquad + \left[\tfrac{17}{64} m^2 + \tfrac{11193}{128} m^3\right] e^2\gamma^2 \\
& + \left[\tfrac{31}{32} m^2 + \tfrac{795}{128} m^3\right] e'^2\gamma^2 - \left[2 + \tfrac{9}{32} m^2 + \tfrac{483}{64} m^3\right] \gamma^4 \\
& \qquad + \left[\tfrac{35}{32} + \tfrac{135}{64} m\right] e^4\gamma^2 - \tfrac{135}{4} me^2\gamma^4 \\
& + \left[\tfrac{109}{8} m^2 + \tfrac{321}{8} m^3 + \left(\tfrac{5}{2} - \tfrac{135}{8} m\right) e^2 + \tfrac{135}{4} m\gamma^2\right] \gamma^2 e \cos l \\
& + \left[-\tfrac{5}{2} + \tfrac{135}{16} m\right] \gamma^2 e^2 \cos 2l - 2\gamma^2 \cos 2F \\
& \qquad + 4\gamma^2 e \cos (2F - l) - 4\gamma^2 e \cos (2F + l) \\
& + \left[-\tfrac{3}{2} m - \tfrac{3}{8} m^2 + \tfrac{273}{128} m^3 + 3me^2 + \tfrac{15}{4} me'^2\right] \gamma^2 \cos 2D \\
& - \tfrac{7}{2} m\gamma^2 e' \cos (2D - l') + \tfrac{3}{2} m\gamma^2 e' \cos (2D + l') \\
& + \left[3m + \tfrac{39}{4} m^2\right] \gamma^2 e \cos (2D - l) - 3m\gamma^2 e \cos (2D + l) \\
& \qquad + \tfrac{3}{2} m\gamma^2 \sin (2D - 2F)
\end{aligned}
$$

By multiplying the series given for $\frac{a^3}{r^3}$ by that for $1 - 3\sin^2 U$, and retaining only the non-periodic terms, we obtain the following expression for N (we prefer to write the portion which arises from elliptic values of the coordinates in its finite form),

$$
\begin{aligned}
N = & (1 - e^2)^{-3/2}(1 - 6\gamma^2 + 6\gamma^4) + \tfrac{1}{2} m^2 - \tfrac{9}{32} m^4 + \tfrac{55}{16} m^5 + \tfrac{2159}{96} m^6 + \tfrac{1997}{24} m^7 \\
& + \left[\tfrac{483}{128} m^2 + \tfrac{8595}{256} m^3 + \tfrac{814965}{4096} m^4 + \tfrac{3695598}{4096} m^5\right] e^2 \\
& + \left[\tfrac{3}{4} m^2 + \tfrac{153}{64} m^4 + \tfrac{1485}{32} m^5\right] e'^2 + \left[\tfrac{41}{32} m^2 - \tfrac{69}{64} m^3 - \tfrac{9273}{1024} m^4 - \tfrac{67819}{3072} m^5\right] \gamma^2 \\
& + \left[\tfrac{1845}{64} m^2 + \tfrac{51615}{256} m^3\right] e^4 + \left[\tfrac{713}{64} m^2 + \tfrac{3369}{64} m^3\right] e^2\gamma^2 + \left[-\tfrac{329}{32} m^2 + \tfrac{185}{64} m^3\right] \gamma^4 \\
& + \tfrac{15}{16} m^2 e'^4 + \left[\tfrac{3333}{128} m^2 + \tfrac{185055}{512} m^3\right] e^2 e'^2 + \left[\tfrac{79}{32} m^2 + \tfrac{459}{128} m^3\right] e'^2\gamma^2 \\
& - \left[\tfrac{145}{32} + \tfrac{135}{32} m\right] e^4\gamma^2 + \left[\tfrac{195}{8} - \tfrac{1041}{32} m\right] e^2\gamma^4 \\
& \qquad + \left[\tfrac{963}{512} m^2 + \tfrac{4095}{256} m^3\right] \frac{a^2}{a'^2} + \left[\tfrac{75}{32} - \tfrac{875}{32} m\right] e'^2 \frac{a^2}{a'^2}
\end{aligned}
$$

The expression, given for this quantity in *Astronomical Papers*, Vol. IV, p. 515, and intended to be exact to terms of the sixth order, is rendered erroneous by a misprint and some omissions in Pontécoulant.

The principal term of nutation arises from the term

$$\frac{m}{r^3} \sin \omega' \cos \omega' \sin 2U \sin (V + \psi)$$

of $\frac{m}{r^3} \sin \delta$, and, using always Delaunay's notation, has the argument $\psi + h$. Hence we must find the coefficient of $\cos (\psi + h)$ in $\frac{a^3}{r^3} \sin 2U \sin (V + \psi)$, which we will denote by N'. The means of deriving this coefficient I have given (*Astronomical Papers*, Vol. III, p. 233). We have only to sum the parts of the term (39), and double and reverse the sign of the whole to have the value of N'. But the approximation is carried only to terms of the fifth order inclusive. It seems desirable to go one order further, and therefore I propose to complete the nine sources of terms worked out pp. 216-229 by the addition of terms which are of the sixth order. It is necessary then to add to the development of R at p. 216 the terms in three dimensions of $\delta \mathrm{r}$, δV and δU. However, it is discovered that two only of these contribute anything to the desired coefficient, viz.:

$$\frac{1}{2} \frac{d^3 R_0}{d\mathrm{r}^2 dU} \delta \mathrm{r}^2 \delta U + \frac{d^3 R_0}{d\mathrm{r}\, dV dU} \delta \mathrm{r} \delta V \delta U$$

We now repeat the nine divisions of pp. 217-229 with the necessary extensions, so far as they are needed for the computation of this term:

I. For the two factors it suffices to put

$$\begin{aligned}
\frac{dR_0}{d\mathrm{r}} &= [-3\gamma + \tfrac{15}{2}\gamma^3 - \tfrac{3}{2}\gamma e^2] \cos (\psi + h) - 3\gamma e \cos (\psi + h \pm l) \\
&\quad + 3\gamma \cos (\psi + h + 2F) - 3\gamma e \cos (\psi + h + 2F - l) \\
\delta \mathrm{r} &= (\tfrac{1}{6} + \tfrac{1}{4} e'^2)\, m^2 - \tfrac{179}{288} m^4 - \tfrac{97}{48} m^5 - (\tfrac{7}{12} m^2 + \tfrac{285}{64} m^3)\, e \cos l \\
&\quad + (-5e^2 + \tfrac{135}{8} e^2 m + 2m^2 - 3m^3)\, \gamma^2 \cos 2F \\
&\quad + (-\tfrac{5}{2} + \tfrac{135}{16} m)\, \gamma^2 e \cos (2F - l)
\end{aligned}$$

The coefficient of $\cos (\psi + h)$ in the product is

$$-\tfrac{15}{4}\gamma^3 e^2 + \tfrac{405}{32}\gamma^3 e^2 m + (-\tfrac{1}{2}\gamma + \tfrac{17}{4}\gamma^3 + \tfrac{3}{2}\gamma e^2 - \tfrac{3}{4}\gamma e'^2)\, m^2 + (-\tfrac{9}{2}\gamma^3 + \tfrac{855}{64}\gamma e^2)\, m^3 + \tfrac{179}{96}\gamma m^4 + \tfrac{97}{16}\gamma m^5$$

II. For the two factors it suffices to put

$$\begin{aligned}
\frac{dR_0}{dV} &= -\gamma \sin (\psi + h + 2F) + \tfrac{1}{2}\gamma e \sin (\psi + h + 2F - l) \\
\delta V &= [-\tfrac{25}{4}\gamma^2 e^2 + \tfrac{675}{32}\gamma^2 e^2 m + \tfrac{11}{4}\gamma^2 m^2 - \tfrac{231}{64}\gamma^2 m^3] \sin 2F \\
&\quad + [-5\gamma^2 e + \tfrac{135}{8}\gamma^2 e m] \sin (2F - l)
\end{aligned}$$

The coefficient of $\cos (\psi + h)$ in the product is

$$\tfrac{15}{8}\gamma^3 e^2 - \tfrac{405}{64}\gamma^3 e^2 m - \tfrac{11}{8}\gamma^3 m^2 + \tfrac{231}{128}\gamma^3 m^3$$

III. For the two factors it suffices to put

$$\frac{dR_0}{dU} = -\tfrac{5}{2}e\sin(\psi+h+F+l) - \tfrac{1}{2}e\sin(\psi+h+F-l) - \tfrac{5}{8}e^2\sin(\psi+h+F-2l)$$

$$\begin{aligned}\delta U = {} & [-\tfrac{1}{2}\gamma em^2 - \tfrac{21}{8}\gamma em^3]\sin(F+l) + [-5\gamma^3 e + \tfrac{5}{4}\gamma e^3 + (\tfrac{185}{8}\gamma^3 e - \tfrac{185}{32}\gamma e^3)m + \tfrac{189}{32}\gamma em^2 \\ & + \tfrac{375}{32}\gamma em^3]\sin(F-l) + [-\tfrac{5}{4}\gamma e^2 + \tfrac{185}{32}\gamma e^2 m]\sin(F-2l)\end{aligned}$$

The coefficient of $\cos(\psi + h)$ in the product is

$$\tfrac{5}{4}\gamma^3 e^2 + \tfrac{5}{64}\gamma e^4 - (\tfrac{185}{32}\gamma^3 e^2 + \tfrac{185}{512}\gamma e^4)m - \tfrac{109}{128}\gamma e^2 m^2 + \tfrac{45}{128}\gamma e^2 m^3$$

IV. For the two factors it suffices to put

$$\tfrac{1}{2}\frac{d^2R_0}{d\mathrm{r}^2} = -3\gamma\cos(\psi+h) - \tfrac{3}{2}\gamma e\cos(\psi+h\pm l)$$

$$\delta\mathrm{r}^2 = \tfrac{225}{128}e^2m^2 + \tfrac{8765}{256}e^2m^3 + \tfrac{19}{36}m^4 + \tfrac{19}{6}m^5 + \tfrac{15}{8}em^3\cos l$$

The coefficient of $\cos(\psi + h)$ in the product is

$$-\tfrac{675}{128}\gamma e^2m^2 - \tfrac{12015}{256}\gamma e^2m^3 - \tfrac{19}{12}\gamma m^4 - \tfrac{19}{2}\gamma m^5$$

V. For the two factors it suffices to put

$$\tfrac{1}{2}\frac{d^2R_0}{dV^2} = \tfrac{1}{2}\gamma\cos(\psi+h) + \tfrac{3}{4}\gamma e\cos(\psi+h\pm l) - \tfrac{1}{2}\gamma\cos(\psi+h+2F)$$

$$\begin{aligned}\delta V^2 = {} & (\tfrac{225}{32}e^2 + \tfrac{9}{2}e'^2)m^2 + (\tfrac{8715}{128}e^2 - \tfrac{33}{32}\gamma^2)m^3 + \tfrac{121}{128}m^4 + \tfrac{649}{96}m^5 \\ & + \tfrac{165}{32}em^3\cos l + \tfrac{99}{32}\gamma^2m^3\cos 2F\end{aligned}$$

The coefficient of $\cos(\psi + h)$ in the product is

$$(\tfrac{225}{64}\gamma e^2 + \tfrac{9}{4}\gamma e'^2)m^2 + (\tfrac{9705}{256}\gamma e^2 - \tfrac{165}{128}\gamma^3)m^3 + \tfrac{121}{256}\gamma m^4 + \tfrac{649}{192}\gamma m^5$$

VI. For the two factors it suffices to put

$$\tfrac{1}{2}\frac{d^2R_0}{dU^2} = -2\gamma\cos(\psi+h+2F) + 2\gamma\cos(\psi+h)$$

$$\delta U^2 = \tfrac{9}{32}\gamma^2m^2 + \tfrac{75}{64}\gamma^2m^3 + \tfrac{33}{32}\gamma^2m^3\cos 2F$$

The coefficient of $\cos(\psi + h)$ in the product is

$$\tfrac{9}{16}\gamma^3m^2 + \tfrac{21}{16}\gamma^3m^3$$

VII. For the two factors it suffices to put

$$\frac{d^2R_0}{d\mathrm{r}\,dV} = -3\gamma\sin(\psi+h+2F) + 3\gamma e\sin(\psi+h\pm l)$$

$$\delta\mathrm{r}\,\delta V = -\tfrac{75}{128}em^3\sin l - \tfrac{9}{8}\gamma^2m^3\sin 2F$$

The coefficient of $\cos(\psi + h)$ in the product is

$$\tfrac{27}{16}\gamma^3m^3$$

VIII. For the factors it suffices to put

$$\frac{d^2R_0}{d\mathrm{r}dU} = [-3 + 12\gamma^2 + \tfrac{3}{2}e^2]\sin(\psi + h + F) - 6e\sin(\psi + h + F + l)$$
$$- \tfrac{15}{2}\gamma^2\sin(\psi + h - F)$$
$$\delta\mathrm{r} = [\tfrac{15}{4}e^2m + (1 - 2\gamma^2 + \tfrac{189}{16}e^2 - \tfrac{5}{2}e'^2)m^2 + \tfrac{19}{6}m^3 + \tfrac{131}{18}m^4]\cos 2D$$
$$- \tfrac{3}{2}e'm^3\cos l' + \tfrac{7}{2}e'm^2\cos(2D - l')$$
$$+ \tfrac{1}{2}e'm^2\cos(2D + l') + [\tfrac{15}{8}em + \tfrac{187}{32}em^2]\cos(2D - l)$$
$$+ \tfrac{33}{16}em^2\cos(2D + l) - 3\gamma^2m^2\cos(2D - 2F)$$

and the necessary terms of δU may be selected from the expression for U^1 previously given. The product of the two latter factors is

$$\delta\mathrm{r}\delta U = [-\tfrac{45}{64}\gamma e^2m^2 + (-\tfrac{3}{8}\gamma - \tfrac{9}{8}\gamma^3 + \tfrac{117}{64}\gamma e^2 - \gamma e'^2)m^3 - \tfrac{41}{32}\gamma m^4 - \tfrac{3829}{1536}\gamma m^5]\sin F$$
$$- \tfrac{63}{64}\gamma em^3\sin(F + l)$$

And the coefficient of $\cos(\psi + h)$ in the product of the three factors is

$$\tfrac{185}{128}\gamma e^2m^2 + (\tfrac{9}{16}\gamma - \tfrac{68}{32}\gamma^3 - \tfrac{9}{128}\gamma e^2 + \tfrac{3}{2}\gamma e'^2)m^3 + \tfrac{123}{64}\gamma m^4 + \tfrac{3829}{1024}\gamma m^5$$

IX. For the factors it suffices to put

$$\frac{d^2R_0}{dVdU} = [-1 + 4\gamma^2 - \tfrac{1}{2}e^2]\cos(\psi + h + F) - \tfrac{5}{2}e\cos(\psi + h + F + l)$$
$$- \tfrac{1}{2}e\cos(\psi + h + F - l) - \tfrac{5}{2}\gamma^2\cos(\psi + h - F)$$
$$\delta V = [(-\tfrac{3}{4}\gamma^2 + \tfrac{75}{16}e^2)m + (\tfrac{11}{8} - \tfrac{47}{16}\gamma^2 + \tfrac{1101}{64}e^2 - \tfrac{55}{16}e'^2)m^2$$
$$+ \tfrac{59}{12}m^3 + \tfrac{893}{72}m^4]\sin 2D$$
$$- 3e'm\sin l' + \tfrac{77}{16}e'm^2\sin(2D - l') - \tfrac{11}{16}e'm^2\sin(2D + l')$$
$$+ [\tfrac{15}{4}em + \tfrac{263}{16}em^2]\sin(2D - l) + \tfrac{17}{8}em^3\sin(2D + l)$$
$$+ [\tfrac{9}{4}\gamma^2m - \tfrac{11}{2}\gamma^2m^2]\sin(2D - 2F)$$

and as before the necessary terms of δU may be taken from U_1 given previously. The product of the two latter factors is

$$\delta V\delta U = [(\tfrac{9}{16}\gamma^2 + \tfrac{1845}{128}\gamma e^2 + \tfrac{9}{4}\gamma e'^2)m^2 + (\tfrac{33}{64}\gamma - \tfrac{255}{128}\gamma^3 + \tfrac{8288}{64}\gamma e^2 + \tfrac{355}{64}\gamma e'^2)m^3$$
$$+ \tfrac{989}{256}\gamma m^4 + \tfrac{219951}{12288}\gamma m^5]\cos F + \tfrac{87}{16}\gamma em^3\cos(F + l)$$
$$+ [\tfrac{45}{32}\gamma em^2 + \tfrac{195}{16}\gamma em^3]\cos(F - l)$$

And the coefficient of $\cos(\psi + h)$ in the product of the three factors is

$$-(\tfrac{9}{32}\gamma^3 + \tfrac{1985}{256}\gamma e^2 + \tfrac{9}{8}\gamma e'^2)m^2 - (\tfrac{33}{128}\gamma - \tfrac{177}{128}\gamma^3 + \tfrac{19119}{256}\gamma e^2 + \tfrac{355}{128}\gamma e'^2)m^3$$
$$- \tfrac{989}{512}\gamma m^4 - \tfrac{219951}{24576}\gamma m^5$$

X. For the factors it suffices to put

$$\tfrac{1}{2}\frac{d^3R_0}{d\mathrm{r}^2dU} = -3\sin(\psi + h + F), \quad \delta\mathrm{r}^2\delta U = -\tfrac{1}{8}\gamma m^5\sin F$$

The coefficient of $\cos(\psi + h)$ in the product is $\tfrac{3}{16}\gamma m^5$

XI. For the factors it suffices to put

$$\frac{d^3 R_0}{d\mathrm{r}\, dV dU} = -3 \cos(\psi + h + F), \quad \delta\mathrm{r}\, \delta V \delta U = \tfrac{11}{128}\gamma m^5 \cos F$$

The coefficient of $\cos(\psi + h)$ in the product is $-\frac{33}{256}\gamma m^5$

By adding the terms arising from these eleven sources, and multiplying the sum by -2, and joining the result to the part of N' arising from the substitution of elliptic values for the coordinates, we obtain the following expression.

$$\begin{aligned} N' = {} & 2(1-e^2)^{-\frac{3}{2}}\gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}} + [m^2 - \tfrac{39}{64}m^3 - \tfrac{381}{256}m^4 + \tfrac{128023}{12288}m^5]\gamma \\ & + [\tfrac{1949}{128}m^2 + \tfrac{17987}{128}m^3]\, e^2\gamma + [-\tfrac{3}{4}m^2 + \tfrac{163}{64}m^3]\, e'^2\gamma \\ & + [-\tfrac{101}{16}m^2 + \tfrac{281}{64}m^3]\gamma^3 + [\tfrac{5}{4} - \tfrac{135}{32}m]\, e^2\gamma^3 + [-\tfrac{5}{32} + \tfrac{135}{256}m]\, e^4\gamma \end{aligned}$$

For deriving the numerical values of N and N' we put

$$m = 0.07480133, \quad e = 0.0548993, \quad \gamma = 0.04488663,$$
$$e' = 0.01677106, \quad \frac{a}{a'} = 0.002576$$

all from Delaunay, except the last, which is modified to correspond to the value 8″.81 for the solar parallax. Then we have

$$N = 0.99241874 + 279762^2 + 7045^4 + 4949^5 + 2420^6 + 842^7 + 308 = 0.995372$$
$$N' = 0.08972677 + 25115^3 - 1145^4 + 648^5 + 914^6 + 491 = 0.089987$$

The first terms of these expressions are the values which result from the employment of elliptic values for the moon's coordinates; the following terms are the aggregates of the terms of each order, the figure above denoting the order; and the last terms, obtained by induction, are added to complete the series. In Peters's *Numerus Constans Nutationis*, p. 27, 0.99212 and 0.08967 are given as the values of these two quantities.

In seeking readily applicable expressions for P the constant of luni-solar precession and the coefficient of the principal term of the lunar nutation of the obliquity N, we call to mind that the first is usually given for a tropical year, as finding a more general use in the theory of the stars. Moreover, for the constants $\frac{m}{a^3}$ and $\frac{m'}{a'^3}$, putting M for the mass of the earth, we substitute their equivalents in terms of the times of revolution of the moon and sun about the earth. Let us suppose then that T, T^{I}, T^{II}, T^{III} and T^{IV} denote severally the times of the rotation of the earth on its axis, of revolution of the earth about the sun, of the moon about the earth, the

tropical revolution of the moon's nodes and the tropical year. Then we shall have the following expressions for P and N:

$$\mathsf{P} = 1944000'' \frac{T^{\mathrm{IV}}}{T^{\mathrm{I}}}\frac{T}{T^{\mathrm{I}}}\frac{C-A}{C}\left[\frac{m'}{M+m'}(1-e'^2)^{-\frac{3}{2}} + \frac{m}{M+m}\left(\frac{T^{\mathrm{I}}}{T^{\mathrm{II}}}\right)^2 N\right]\cos\omega'$$

$$\mathsf{N} = \tfrac{3}{2}\frac{m}{M+m}\frac{TT^{\mathrm{III}}}{T^{\mathrm{II}^2}}\frac{C-A}{C}N'\cos\omega'$$

The first factor of P is simply 3π in sexagesimal seconds of arc. All the quantities entering into the right members of these equations are known with precision, except the two ratios $\frac{C-A}{C}$ and $\frac{m}{M+m}$. Let us deduce numerical expressions for P and N for the epoch 1850. We assume

$$\frac{T}{T^{\mathrm{I}}} = \frac{1}{366.25636}, \quad \frac{T^{\mathrm{IV}}}{T^{\mathrm{I}}} = \frac{365.2422}{365.25636}, \quad \frac{T^{\mathrm{II}}}{T^{\mathrm{I}}} = 0.07480133$$

$$\frac{T^{\mathrm{III}}}{T^{\mathrm{II}}} = \frac{6798.353}{27.321661}, \quad \frac{m'}{M+m'} = \frac{327214}{327215}, \quad \omega' = 23^\circ 27' 31''.65$$

For convenience, writing the common logarithms of the numerical factors in brackets, we have

$$\mathsf{P} = [3.6876097]\frac{C-A}{C} + [5.9375945]\frac{m}{M+m}\frac{C-A}{C}$$

$$\mathsf{N} = [5.3654318]\frac{m}{M+m}\frac{C-A}{C}$$

These expressions give the quantities immediately in seconds of arc. When it is desired to get $\frac{C-A}{C}$ and $\frac{m}{M+m}$ from given values of P and N, the following formulas serve:

$$\frac{C-A}{C} = [6.3123903]\,\mathsf{P} - [6.8845530]\,\mathsf{N}$$

$$\frac{m}{M+m} = \frac{[8.3221779]\,\mathsf{N}}{\mathsf{P} - [0.05721627]\,\mathsf{N}}$$

As an illustration let us take Struve's value of the luni-solar precession and Peter's value of the nutation constant. Reduced to the epoch 1850, they may be set down at $50''.38227$ and $9''.22355$. Then,

$$\frac{C-A}{C} = 0.003272995, \quad \frac{m}{M+m} = \frac{1}{82.31500}$$

MEMOIR No. 54.

On Intermediate Orbits.

(Annals of Mathematics, Vol. VIII, pp. 1–20, 1893.)

The assumption of the Keplerian ellipse as the first approximation to the motion of a planet leads to some inconveniences, the worst of which, perhaps, is that the mean longitude at the epoch suffers a perturbation proportional to the time, and thus the mean motion in longitude is not the same in the perturbed and unperturbed orbits. In the case of Saturn this perturbation amounts to 110″ per year. This inconvenience would be avoided if, developing the perturbative function in a series proceeding according to the cosines of multiples of the angle H contained at the sun by the radii of the planets thus

$$R = \varphi_0(r, r') + \varphi_1(r, r') \cos H + \varphi_2(r, r') \cos 2H + \ldots,$$

we should suppose $\Phi_0(r, r')$ annexed to the potential of the attraction exerted by the sun and the resulting differential equations integrated and the expressions of the coordinates thus deduced regarded as the first approximation. Then the following approximations could be obtained by the method of variation of the elements or otherwise by attributing to R the value

$$R = \varphi_1(r, r') \cos H + \varphi_2(r, r') \cos 2H + \ldots$$

In this way of approaching the question, the first approximation to the motion of a system of planets revolving about their central body would involve a potential function containing the radii of the planets alone, without their longitudes or latitudes; thus the force acting on each planet would be directed along its radius, but would be a function of all the radii. The orbits are then all plane curves whose planes pass through the central body, and thus we know precisely what functions the latitudes and the reduced longitudes are of the orbit longitudes. Also there is equable description of areas by the radii, and when the latter are known functions of the time the determination of the orbit longitudes is reduced to quadratures. Thus the question is narrowed to the finding of the radii as functions of the time.

Let $r, r', \ldots. v, v', \ldots.$ denote the radii and orbit longitudes of a system of planets, $m, m', \ldots.$ their masses and M the mass of the central body; $\mu = M + m$, $\mu' = M + m', \ldots.$; $R, R', \ldots.$ the perturbative functions for $m, m', \ldots.$ Then the differential equations for determining the radii are

$$\frac{d^2r}{dt^2} - r\frac{dv^2}{dt^2} + \frac{\mu}{r^2} = \frac{\partial R}{\partial r},$$
$$\frac{d^2r'}{dt^2} - r'\frac{dv'^2}{dt^2} + \frac{\mu'}{r'^2} = \frac{\partial R'}{\partial r'},$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

But, $h, h', \ldots.$ being constants, we have

$$\frac{dv}{dt} = \frac{h}{r^2}, \quad \frac{dv'}{dt} = \frac{h'}{r'^2}, \ldots.$$

The functions $R, R', \ldots$ are to be reduced to

$$R = m'\varphi(r, r') + m''\varphi(r, r'') + \ldots,$$
$$R' = m\varphi(r', r) + m''\varphi(r', r'') + \ldots,$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

where we have the relation $\phi(x, y) = \phi(y, x)$. To evaluate the function ϕ, let the symbol $\mathfrak{M}$ denote the operation of taking the arithmetico-geometrical mean, then

$$\varphi(r, r') = \frac{1}{\mathfrak{M}(r' + r, r' - r)} = \frac{1}{\mathfrak{M}(r', \sqrt{r'^2 - r^2})},$$

where it is supposed that $r' > r$. The two quantities under the symbol $\mathfrak{M}$ may be brought still nearer to equality by continuing the operation, and writing $\sqrt{r'} = \alpha$, $\sqrt[4]{r'^2 - r^2} = \beta$, we have

$$\varphi(r, r') = \frac{1}{\mathfrak{M}[\frac{1}{2}(\alpha^2 + \beta^2), \alpha\beta]} = \frac{1}{\mathfrak{M}[\frac{1}{4}(\alpha + \beta)^2, \sqrt{\frac{1}{2}\alpha\beta(\alpha^2 + \beta^2)}]}.$$

And if r and r' are not too near each other a sufficiently approximate value is

$$\varphi(r, r') = \frac{2\sqrt[4]{2}}{(\alpha + \beta)\sqrt[4]{\alpha\beta(\alpha^2 + \beta^2)}}.$$

If we write

$$\Omega = \frac{\mu m}{r} + \frac{\mu' m'}{r'} + \ldots - \tfrac{1}{2}\frac{mh^2}{r^2} - \tfrac{1}{2}\frac{m'h'^2}{r'^2} - \ldots$$
$$+ mm'\varphi(r, r') + mm''\varphi(r, r'') + m'm''\varphi(r', r'') + \ldots,$$

the differential equations determining $r, r', \ldots.$ become

$$m\frac{d^2r}{dt^2} = \frac{\partial \Omega}{\partial r}, \quad m'\frac{d^2r'}{dt^2} = \frac{\partial \Omega}{\partial r'}, \ldots.$$

From which we derive the integral equation

$$m\frac{dr^2}{dt^2} + m'\frac{dr'^2}{dt^2} + \ldots = 2(\Omega + C),$$

C being the arbitrary constant. The left member of this cannot be a negative quantity. Consequently, if we construct in a space of n dimensions, n being the number of planets, the surface whose equation is

$$\Omega + C = 0,$$

the representative point P, whose coordinates are the values of the variables $r, r', \ldots$, must lie on the positive side of this surface. In such a system of planets as our solar system composed of the eight major planets, at least one fold of this surface is closed; and it is within this that the representative point P always falls. Thus, in this case, we are able to set definite inferior and superior limits to $r, r', \ldots$, which, however, in general are not the *minima-minimorum* or *maxima-maximorum* values of the variables, since the point P only attains the surface when all the planets are together on the lines of their apsides.

A particular solution of the foregoing system of differential equations can be obtained in the following way: Put

$$\frac{\partial \Omega}{\partial r} = 0, \quad \frac{\partial \Omega}{\partial r'} = 0, \ldots,$$

and solve these equations, regarding $r, r', \ldots$ as the unknowns. Let

$$r = a, \quad r' = a', \ldots$$

be a system of values satisfying them. Then the last equations evidently satisfy the differential equations. When this process is applied to the solar system the values found for the a do not greatly differ from the mean distances of the several planets from the sun. The excessive smallness of the m relatively to M brings it about that there is but one real solution; and quite approximately, by neglecting in Ω all the terms dependent on the functions ϕ, we get

$$a = \frac{h^2}{\mu}, \quad a' = \frac{h'^2}{\mu'}, \ldots.$$

By substituting these values in the terms arising from these functions we can arrive at more exact values for the a; and this process can be repeated as often as is deemed necessary. Evidently all this comes to the adjustment of the planets at such distances from the sun that for the given values of the m and the h the centripetal force may be equal to the centrifugal force in each case, thus rendering it possible that all the orbits may be circles.

Desiring now to find inferior and superior limits to the particular variable $r^{(i)}$, in the surface $\Omega + C = 0$ we suppose this variable to be a function of all the rest, and thus that when it arrives at a maximum or minimum value we have

$$\frac{\partial r^{(i)}}{\partial r} = 0, \quad \frac{\partial r^{(i)}}{\partial r'} = 0, \ldots,$$

except that $\dfrac{\partial r^{(i)}}{\partial r^{(i)}} = 1$. In order to find limits for the values of $r^{(i)}$ we take the group of equations

$$\frac{\partial \Omega}{\partial r} = 0, \quad \frac{\partial \Omega}{\partial r'} = 0, \ldots,$$

and in it replace the equation $\dfrac{\partial \Omega}{\partial r^{(i)}} = 0$ by the equation $\Omega + C = 0$. Solving these equations, regarding $r, r', \ldots$ as the unknowns, we shall always, in cases like our solar system, be able to find two sets of corresponding real values for these quantities such that the two values of $r^{(i)}$ shall contain between them the observed value of this radius from which the value of C was derived. And no other roots can be found which satisfy this condition. In ordinary cases these roots are non-multiple and there is but one solution to the problem.

For the sake of illustration, limiting ourselves to two planets, we may take a hypothetical case, suggested by Jupiter and Saturn. Assume as the values of the planetary masses

$$m = \frac{1}{1047.355}, \quad m' = \frac{1}{3501.6}.$$

For Leverrier's Tables for the time 1875, Jan. 1.0, we get

$$\log r = 0.7367630, \quad \log r' = 0.9953504;$$

and, with the Julian year as the unit of time,

$$\log \frac{dr}{dt} = 8.0841634n, \quad \log \frac{dr'}{dt} = 8.8776073.$$

From the same source the values of the osculating elements necessary for the computations are

$$\log a = 0.7162505, \ \log e = 8.6865018, \ \log a' = 0.9794334, \ \log e' = 8.7291462.$$

Accommodated to the units of length and time we employ,

$$\log \mu = 1.5967563, \ \log \mu' = 1.5964658, \ \log m = 8.5762479, \ \log m' = 8.0520753.$$
$$\log h^2 = 2.3119809, \ \log h'^2 = 2.5746499.$$

When these numerical values are substituted in the equation $\Omega + C = 0$, t is found that $C = -0.166513246$. Employing brackets to denote numbers corresponding to common logarithms, the equation $\Omega + C = 0$ becomes in this case

$$[0.1730042]\, r^{-1} - [0.5871988]\, r^{-2} + [9.6485411]\, r'^{-1} - [0.3256952]\, r'^{-2}$$
$$+ [6.6283232]\, r'^{-1} \psi\left(\frac{r}{r'}\right) = 0.166513246\,.$$

And the equations $\dfrac{\partial \Omega}{\partial r} = 0$, $\dfrac{\partial \Omega}{\partial r'} = 0$ become

$$2\,[0.5871988]\, r^{-1} - [0.1730042] + [6.6283232] \frac{r^2}{r'^2} \psi'\left(\frac{r}{r'}\right) = 0\,,$$
$$2\,[0.3256952]\, r'^{-2} - [9.6485411] - [6.6283232] \left[\psi\left(\frac{r}{r'}\right) + \frac{r}{r'} \psi'\left(\frac{r}{r'}\right)\right] = 0\,,$$

where ψ denotes the same function as Laplace's $b_{\frac{1}{2}}^{(0)}$ is of a.

The solution of the first and third of these equations gives inferior and superior limits for r, the solution of the first and second the same for r'. Employing the tentative process we obtain as the corresponding values of the logarithms of the radii in the four solutions

$$\log r = 0.6926819, \quad 0.7390339, \quad 0.7152497, \quad 0.7152334;$$
$$\log r' = 0.9776575, \quad 0.9776183, \quad 0.9238410, \quad 1.0390527.$$

The solution of the second and third equations gives the values of the radii for which the planets could describe uniformly circles. These are

$$\log a = 0.7152396\,, \quad \log a' = 0.9776403\,.$$

If we assume that the maximum eccentricities of Jupiter and Saturn are respectively 0.0594902 and 0.0845677, the minimum and maximum values of $\log r$ and $\log r'$ are severally

$$\log r = 0.6895907\,, \quad \log r = 0.7413343\,, \quad \log r' = 0.9411223\,, \quad \log r' = 1.0147527\,.$$

The differences between the former values and these are, of course, due to the neglect of the terms of R involving the angle H.

For the purpose of graphically exhibiting the numerical results just obtained we may suppose a system of rectangular axes to be drawn with r and r' as coordinates. The four sets of corresponding values of these variables give four points on the oval which constitutes one of the branches of the curve having $\Omega + C = 0$ as its equation. These suffice to roughly indicate the course of the oval within which the representative point P, exhibiting the simultaneous values of the radii, always keeps. The position for which both radii are constant is well towards the centre of the oval. If we

draw the right line bisecting the right angle at the origin formed by the axes of coordinates, the intersection of this line or its nonintersection with the oval will show whether there is intrusion or not of the spheres of the planets on each other. In the case here treated there is no intrusion; each planet maintains its character as exterior or interior; consequently the distance between them can never vanish, and stability of motion is assured.

The given example shows that there must exist a large domain in which it is possible to expand the function Ω in a series of powers and products of the deviations of the radii from their mean values. Here we will limit the exposition to the case of two planets, as the formulæ given can be readily extended when there are more. Let us put

$$r = a + \frac{x}{\sqrt{m}}, \quad r' = a' + \frac{x'}{\sqrt{m'}},$$

where the second terms are small relatively to the first. The differential equations then become

$$\frac{d^2x}{dt^2} = \frac{\partial \Omega}{\partial x}, \quad \frac{d^2x'}{dt^2} = \frac{\partial \Omega}{\partial x'}.$$

In developing Ω in a series of ascending powers and products of x and x' we will stop with terms of three dimensions. The constant term can be omitted as its retention would only serve to modify the value of the arbitrary constant C. As a and a' are corresponding roots of the equations $\frac{\partial \Omega}{\partial r} = 0, \frac{\partial \Omega}{\partial r'} = 0$, the terms of one dimension vanish. Also the term xx' can be supposed absent, for, if present, it could be removed by a linear and orthogonal transformation of variables. We write therefore

$$\Omega = -\tfrac{1}{2}a^2x^2 - \tfrac{1}{2}a'^2x'^2 + \tfrac{1}{3}cx^3 + bx^2x' + b'xx'^2 + \tfrac{1}{3}c'x'^3,$$

where a, a', b, b', c, c', are constants.*

Adopting two constants n and n' at present left indeterminate, but to be determined hereafter so as to fulfil certain conditions, the differential equations can be written

$$\frac{d^2x}{dt^2} + n^2x = (n^2 - a^2)\,x + cx^2 + 2bxx' + b'x'^2,$$

$$\frac{d^2x'}{dt^2} + n'^2x' = (n'^2 - a'^2)\,x' + c'x'^2 + 2b'xx' + bx^2.$$

* The reader is asked not to confound these a and a' with the quantities previously donated by the same letters.

Before integrating these it will be well to ascertain superior and inferior limits for x and x'. Those for x are determined by the equations

$$C - \tfrac{1}{2}a^2x^2 - \tfrac{1}{2}a'^2x'^2 + \tfrac{1}{3}cx^3 + bx^2x' + b'xx'^2 + \tfrac{1}{3}c'x'^3 = 0\,,$$
$$-a'^2x' + c'x'^2 + 2b'xx' + bx^2 = 0\,.$$

If we put $\eta = \pm \frac{\sqrt{2C}}{a}$, this will be a small quantity of the same order as x and the limiting values of x can be developed in an infinite series of powers of η. Thus

$$x = \eta + \tfrac{1}{3}\frac{c}{a^2}\eta^2 + \left[\tfrac{1}{2}\frac{b^2}{a^2a'^2} + \tfrac{5}{18}\frac{c^2}{a^4}\right]\eta^3 + \left[\frac{b^2b'}{a^2a'^4} + \frac{b^2c}{a^4a'^2} + \tfrac{8}{27}\frac{c^3}{a^6}\right]\eta^4 + \ldots,$$

where we take η positively for the superior limit and negatively for the inferior. The corresponding value of x' can be derived from the equation

$$x' = \frac{b}{a'^2}x^2 + 2\frac{bb'}{a'^4}x^3 + \frac{b^2c'}{a'^6}x^4 + \ldots.$$

The limits for x', by putting $\eta = \pm \frac{\sqrt{2C}}{a'}$, can be obtained from those of x by removing the accent from the symbols which there have it and applying it to those which are destitute of it.

By neglecting the right-hand members of the differential equations as last written their integrals are simply

$$x = e\cos\varphi\,, \quad x' = e'\cos\varphi'\,,$$

e and e' being arbitrary constants and ϕ and ϕ' are arguments increasing proportionably to t so that $\frac{d\phi}{dt} = n$, $\frac{d\phi'}{dt} = n'$. By substituting these expressions for x and x',

$$cx^2 + 2bxx' + b'x'^2 = \tfrac{1}{2}(ce^2 + b'e'^2) + \tfrac{1}{2}ce^2\cos 2\varphi + bee'\cos(\varphi + \varphi')$$
$$+ bee'\cos(\varphi - \varphi') + \tfrac{1}{2}b'e'^2\cos 2\varphi'\,,$$
$$c'x'^2 + 2b'xx' + bx^2 = \tfrac{1}{2}(c'e'^2 + be^2) + \tfrac{1}{2}c'e'^2\cos 2\varphi' + b'ee'\cos(\varphi + \varphi')$$
$$+ b'ee'\cos(\varphi - \varphi') + \tfrac{1}{2}be^2\cos 2\varphi\,.$$

Whence it follows that x contains the additional terms of the second order with respect to e and e'

$$x = \frac{1}{2n^2}(ce^2 + b'e'^2) - \frac{1}{6n^2}ce^2\cos 2\varphi + \frac{bee'}{n^2 - (n + n')^2}\cos(\varphi + \varphi')$$
$$+ \frac{bee'}{n^2 - (n - n')^2}\cos(\varphi - \varphi') + \tfrac{1}{2}\frac{b'e'^2}{n^2 - 4n'^2}\cos 2\varphi'\,,$$

and x' terms which are obtained from those of x by simply interchanging the accents.

Pushing the approximation to terms of the third order, x^2 is found to contain the following terms of this order:

$$x^2 = [\tfrac{5}{6}ce^2 + b'e'^2]\frac{e}{n^2}\cos\varphi + \frac{2be^2e'}{4n^2 - n'^2}\cos\varphi' - \tfrac{1}{6}\frac{ce^3}{n^2}\cos 3\varphi$$
$$+ \frac{be^2e'}{n^2 - (n + n')^2}\cos(2\varphi + \varphi') + \frac{be^2e'}{n^2 - (n - n')^2}\cos(2\varphi - \varphi')$$
$$+ \tfrac{1}{2}\frac{b'ee'^2}{n^2 - 4n'^2}\cos(\varphi + 2\varphi') + \tfrac{1}{2}\frac{b'ee'^2}{n^2 - 4n'^2}\cos(\varphi - 2\varphi').$$

The similar terms of x'^2 are obtained from those of x^2 by interchanging the accents. The terms of the third order of xx' are

$$xx' = \left[\tfrac{1}{2}\frac{c'e'^2 + be^2}{n'^2} + \frac{b(e'^2 - \frac{1}{4}e^2)}{4n^2 - n'^2}\right]e\cos\varphi + \left[\tfrac{1}{2}\frac{ce^2 + b'e'^2}{n^2} + \frac{b'(e^2 - \frac{1}{4}e'^2)}{4n'^2 - n^2}\right]e'\cos\varphi'$$
$$+ \tfrac{1}{4}\frac{be^3}{n'^2 - 4n^2}\cos 3\varphi + \tfrac{1}{4}\frac{b'e'^3}{n^2 - 4n'^2}\cos 3\varphi'$$
$$- \left[\frac{b'}{2n' + n} + \frac{c}{6n}\right]\frac{e^2e'}{2n}\cos(2\varphi + \varphi') + \left[\frac{b'}{2n' - n} - \frac{c}{6n}\right]\frac{e^2e'}{2n}\cos(2\varphi - \varphi')$$
$$- \left[\frac{b}{2n + n'} + \frac{c'}{6n'}\right]\frac{ee'^2}{2n'}\cos(\varphi + 2\varphi') + \left[\frac{b}{2n - n'} - \frac{c'}{6n'}\right]\frac{ee'^2}{2n'}\cos(\varphi - 2\varphi').$$

The coefficients $n^2 - a^2$ and $n'^2 - a'^2$ are of the second order, hence limiting ourselves to terms of the third order

$$(n^2 - a^2)\,x = (n^2 - a^2)\,e\cos\varphi, \quad (n'^2 - a'^2)\,x' = (n'^2 - a'^2)\,e'\cos\varphi'.$$

The constants e and e' being arbitrary, the coefficients of $\cos\phi$ in x and of $\cos\phi'$ in x' need receive no corrections in the following approximations. Hence the coefficient of $\cos\phi$ in the right member of the first differential equation should vanish, as also that of $\cos\phi'$ in the second. This gives us the two equations

$$n^2 - a^2 + \left[\tfrac{5}{6}\frac{c^2}{n^2} + \frac{b^2}{n'^2} - \tfrac{1}{2}\frac{b^2}{4n^2 - n'^2}\right]e^2$$
$$+ \left[\frac{b'c}{n^2} + \frac{bc'}{n'^2} + \frac{2b^2}{4n^2 - n'^2} + \frac{2b'^2}{4n'^2 - n^2}\right]e'^2 = 0,$$
$$n'^2 - a'^2 + \left[\tfrac{5}{6}\frac{c'^2}{n'^2} + \frac{b'^2}{n^2} - \tfrac{1}{2}\frac{b'^2}{4n'^2 - n^2}\right]e'^2$$
$$+ \left[\frac{b'c}{n^2} + \frac{bc'}{n'^2} + \frac{2b^2}{4n^2 - n'^2} + \frac{2b'^2}{4n'^2 - n^2}\right]e^2 = 0,$$

of which the second is obtained from the first by interchange of accents. To the degree of approximation to which they are pushed they serve to determine the values of n and n' as functions of the constants a, a', b, b', c, c', e, e'. As the next terms which would appear in these equations through the following approximations are of the fourth order with reference to e and e',

the errors of n and n', as determined from these equations, are of the same order.

The remaining terms of the third order in the right member of the first differential equation are

$$\left[\tfrac{1}{2}\frac{b^2}{n'^2-4n^2}-\tfrac{1}{6}\frac{c^2}{n^2}\right]e^3\cos 3\varphi+\left[\tfrac{1}{2}\frac{bb'}{n^2-4n'^2}-\tfrac{1}{6}\frac{b'c'}{n'^2}\right]e'^3\cos 3\varphi'$$
$$-\left[\frac{bc}{2nn'+n'^2}+\tfrac{1}{6}\frac{bc}{n^2}+\frac{bb'}{2nn'+n^2}+\tfrac{1}{2}\frac{bb'}{4n^2-n'^2}\right]e^2e'\cos(2\varphi+\varphi')$$
$$+\left[\frac{bc}{2nn'-n'^2}-\tfrac{1}{6}\frac{bc}{n^2}+\frac{bb'}{2nn'-n^2}-\tfrac{1}{2}\frac{bb'}{4n^2-n'^2}\right]e^2e'\cos(2\varphi-\varphi')$$
$$-\left[\tfrac{1}{2}\frac{b'c}{4n'^2-n^2}+\tfrac{1}{6}\frac{bc'}{n'^2}+\frac{b^2}{2nn'+n'^2}+\frac{b'^2}{2nn'+n^2}\right]ee'^2\cos(\varphi+2\varphi')$$
$$-\left[\tfrac{1}{2}\frac{b'c}{4n'^2-n^2}+\tfrac{1}{6}\frac{bc'}{n'^2}-\frac{b^2}{2nn'-n'^2}-\frac{b'^2}{2nn'-n^2}\right]ee'^2\cos(\varphi-2\varphi').$$

The similar terms of the right member of the second differential equation are obtained by interchange of accents in the preceding expression. The terms of the third order of x are then

$$\left[\frac{c^2}{n^2}+\frac{3b^2}{4n^2-n'^2}\right]\frac{e^3}{48n^2}\cos 3\varphi+\tfrac{1}{6}\left[\frac{b'c'}{n'^2}+\frac{3bb'}{4n'^2-n^2}\right]\frac{e'^3}{9n'^2-n^2}\cos 3\varphi'$$
$$+\left[\frac{bc}{2nn'+n'^2}+\tfrac{1}{6}\frac{bc}{n^2}+\frac{bb'}{2nn'+n^2}+\tfrac{1}{2}\frac{bb'}{4n^2-n'^2}\right]\frac{e^2e'}{3n^2+4nn'+n'^2}\cos(2\varphi+\varphi')$$
$$-\left[\frac{bc}{2nn'-n'^2}-\tfrac{1}{6}\frac{bc}{n^2}+\frac{bb'}{2nn'-n^2}-\tfrac{1}{2}\frac{bb'}{4n^2-n'^2}\right]\frac{e^2e'}{3n^2-4nn'+n'^2}\cos(2\varphi-\varphi')$$
$$+\left[\tfrac{1}{2}\frac{b'c}{4n'^2-n^2}+\tfrac{1}{6}\frac{bc'}{n'^2}+\frac{b^2}{2nn'+n'^2}+\frac{b'^2}{2nn'+n^2}\right]\frac{ee'^2}{4nn'+4n'^2}\cos(\varphi+2\varphi')$$
$$-\left[\tfrac{1}{2}\frac{b'c}{4n'^2-n^2}+\tfrac{1}{6}\frac{bc'}{n'^2}-\frac{b^2}{2nn'-n'^2}-\frac{b'^2}{2nn'-n^2}\right]\frac{ee'^2}{4nn'-4n'^2}\cos(\varphi-2\varphi').$$

The similar terms in x' are obtained from these by interchange of accents.

On many occasions it will be interesting to know the equivalent of C, the arbitrary constant attached to the integral equation, in terms of a, a', b, b', c, c', e, e'. To obtain this we compute the non-periodic terms of x^2, $\frac{dx^2}{dt^2}$, x^3, x^2x'. The non-periodic terms of x'^2, $\frac{dx'^2}{dt^2}$, x'^3, x'^2x can be derived from these by interchange of accents. Limiting ourselves to these terms we have

$$x^2=\tfrac{1}{2}e^2+\tfrac{19}{72}\frac{c^2}{n^4}e^4+\left[\frac{4n^2+n'^2}{n'^2(4n^2-n'^2)^2}b^2+\tfrac{1}{2}\frac{b'c}{n^4}\right]e^2e'^2+\left[\tfrac{1}{8}\frac{b'^2}{(4n'^2-n^2)^2}+\frac{b'^2}{4n^4}\right]e'^4,$$
$$\frac{dx^2}{dt^2}=\tfrac{1}{2}n^2e^2+\frac{c^2}{18n^2}e^4+\frac{4n^4-3n^2n'^2+n'^4}{n'^2(4n^2-n'^2)^2}b^2e^2e'^2+\tfrac{1}{2}\frac{n'^2}{(4n'^2-n^2)^2}b'^2e'^4,$$
$$x^3=\frac{5}{8n^2}ce^4+\frac{3}{4n^2}b'e^2e'^2,$$
$$x^2x'=\tfrac{1}{8}\frac{8n^2-3n'^2}{n'^2(4n^2-n'^2)}be^4+\left[\frac{b}{4n^2-n'^2}+\frac{c'}{4n'^2}\right]e^2e'^2.$$

Substituting these values in the integral equation and eliminating n^2 and n'^2 by means of the two equations which determine them, we find

$$
\begin{aligned}
C = \tfrac{1}{2}(a^2e^2 + a'^2e'^2) &- \left[\frac{96a^4 - 68a^2a'^2 + 9a'^4}{16a'^2(4a^2 - a'^2)^2}b^2 + \frac{37}{144}\frac{c^2}{a^2}\right]e^4 \\
&+ \left[\frac{8a^4 - 18a^2a'^2 + 5a'^4}{2a'^2(4a^2 - a'^2)^2}b^2 + \frac{8a'^4 - 18a^2a'^2 + 5a^4}{2a^2(4a'^2 - a^2)^2}b'^2 - \tfrac{3}{4}\frac{b'c}{a^2} - \tfrac{3}{4}\frac{bc'}{a'^2}\right]e^2e'^2 \\
&- \left[\frac{96a'^4 - 68a^2a'^2 + 9a^4}{16a^2(4a'^2 - a^2)^2}b'^2 + \frac{37}{144}\frac{c'^2}{a'^2}\right]e'^4.
\end{aligned}
$$

It is necessary to notice the effect of the vanishing of the divisors introduced by the integration. In the terms of the second order if $n = 2n'$ the coefficient of $\cos 2\phi'$ in x and the coefficient of $\cos(\phi - \phi')$ in x' apparently become infinite. Also, if $n' = 2n$, the coefficient of $\cos(\phi - \phi')$ in x and the coefficient of $\cos 2\phi$ in x' are in like case. This evidently is the analytical warning that in these cases the arguments 2ϕ, $2\phi'$, $\phi - \phi'$ cannot be distinguished from ϕ and ϕ'. Thus the coefficients of their cosines should have some indetermination. But let us see what conditions the quantities a, a', b, b', c, c', e, e' must satisfy that either of these relations may have place. We shall not allow that any of the quantities a, a', n, n' can be infinite. Then, from the equations determining n and n', it is plain that, in the first case, either $b' = 0$, or $2e^2 - e'^2 = 0$, and in the second case, either $b = 0$, or $e^2 - 2e'^2 = 0$. Thus all the mentioned coefficients in x and x', instead of becoming infinite in these cases, really take the form $\frac{0}{0}$.

The particular supposition of $e = 0$, $e' = 0$ makes the first approximation vanish, and it is plain from the differential equations that $x = 0$, $x' = 0$ form a particular solution of them in which n and n' are quite indeterminate. This case therefore does not demand further consideration. But suppose that, in the first case, $b' = 0$ or, in the second, $b = 0$. Then, in Ω, the term $b'xx'^2$ or the term bx^2x' disappears, and the terms in x and x' having indeterminate coefficients should also disappear. However, the relation $n = 2n'$ or $n' = 2n$, without being exactly fulfilled, may be so very nearly. Let us take the first. If, in the equations determining n and n' we make $n^2 = 4n'^2$ and $n'^2 = a'^2$ except in the first terms and those which involve the divisor $4n'^2 - n^2$, we shall have quadratic equations for the determination of n^2 and n'^2. Thus, putting

$$
\begin{aligned}
\alpha'^2 &= a'^2 - \left(\tfrac{5}{6}c'^2 + \tfrac{1}{4}b'^2\right)\frac{e'^2}{a'^2} - \left(\tfrac{1}{4}b'c + bc' + \tfrac{2}{15}b^2\right)\frac{e^2}{a'^2}, \\
\beta &= \tfrac{1}{4}a^2 - a'^2 + \left(\tfrac{5}{6}c'^2 + \tfrac{1}{4}b'^2 - \tfrac{1}{16}b'c - \tfrac{1}{4}bc' - \tfrac{1}{80}b^2\right)\frac{e'^2}{a'^2} \\
&\quad + \left(\tfrac{1}{4}b'c + bc' - \tfrac{13}{120}b^2 - \tfrac{5}{96}c^2\right)\frac{e^2}{a'^2}
\end{aligned}
$$

the solution of these quadratic equations will give

$$n^2 = 4a'^2 \pm 4\sqrt{\beta^2 - b'^2(2e^2 - e'^2)}\,,$$
$$n'^2 = a'^2 - \tfrac{1}{2}\beta \pm \tfrac{1}{2}\sqrt{\beta^2 - b'^2(2e^2 - e'^2)}\,,$$
$$4n'^2 - n^2 = -2\beta \mp 2\sqrt{\beta^2 - b'^2(2e^2 - e'^2)}\,.$$

The ambiguity of the sign before the radical is determined from the following considerations. When β is a somewhat large positive quantity it is evident that we ought to take the upper sign, as always approximately $n'^2 = a'^2$, and then $n^2 = 4a'^2 + 4\beta$, thus n^2 is larger than $4n'^2$. Hence, if we begin with n^2 larger than $4n'^2$ and keep diminishing β, continuity demands that the upper sign be maintained. And we may persist in diminishing β until, if $2e^2 - e'^2$ is a positive quantity, $\beta = +\sqrt{b'^2(2e^2 - e'^2)}$ when $n'^2 = a'^2 - \frac{1}{2}\sqrt{b'^2(2e^2 - e'^2)}$ and $n^2 = 4a'^2$; thus n^2 is still larger than $4n'^2$. But if β is a somewhat large negative quantity it is evident that the lower sign must be attributed to the radical; and then approximately $n'^2 = a'^2$, $n^2 = 4a'^2 + 4\beta$; that is, here n^2 is less than $4n'^2$. Then β may be numerically diminished until $\beta = -\sqrt{b'^2(2e^2 - e'^2)}$, when $n'^2 = a'^2 + \frac{1}{2}\sqrt{b'^2(2e^2 - e'^2)}$ and $n^2 = 4a'^2$; and n^2 is still less than $4n'^2$. On the other hand if $2e^2 - e'^2$ is a negative quantity, β in both cases can be numerically diminished until it vanishes, and then, in the first case, we have $n'^2 = a'^2 + \frac{1}{2}\sqrt{b'^2(e'^2 - 2e^2)}$, $n^2 = 4a'^2 + 4\sqrt{b'^2(e'^2 - 2e^2)}$, and, in the second case, $n'^2 = a'^2 - \frac{1}{2}\sqrt{b'^2(e'^2 - 2e^2)}$, $n^2 = 4a'^2 - 4\sqrt{b'^2(e'^2 - 2e^2)}$.

From all this it is apparent that we cannot have $4n'^2 - n^2 = 0$ unless two conditions are satisfied. The first is $\beta = 0$, and the second either $b' = 0$ or $2e^2 - e'^2 = 0$. If the second condition is not fulfilled the limits towards which tend the coefficients $\frac{1}{2}\dfrac{b'e'^2}{n^2 - 4n'^2}$ and $\dfrac{b'ee'}{n'^2 - (n - n')^2}$, severally belonging to $\cos 2\phi'$ in x and $\cos(\phi - \phi')$ in x', can be written so as to cover all the cases

$$\pm\tfrac{1}{2}\frac{e'^2}{\sqrt{\pm(2e^2 - e'^2)}} \text{ and } \pm\frac{ee'}{\sqrt{\pm(2e^2 - e'^2)}}\,.$$

The ambiguous sign within the radical must be taken so as to render the quantity following positive; that without the radical depends at once on the sign of b' and on whether β has been supposed to approach the limit from positive or negative values. These limits are independent of b', and it will be perceived that instead of being of the second order with respect to e and e' they are of the same order as the first terms of x and x'. The second supposition of the second condition viz. $2e^2 - e'^2 = 0$ renders these

limits infinite; but it is probable that this conclusion would be modified if terms of the fourth order were included in the equations determining n and n'.

There is no need of giving the similar investigation for the assumption $n' = 2n$. All the results of the latter are obtained from those of the former assumption by simply interchanging the accents. Thus the limits of the coefficient of $\cos 2\phi$ in x' and the coefficient of $\cos(\phi - \phi')$ in x are severally

$$\pm \frac{1}{4} \frac{e^2}{\sqrt{\pm (2e'^2 - e^2)}} \text{ and } \pm \frac{ee'}{\sqrt{\pm (2e'^2 - e^2)}}.$$

In all this we have assumed that the quantity under the radical sign in the expressions for n^2 and n'^2 must not be negative. This is simply to insure stability of motion. If the constants of the problem are such as to make it negative, or if either n^2 and n'^2 come out negative, the problem has still a real solution but the motion is unstable.

The question of periodic solutions is intimately connected with the vanishing of integrating divisors. If we suppose that $in + i'n' = 0$, where i and i' are integers prime to each other, by adopting an argument ψ such that $\frac{d\phi}{dt} = i' \frac{d\psi}{dt}$ and $\frac{d\phi'}{dt} = -i \frac{d\psi}{dt}$, the series for x and x' reduce to the form

$$x \text{ or } x' = a_0 + a_1 \cos \psi + \beta_1 \sin \psi + a_2 \cos 2\psi + \beta_2 \sin 2\psi + \ldots.$$

The variables then return to the same values after ψ has gone through a circumference as also do their differential coefficients. If we conceive a system of rectangular coordinates for graphically exhibiting the values of x and x', the representative point P, in its motion, will describe a certain curve which, if the variables are confined within limits, will lie within a limited portion of the plane. This curve may have multiple points; let us see what is the condition necessary and sufficient that this may have place. The motion of x and x' being a constant oscillation between minimum and maximum values, the curve must be either a spiral having no multiple points, in which case there would be a progressive diminution or augmentation of the minimum and maximum values of x and x', or there must be an infinite number of multiple points. The periodic solution is the case which lies between these two classes of solutions. We may conceive the angle of intersection at the multiple point to constantly diminish through variation of the arbitrary constants of the problem until it vanishes, when the periodic solution is reached.

Let us suppose that the solution of the system of differential equations

$$\frac{d^2x}{dt^2} = \frac{\partial \Omega}{\partial x}, \quad \frac{d^2x'}{dt^2} = \frac{\partial \Omega}{\partial x'},$$

admitting the integral

$$\frac{1}{2}\frac{dx^2+dx'^2}{dt^2}=\Omega+C\,,$$

has been discovered and that it may be written

$$x=\mathrm{f}_1(t)\,,\quad x'=\mathrm{f}_2(t)\,.$$

Let t' denote another value of the independent variable t; then, if the equations

$$\mathrm{f}_1(t')=\mathrm{f}_1(t),\quad \mathrm{f}_2(t')=\mathrm{f}_2(t)\,,$$

t and t' being regarded as the unknowns, are satisfied by real values of t and t', the curve under discussion will have multiple points; but, if the roots of these equations are all imaginary, the curve will be a spiral. It appears that when x and x' are contained within finite limits the roots t and t' are all real and infinite in number; and thus that, in general, there are an infinite number of multiple points.

Let us now suppose that the arbitrary constants are adjusted so that, in addition to the equations $\mathrm{f}_1(t')=\mathrm{f}_1(t)$, $\mathrm{f}_2(t')=\mathrm{f}_2(t)$ being satisfied by certain values of t and t', the equation

$$\frac{\mathrm{f}_2'(t')}{\mathrm{f}_1'(t')}=\frac{\mathrm{f}_2'(t)}{\mathrm{f}_1'(t)}$$

is satisfied by the same values. (Accents have been used to denote differentiation of the form of f.) Then it is evident that the last equation is the condition of the presence of a periodic solution; and if we put $t'-t=P$, the equations

$$\mathrm{f}_1(t+P)=\mathrm{f}_1(t)\,,\quad \mathrm{f}_2(t+P)=\mathrm{f}_2(t)\,,$$

are satisfied for all values of t.

A periodic solution must have at least one arbitrary constant less than in the general case. But, for the differential equations we are treating, there is usually a periodic solution in which there are only two arbitrary constants. This happens when both variables arrive simultaneously at their maximum or minimum values. Integrating the differential equations by the application of Maclaurin's Theorem, we should have, in general,

$$x=a+bt+\frac{\partial\Omega}{\partial x}\frac{t^2}{2}+\ldots,$$
$$x'=a'+b't+\frac{\partial\Omega}{\partial x'}\frac{t^2}{2}+\ldots,$$

where a, a', b, b', are the arbitrary constants, and in the derivatives of Ω it is understood that x and x' are to be replaced severally by a and a'. In the

case where both variables arrive simultaneously at a maximum or minimum value, and we count t from this epoch, not only $b = 0$, $b' = 0$, but also all the coefficients of the odd powers of t in x and x', as is easy to see from the successive differentiation of the derivatives of Ω with respect to the time. We may therefore put

$$x = a + a_1 t^2 + a_2 t^4 + a_3 t^6 + \dots = \mathrm{f}_1(t),$$
$$x' = a' + a'_1 t^2 + a'_2 t^4 + a'_3 t^6 + \dots = \mathrm{f}_2(t).$$

In the previous equations for determining the times of the representative point P passing through multiple points, we may suppose $t' = -t$, and the equations $\mathrm{f}_1(-t) = \mathrm{f}_1(t)$, $\mathrm{f}_2(-t) = \mathrm{f}_2(t)$ are fulfilled for all values of t. Also the equation showing the presence of a periodic solution is, in this case, an identity. This is explained by the circumstance that the point P having attained the limiting curve whose equation is $\Omega + C = 0$ returns on its previous path. Hence the condition necessary for the existence of a periodic solution in this case is that when the path is sufficiently prolonged backward it should again meet the limiting curve for real velocities. The motion of P will then be a swinging back and forth on this arc. This is tantamount to saying that the values of a and a' must be so adjusted that a value t_1 for t can be found which will make x and x' have such values that they satisfy the equation $\Omega + C = 0$.

It may be of interest to know the earlier terms of the expansions of x and x' in powers of t^2. For the sake of brevity we put

$$2\Psi = \left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial x'}\right)^2.$$

Then

$$x = a + \frac{\partial \Omega}{\partial x}\frac{t^2}{1.2} + \frac{\partial \Psi}{\partial x}\frac{t^4}{1.2.3.4}$$
$$+ \left\{3\left[\frac{\partial \Omega}{\partial x}\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial \Omega}{\partial x'}\frac{\partial^2 \Psi}{\partial x \partial x'}\right] - 2\left[\frac{\partial^2 \Omega}{\partial x^2}\frac{\partial \Psi}{\partial x} + \frac{\partial^2 \Omega}{\partial x \partial x'}\frac{\partial \Psi}{\partial x'}\right]\right\}\frac{t^6}{6!} + \dots$$
$$x' = a' + \frac{\partial \Omega}{\partial x'}\frac{t^2}{1.2} + \frac{\partial \Psi}{\partial x'}\frac{t^4}{1.2.3.4}$$
$$+ \left\{3\left[\frac{\partial \Omega}{\partial x'}\frac{\partial^2 \Psi}{\partial x'^2} + \frac{\partial \Omega}{\partial x}\frac{\partial^2 \Psi}{\partial x \partial x'}\right] - 2\left[\frac{\partial^2 \Omega}{\partial x'^2}\frac{\partial \Psi}{\partial x'} + \frac{\partial^2 \Omega}{\partial x \partial x'}\frac{\partial \Psi}{\partial x}\right]\right\}\frac{t^6}{6!} + \dots$$

where in the partial derivatives it is supposed that x and x' are replaced severally by a and a'.

Evidently, for these series proceeding according to ascending powers of t^2, we may substitute others proceeding according to cosines of multiples of an angle proportional to t, or x and x' may have the form

$$x \text{ or } x' = \alpha_0 + \alpha_1 \cos nt + \alpha_2 \cos 2nt + \dots.$$

But here we are not entitled to assume that the coefficients rapidly diminish. To bring about such a relation as $n = n'$ it may be necessary for the arbitrary constants we have denoted by e and e' to be of the zero order of magnitude.

If in the expansion of Ω according to powers and products of x and x' we omit all the terms involving both variables the integration of the differential equations is reduced to quadratures. Let us write in a more general way this expansion of Ω

$$\begin{aligned}\Omega = {} & a_{20}x^2 + a_{11}xx' + a_{02}x'^2 \\ & + a_{30}x^3 + a_{21}x^2x' + a_{12}xx'^2 + a_{03}x'^3 \\ & + a_{40}x^4 + a_{31}x^3x' + a_{22}x^2x'^2 + a_{13}xx'^3 + a_{04}x'^4 \\ & + \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{aligned}$$

It must be noticed that the coefficients of the terms involving both x and x' are of the order of the planetary masses while those which multiply the simple powers of x or x' are of the order of the central mass. If we put

$$\begin{aligned}\Omega_0 = {} & a_{20}x^2 + a_{30}x^3 + a_{40}x^4 + \ldots \\ & + a_{02}x'^2 + a_{03}x'^3 + a_{04}x'^4 + \ldots\end{aligned}$$

the differential equations resulting from this expression of the potential function are

$$\frac{d^2x}{dt^2} = 2a_{20}x + 3a_{30}x^2 + 4a_{40}x^3 + \ldots,$$
$$\frac{d^2x'}{dt^2} = 2a_{02}x' + 3a_{03}x'^2 + 4a_{04}x'^3 + \ldots.$$

From these are immediately derived the integrals

$$\frac{dx^2}{dt^2} = C + a_{20}x^2 + a_{30}x^3 + a_{40}x^4 + \ldots.$$
$$\frac{dx'^2}{dt^2} = C' + a_{02}x'^2 + a_{03}x'^3 + a_{04}x'^4 + \ldots,$$

and thence the final integrals

$$t + c = \int \frac{dx}{\sqrt{(C + a_{20}x^2 + a_{30}x^3 + a_{40}x^4 + \ldots)}},$$
$$t + c' = \int \frac{dx'}{\sqrt{(C' + a_{02}x'^2 + a_{03}x'^3 + a_{04}x'^4 + \ldots)}}.$$

The polynomials under the radical sign, on being equated to 0, will have each two real roots, which, on account of the smallness of x and x', will be, severally, in the neighborhood of the values

$$x = +\sqrt{-\frac{C}{a_{20}}}, \quad x = -\sqrt{-\frac{C}{a_{20}}}, \quad x' = +\sqrt{-\frac{C'}{a_{02}}}, \quad x' = -\sqrt{-\frac{C'}{a_{02}}}.$$

The exact values of these can readily be found by trial; let them be denoted severally, by a, b, a′, b′. Then we can write

$$t + c = \int \frac{dx}{\sqrt{[(\mathrm{a} - x)(x - \mathrm{b})(A_0 + A_1 x + A_2 x^2 + \ldots)]}},$$
$$t + c' = \int \frac{dx'}{\sqrt{[(\mathrm{a}' - x')(x' - \mathrm{b}')(A'_0 + A'_1 x' + A'_2 x'^2 \ldots)]}}.$$

The limiting values of x are a and b, and those of x', a′ and b′. Within these limits the last factors under the radical sign are always positive and do not vary much, and thus x and x' are periodic functions of t, the periods being given by the formulæ

$$T = 2\int_{\mathrm{b}}^{\mathrm{a}} \frac{dx}{\sqrt{[(\mathrm{a} - x)(x - \mathrm{b})(A_0 + A_1 x + A_2 x^2 + \ldots)]}}$$
$$T' = 2\int_{\mathrm{b}'}^{\mathrm{a}'} \frac{dx'}{\sqrt{[(\mathrm{a}' - x')(x' - \mathrm{b}')(A'_0 + A'_1 x' + A'_2 x'^2 + \ldots)]}}.$$

By adopting, in place of x and x', variables ψ and ψ' such that

$$2x = (\mathrm{a} + \mathrm{b}) - (\mathrm{a} - \mathrm{b}) \cos \psi, \quad 2x' = (\mathrm{a}' + \mathrm{b}') - (\mathrm{a}' - \mathrm{b}') \cos \psi',$$

the integrations are considerably simplified. The motion of ψ and ψ' is unlimited like that of t. Let

$$[A_0 + A_1 x + A_2 x^2 + \ldots]^{-\frac{1}{2}} = \frac{1}{n}[1 + B_1 \cos \psi + 2B_2 \cos 2\psi + \ldots],$$
$$[A'_0 + A'_1 x' + A'_2 x'^2 + \ldots]^{-\frac{1}{2}} = \frac{1}{n'}[1 + B'_1 \cos \psi' + 2B'_2 \cos 2\psi' + \ldots].$$

Then

$$n(t + c) = \psi + B_1 \sin \psi + B_2 \sin 2\psi + \ldots,$$
$$n'(t + c') = \psi' + B'_1 \sin \psi' + B'_2 \sin 2\psi' + \ldots.$$

The values of ψ and ψ' can be derived from these transcendental equations and have the forms

$$\psi = n\,(t + c) + E_1 \cos n\,(t + c) + E_2 \cos 2\,(nt + c) + \ldots,$$
$$\psi' = n'\,(t + c') + E'_1 \cos n'\,(t + c') + E'_2 \cos 2\,n't + c') + \ldots.$$

From these we can pass to the values of $\cos\psi$ and $\cos\psi'$, and thence to those of x and x', which have the form

$$x = \tfrac{1}{2} G_0 + G_1 \cos \varphi + G_2 \cos 2\varphi + \ldots,$$
$$x' = \tfrac{1}{2} G'_0 + G'_1 \cos \varphi' + G'_2 \cos 2\varphi + \ldots,$$

ϕ and ϕ' being put for $nt + c$ and $n't + c'$.

The reversion of the periodic series can be avoided by the use of definite integrals, and we can pass directly to the values of the G_i and G'_i. For we have

$$G_i = \frac{2}{\pi}\int_0^\pi x \cos i\varphi d\varphi,$$

with a similar formula for G'_i. By adopting ψ as the independent variable this becomes

$$G_i = \frac{2}{\pi}\int_0^\pi [(\mathrm{a}+\mathrm{b}) - (\mathrm{a}-\mathrm{b})\cos\psi] \cos[i\psi + iB_1 \sin\psi + iB_2 \sin 2\psi + \ldots]$$
$$[1 + B_1\cos\psi + 2B_2\cos 2\psi + \ldots]\, d\psi.$$

In the particular case of $i = 0$, this reduces to

$$G_0 = \mathrm{a} + \mathrm{b} - \tfrac{1}{2}(\mathrm{a}-\mathrm{b}) B_1.$$

Proposing now to take into account the omitted terms of Ω, which we will denote as Ω_1, let the values of x and x', just determined, be denoted as x_0 and x'_0, their corrections as δx and $\delta x'$, the differential equations for the latter will be

$$\frac{d^2\delta x}{dt^2} = [2a_{20} + 3a_{30}(x + x_0) + 4a_{40}(x^2 + xx_0 + x_0^2) + \ldots]\,\delta x + \frac{\partial \Omega_1}{\partial x},$$

$$\frac{d^2\delta x'}{dt^2} = [2a_{02} + 3a_{03}(x' + x'_0) + 4a_{04}(x'^2 + x'x'_0 + x_0'^2) + \ldots]\,\delta x' + \frac{\partial \Omega_1}{\partial x'}.$$

These can be integrated by successive approximations, introducing into the right members the values of x and x' from the previous approximation, beginning with $x = x_0$, $x' = x'_0$. In order to prevent t from getting outside of the functional sign cos, we adopt the following device: Let n and n' be two indeterminate constants, whose values come out only at the end of the approximation, but which are equivalent to the rigorous rates of motion of the arguments ϕ and ϕ'. We add $n^2\delta x$ to each member of the first equation and $n'^2\delta x'$ to each member of the second. Then, putting

$$Q = [n^2 + 2a_{20} + 3a_{30}(x + x_0) + 4a_{40}(x^2 + xx_0 + x_0^2) + \ldots]\,\delta x + \frac{\partial \Omega_1}{\partial x},$$

and Q' for the similar quantity in the second equation, we have

$$\frac{d^2\delta x}{dt^2} + n^2\delta x = Q, \quad \frac{d^2\delta x'}{dt^2} + n'^2\delta x' = Q'.$$

In any stage of the approximation we shall have

$$Q = \Sigma K_{i,i'} \cos(i\varphi + i'\varphi'), \quad Q' = \Sigma K'_{i,i'} \cos(i\varphi + i'\varphi'),$$

from which

$$\delta x = \Sigma . [n^2 - (in + i'n')^2]^{-1} K_{i,i'} \cos(i\varphi + i'\varphi'),$$
$$\delta x' = \Sigma . [n'^2 - (in + i'n')^2]^{-1} K'_{i,i'} \cos(i\varphi + i'\varphi').$$

Then the equations for determining n and n' are

$$K_{1,0} = 0, \quad K'_{0,1} = 0.$$

The terms of Ω in x^2x' and xx'^2 can be removed from it by a linear transformation, but at the expense of reintroducing a term in xx', as also a term involving the product of the velocities into the expression for the living force. For substituting

$$x = y + ly', \quad x' = y' + l'y,$$

if l and l' are so determined that

$$3a_{30}l + a_{21}(1 + 2ll') + a_{12}l'(2 + ll') + 3a_{03}l'^2 = 0,$$
$$3a_{30}l^2 + a_{21}l(2 + ll') + a_{12}(1 + 2ll') + 3a_{03}l' = 0,$$

the terms in Ω involving y^2y' and yy'^2 vanish. And if we put

$$\begin{aligned}
B_{20} &= a_{20} + a_{11}l' + a_{02}l'^2,\\
B_{11} &= 2a_{20}l + a_{11}(1 + ll') + 2a_{02}l',\\
B_{02} &= a_{20}l^2 + a_{11}l + a_{02},\\
B_{30} &= a_{30} + a_{21}l' + a_{12}l'^2 + a_{03}l'^3,\\
B_{03} &= a_{30}l^3 + a_{21}l^2 + a_{12}l + a_{03},
\end{aligned}$$

we have

$$\Omega = B_{20}y^2 + B_{11}yy' + B_{02}y'^2 + B_{30}y^3 + B_{03}y'^3 + \ldots,$$

and the expression of the living force becomes

$$T = \tfrac{1}{2}(1 + l'^2)\frac{dy^2}{dt^2} + (l + l')\frac{dy}{dt}\frac{dy'}{dt} + \tfrac{1}{2}(1 + l^2)\frac{dy'^2}{dt^2},$$

and the differential equations of the problem are

$$(1 + l'^2)\frac{d^2y}{dt^2} + (l + l')\frac{d^2y'}{dt^2} = 2B_{20}y + 3B_{30}y^2 + B_{11}y',$$
$$(1 + l^2)\frac{d^2y'}{dt^2} + (l + l')\frac{d^2y}{dt^2} = 2B_{02}y' + 3B_{03}y'^2 + B_{11}y.$$

The elimination of l or l' between the equations which determine them gives rise to an equation of the fifth degree, which consequently, always has

a real root. As a_{21} and a_{12} are of the order of the masses of the planets, approximate values for l and l' are $l = -\frac{a_{21}}{3a_{30}}$, $l' = -\frac{a_{12}}{3a_{03}}$.

In integrating the preceding equations one may follow either of two courses; first, we may neglect the terms in y^2 and y'^2, which will afterwards be regarded as perturbative, and then we shall have linear equations with constant coefficients easily integrable; or second, we may regard the last terms of both members of each equation as perturbative, and the equations restricted to the remaining terms are integrable by quadratures, as has been shown.

MEMOIR No. 55.

Literal Expression for the Motion of the Moon's Perigee.

(Annals of Mathematics, Vol. IX, pp. 31–41, 1894.)

The earlier investigators of the Lunar Theory contented themselves with giving numerical values for this quantity. The results of Clairaut, D'Alembert, Euler, Laplace and Damoiseau are of this nature. Beyond the rudest approximation, Plana was the first to give the value in a literal form. This was nearly reproduced by Pontécoulant. The salient portion, which is a function of the ratio of the month to the year, is given by these two authors to the order of the seventh power of this ratio. Delaunay, in his treatment of the subject, (*Comptes Rendus*, t. LXXIV, p. 19) has added the two following terms, viz., those which involve the eighth and ninth powers of the mentioned ratio. The correctness of these has, however, been called in question by M. Andoyer (*Annales de la Faculté des Sciences de Toulouse*, t. VI) who has also given other values for them. They may be seen in Tisserand's *Mécanique Céleste*, t. III, p. 412. I have not been able to consult M. Andoyer's memoir, and do not know what method he used in obtaining his results. The comparison I made at the end of my memoir "On the part of the motion of the Lunar Perigee, etc." (Acta Math., Vol. VIII) of Delaunay's series with my numerical value, indicated with some probability, that the newly added terms were, one or both, too large; which corresponds with what M. Andoyer has found. In this state of matters I have thought it would not be without interest to test the validity of M. Andoyer's corrections; and I have determined to add two more terms to the series, viz. those factored by m^{10} and m^{11}.

For this purpose, I shall employ the method of my above-mentioned memoir. There the computation was carried out in the numerical fashion, here it is proposed to give algebraic developments. It is there shown that the determination of the lunar inequalities of the type of the evection and the motion of the perigee depend on, at least as far as a first approximation is concerned, the integration of the linear differential equation of the second order

$$D^2\mathrm{w} = \theta\mathrm{w}\,,$$

where w is the unknown, Θ a periodic function of double the mean angular distance of the Moon from the Sun, involving only cosines, and D is an operator such that $D(\alpha\zeta^\nu) = \nu\alpha\zeta^\nu$, ζ being the trigonometrical exponential corresponding to the mentioned mean angular distance. The motion of the perigee depends solely on the coefficients of Θ, and these can be found when we know the coefficients of the inequalities of the type of the variation. I have given the latter in a literal form to the 9th order inclusive (Amer. Jour. Math., Vol. I, pp. 142–143).

We adopt a moving system of rectangular coordinates, the origin being at the centre of the Earth, the axis of x constantly passing through the centre of the Sun, and, in place of x and y, we use the imaginary coordinates

$$u = x + y\sqrt{-1}, \quad s = x - y\sqrt{-1}.$$

Then

$$u = \Sigma \mathrm{a}_i \zeta^{2i+1}, \quad s = \Sigma \mathrm{a}_i \zeta^{-2i-1},$$

where, in the summation, i receives all integral values from $-\infty$ to $+\infty$, zero included, and the a_i are constants, being each equivalent to the same constant multiplied by a function of the ratio of the month to the year, which is of the $|2i|$th order with respect to this parameter. For simplicity in writing, then, we assume that the value of a_0 is unity; consequently, as written, a_i always denotes $\frac{\mathrm{a}_i}{\mathrm{a}_0}$.

In pushing the development of Θ to the degree of approximation we desire, the values of the a_i given (Amer. Jour. Math., Vol. I, pp. 142–143) generally suffice; but it will be perceived from the approximate expression for Θ (Motion of Lunar Perigee, p. 13) that it will be necessary for the determination of c, the ratio of the anomalistic to the synodic month, to the 11th order inclusive, that we should know the term factored by m^{10} in $\mathrm{a}_1 + \mathrm{a}_{-1}$; it is not necessary, however, that a_1 and a_{-1} separately should be known to this degree of approximation. Hence, we now proceed to obtain this term From the equations given (Am. Jour. Math., Vol. I, p. 137) by neglecting all terms whose order exceeds the 10th we derive

$$\begin{aligned} \mathrm{a}_1 + \mathrm{a}_{-1} = &-\frac{3\,(2+\mathrm{m})\,\mathrm{m}^2}{6-4\mathrm{m}+\mathrm{m}^2}(1 + 2\mathrm{a}_1\mathrm{a}_{-1}) - \frac{3\,(1-\mathrm{m})\,\mathrm{m}^2}{6-4\mathrm{m}+\mathrm{m}^2}(\mathrm{a}_{-1}^2 + 2\mathrm{a}_{-2} + 2\mathrm{a}_1\mathrm{a}_{-3}) \\ &- \frac{22-4\mathrm{m}+\mathrm{m}^2}{6-4\mathrm{m}+\mathrm{m}^2}(\mathrm{a}_1\mathrm{a}_2 + \mathrm{a}_{-1}\mathrm{a}_{-2}) - 9\mathrm{a}_2\mathrm{a}_3. \end{aligned}$$

By substituting in the right member of this the values of the a_i in powers of m it is found that the term in m^{10} in $\mathrm{a}_1 + \mathrm{a}_{-1}$ is

$$+\frac{1605921808447}{2^{18}.3^8.5^3}\mathrm{m}^{10}.$$

We have

$$\theta = -\left[\frac{x}{r^3} + \mathrm{m}^2\right] + 2\left[D \log \sqrt{\frac{Du}{Ds}} + \mathrm{m}\right]^2 - [D \log \sqrt{DuDs}]^2$$
$$- D^2 [\log \sqrt{DuDs}].$$

The first term of this can be developed from the formula

$$\frac{x}{r^3} + \mathrm{m}^2 = \frac{D^2u + 2\mathrm{m}Du + \frac{3}{2}\mathrm{m}^2 s}{u} + \tfrac{5}{2}\mathrm{m}^2 = 1 + 2m + \tfrac{5}{2}\mathrm{m}^2 + \Sigma\,.\, R_i \zeta^{2i},$$

where we have $R_{-i} = R_i$. The equations which determine the values of the R_i, R_0 with an error of the 12th order, R_1 with one of the 11th order, R_2 with one of the 9th order, and R_3 with one of the 7th order, are

$$R_0 + (\mathrm{a}_1 + \mathrm{a}_{-1}) R_1 + (\mathrm{a}_2 + \mathrm{a}_{-2}) R_2 = \tfrac{3}{2}\mathrm{m}^2\mathrm{a}_{-1},$$
$$\mathrm{a}_{-1}R_0 + (1 + \mathrm{a}_{-2}) R_1 + (\mathrm{a}_1 + \mathrm{a}_{-3}) R_2 + \mathrm{a}_2 R_3 = -4\mathrm{m}\mathrm{a}_{-1} + \tfrac{3}{2}\mathrm{m}^2,$$
$$\mathrm{a}_{-2}R_0 + (\mathrm{a}_{-1} + \mathrm{a}_{-3}) R_1 + R_2 + \mathrm{a}_1 R_3 = 8(1 - \mathrm{m})\,\mathrm{a}_{-2} + \tfrac{3}{2}\mathrm{m}^2\mathrm{a}_1,$$
$$\mathrm{a}_{-1}R_2 + R_3 = 24\mathrm{a}_{-3} + \tfrac{3}{2}\mathrm{m}^2\mathrm{a}_2.$$

Solving these by successive approximations in using the known literal values of the a_i, we get

$$R_0 = -\frac{9}{2^5}\mathrm{m}^4 + 4\mathrm{m}^5 + \frac{34}{3}\mathrm{m}^6 + 15\mathrm{m}^7 + \frac{2704801}{2^{13}.3^3}\mathrm{m}^8 + \frac{122957}{2^5.3^4.5}\mathrm{m}^9$$
$$+ \frac{1260881}{2^9.3^4.5}\mathrm{m}^{10} - \frac{291394307}{2^7.3^6.5^3}\mathrm{m}^{11},$$
$$R_1 = \frac{3}{2}\mathrm{m}^2 + \frac{19}{2^2}\mathrm{m}^3 + \frac{20}{3}\mathrm{m}^4 + \frac{43}{3^2}\mathrm{m}^5 + \frac{18709}{2^9.3^3}\mathrm{m}^8 + \frac{759413}{2^{10}.3^4.5}\mathrm{m}^7$$
$$+ \frac{6675059}{2^8.3^5.5^2}\mathrm{m}^8 - \frac{41991161}{2^7.3^6.5^3}\mathrm{m}^9 - \frac{4528083484913}{2^{17}.3^7.5^4}\mathrm{m}^{10},$$
$$R_2 = \frac{33}{2^4}\mathrm{m}^4 + \frac{2937}{2^6.5}\mathrm{m}^5 + \frac{23051}{2^4.3.5^2}\mathrm{m}^6 + \frac{97051}{2^5.5^3}\mathrm{m}^7 + \frac{334413271}{2^{10}.3^3.5^4}\mathrm{m}^8,$$
$$R_3 = \frac{1393}{2^9}\mathrm{m}^6.$$

We next attend to the coefficients of $\dfrac{D^2u}{Du} = \Sigma\,.\,U_i\zeta^{2i}$. The formulas given for these (Motion of Lunar Perigee, p. 12) in general suffice; it is necessary, however, to push the development of U_1 and U_{-1} farther, so that terms of the 10th order may be included. Thus their equivalents read

$$U_1 = 2[\mathrm{h}_1 - \mathrm{h}_{-1}\mathrm{h}_2 + \mathrm{h}_1^2\mathrm{h}_{-1} + 2\mathrm{h}_1^3\mathrm{h}_{-1}^2 - \mathrm{h}_1^3\mathrm{h}_{-2} - 3\mathrm{h}_1\mathrm{h}_{-1}^2\mathrm{h}_2 + 2\mathrm{h}_1\mathrm{h}_2\mathrm{h}_{-2} + \mathrm{h}_{-1}^2\mathrm{h}_3 - \mathrm{h}_{-2}\mathrm{h}_3],$$
$$U_{-1} = -2[\mathrm{h}_{-1} - \mathrm{h}_1\mathrm{h}_{-2} + \mathrm{h}_{-1}^2\mathrm{h}_1 + 2\mathrm{h}_{-1}^3\mathrm{h}_1^2 - \mathrm{h}_{-1}^3\mathrm{h}_2 - 3\mathrm{h}_{-1}\mathrm{h}_1^2\mathrm{h}_{-2}$$
$$+ 2\mathrm{h}_{-1}\mathrm{h}_{-2}\mathrm{h}_2 + \mathrm{h}_1^2\mathrm{h}_{-3} - \mathrm{h}_2\mathrm{h}_{-3}].$$

By substituting in these equations the values of the a_i in powers of m, and making the assumption that the coefficient of m^{10} in a_1 is 0, which cannot lead us into error within the limits we set to the approximation, and putting

$$\tfrac{1}{2}(U_i + U_{-i}) = A_i, \quad \tfrac{1}{2}(U_i - U_{-i}) = B_i,$$

we get

$$A_1 = -\frac{5}{2^3}\mathrm{m}^2 - \frac{1}{2\,.\,3}\mathrm{m}^3 + \frac{5}{3^2}\mathrm{m}^4 + \frac{43}{2^2\,.\,3^3}\mathrm{m}^5 - \frac{318575}{2^{11}\,.\,3^4}\mathrm{m}^6 - \frac{2297593}{2^8\,.\,3^5\,.\,5}\mathrm{m}^7$$
$$- \frac{9225887}{2^5\,.\,3^6\,.\,5^2}\mathrm{m}^8 - \frac{3471983789}{2^9\,.\,3^7\,.\,5^3}\mathrm{m}^9 - \frac{12903700736069}{2^{19}\,.\,3^8\,.\,5^4}\mathrm{m}^{10},$$
$$B_1 = \frac{7}{2^2}\mathrm{m}^2 + \frac{19}{2\,.\,3}\mathrm{m}^3 + \frac{53}{2\,.\,3^2}\mathrm{m}^4 + \frac{155}{2^2\,.\,3^3}\mathrm{m}^5 - \frac{12941}{2^{10}\,.\,3^4}\mathrm{m}^6 - \frac{904921}{2^9\,.\,3^5\,.\,5}\mathrm{m}^7$$
$$- \frac{35308207}{2^9\,.\,3^6\,.\,5^2}\mathrm{m}^8 - \frac{2190838913}{2^{10}\,.\,3^7\,.\,5^3}\mathrm{m}^9 + \frac{29589760583167}{2^{18}\,.\,3^8\,.\,5^4}\mathrm{m}^{10},$$
$$A_2 = \frac{265}{2^7}\mathrm{m}^4 + \frac{1067}{2^5\,.\,5}\mathrm{m}^5 + \frac{38261}{2^4\,.\,3^2\,.\,5^2}\mathrm{m}^6 + \frac{755591}{2^8\,.\,3^2\,.\,5^3}\mathrm{m}^7 + \frac{405840581}{2^{10}\,.\,3^4\,.\,5^4}\mathrm{m}^8,$$
$$B_2 = -\frac{3}{2^2}\mathrm{m}^4 - \frac{403}{2^4\,.\,3\,.\,5}\mathrm{m}^5 - \frac{3773}{2^3\,.\,3^2\,.\,5^2}\mathrm{m}^6 - \frac{246139}{2^5\,.\,3^3\,.\,5^3}\mathrm{m}^7 - \frac{1077852389}{2^{12}\,.\,3^4\,.\,5^4}\mathrm{m}^8,$$
$$A_3 = -\frac{1677}{2^{11}}\mathrm{m}^6,$$
$$B_3 = \frac{2431}{2^{10}}\mathrm{m}^6.$$

The coefficients of the function Θ have the following equivalents:

$$\theta_0 = 1 + 2\mathrm{m} - \tfrac{1}{2}\mathrm{m}^2 + 4(A_1^2 + A_2^2) + 2(B_1^2 + B_2^2) - R_0,$$
$$\theta_1 = 4(1+\mathrm{m})\,A_1 - 2B_1 + 4(A_1A_2 + A_2A_3) + 2(B_1B_2 + B_2B_3) - R_1,$$
$$\theta_2 = 4(1+\mathrm{m})\,A_2 - 4B_2 + 2(A_1^2 + 2A_1A_3) - B_1^2 + 2B_1B_3 - R_2,$$
$$\theta_3 = 4A_3 - 6B_3 + 4A_1A_2 - 2B_1B_2 - R_3.$$

When the expressions in powers of m for the quantities A, B, and R are substituted in the preceding equations, the results are

$$\theta_0 = 1 + 2\mathrm{m} - \tfrac{1}{2}\mathrm{m}^2 + \frac{255}{2^6}\mathrm{m}^4 + 19\mathrm{m}^5 + \frac{80}{3}\mathrm{m}^6 + \frac{533}{2\,.\,3^2}\mathrm{m}^7 + \frac{11230225}{2^{13}\,.\,3^3}\mathrm{m}^8$$
$$+ \frac{1576037}{2^7\,.\,3^4}\mathrm{m}^9 + \frac{49359583}{2^9\,.\,3^5}\mathrm{m}^{10} + \frac{720508007}{2^8\,.\,3^6\,.\,5}\mathrm{m}^{11},$$
$$\theta_1 = -\frac{15}{2}\mathrm{m}^2 - \frac{57}{2^2}\mathrm{m}^3 - 11\mathrm{m}^4 - \frac{23}{2\,.\,3}\mathrm{m}^5 - \frac{68803}{2^9\,.\,3^2}\mathrm{m}^6 - \frac{1792417}{2^{10}\,.\,3^3}\mathrm{m}^7$$
$$- \frac{7172183}{2^7\,.\,3^4\,.\,5}\mathrm{m}^8 - \frac{596404499}{2^9\,.\,3^5\,.\,5^2}\mathrm{m}^9 - \frac{2641291011773}{2^{17}\,.\,3^6\,.\,5^3}\mathrm{m}^{10},$$
$$\theta_2 = \frac{111}{2^4}\mathrm{m}^4 + \frac{1397}{2^8}\mathrm{m}^5 + \frac{8807}{2^4\,.\,3\,.\,5}\mathrm{m}^8 + \frac{319003}{2^5\,.\,3^2\,.\,5^2}\mathrm{m}^7 + \frac{252382507}{2^{10}\,.\,3^3\,.\,5^3}\mathrm{m}^8,$$
$$\theta_3 = -\frac{11669}{2^9}\mathrm{m}^8.$$

We employ now the system of equations given (Motion of the Lunar Perigee, p. 14) and for brevity of notation put $[i]$ for $(\mathrm{c}+i)^2 - \Theta_0$. The equations, written to the requisite degree of approximation, are

$$\begin{array}{llllll}
[-3]\,\mathrm{b}_{-3} & -\theta_1\mathrm{b}_{-2} & -\theta_2\mathrm{b}_{-1} & -\theta_3\mathrm{b}_0 & & = 0, \\
-\theta_1\mathrm{b}_{-3} & +[-2]\,\mathrm{b}_{-2} & -\theta_1\mathrm{b}_{-1} & -\theta_2\mathrm{b}_0 & -\theta_3\mathrm{b}_1 & = 0, \\
-\theta_2\mathrm{b}_{-3} & -\theta_1\mathrm{b}_{-2} & +[-1]\,\mathrm{b}_{-1} & -\theta_1\mathrm{b}_0 & -\theta_2\mathrm{b}_1 & -\theta_3\mathrm{b}_2 = 0, \\
-\theta_3\mathrm{b}_{-3} & -\theta_2\mathrm{b}_{-2} & -\theta_1\mathrm{b}_{-1} & +[0]\,\mathrm{b}_0 & -\theta_1\mathrm{b}_1 & -\theta_2\mathrm{b}_2 = 0, \\
 & -\theta_3\mathrm{b}_{-2} & -\theta_2\mathrm{b}_{-1} & -\theta_1\mathrm{b}_0 & +[1]\,\mathrm{b}_1 & -\theta_1\mathrm{b}_2 = 0, \\
 & & -\theta_3\mathrm{b}_{-1} & -\theta_2\mathrm{b}_0 & -\theta_1\mathrm{b}_1 & +[2]\,\mathrm{b}_2 = 0.
\end{array}$$

That the relative degree of importance of the terms of these equations may be perceived, it may be pointed out that the diagonal line of coefficients $[-3], [-2], \ldots, [1], [2]$ are all of the zero order of magnitude except $[-1]$ and $[0]$, the first of which is of the first order, and the second of the third order; but the latter we need not concern ourselves about; and Θ_i is of the $2i$th order. From this it follows that, if we write the quantities b in the order b_{-1}, b_1, b_{-2}, b_2, b_{-3}, leaving out b_0, which is an arbitrary quantity, the first is of the first order, and every succeeding one an order higher, so that b_{-3} is of the fifth order. In order to have the equation determining c, it is necessary to eliminate the 5 mentioned b's from the group of equations. The readiest method of accomplishing this is to proceed by successive approximations using formulas of recursion. To attain the desired degree of accuracy, three approximations are necessary, each of which will give three terms in powers of m in the value of each b involved. When the values of the five b's have been obtained and substituted in the middle equation, after the rejection of the useless factor b_0, we have the following equation serving for the determination of c:

$$
\begin{aligned}
[0] &- \left[\frac{1}{[-1]} + \frac{1}{[1]}\right]\theta_1^2 - \left[\frac{1}{[-1]^2[-2]} + \frac{1}{[1]^2[2]}\right]\theta_1^4 \\
&\quad - 2\left[\frac{1}{[-1][1]} + \frac{1}{[-1][-2]} + \frac{1}{[1][2]}\right]\theta_1^2\theta_2 \\
&- \left[\frac{1}{[-2]} + \frac{1}{[2]}\right]\theta_2^2 - \frac{1}{[-1]^2}\left[\frac{1}{[-1][-2]^2} + \frac{1}{[-2]^2[-3]}\right]\theta_1^6 \\
&- 2\left[\frac{1}{[-1]^2[-2]^2} + \frac{1}{[-1]^2[1][-2]} + \frac{1}{[-1][1]^2[2]}\right. \\
&\quad \left. + \frac{1}{[-1]^2[-2][-3]} + \frac{1}{[-1][-2]^2[-3]}\right]\theta_1^4\theta_2 \\
&- \left[\frac{1}{[-1][-2]^2} + \frac{1}{[-1]^2[1]} + \frac{1}{[-1][1]^2} + \frac{1}{[-1]^2[-3]}\right. \\
&\quad \left. + \frac{2}{[-1][1][-2]} + \frac{2}{[-1][1][2]} + \frac{2}{[-1][-2][-3]}\right]\theta_1^2\theta_2^2 \\
&- 2\left[\frac{1}{[-1][-2][-3]} + \frac{1}{[-1][1][-2]} + \frac{1}{[-1][1][2]}\right]\theta_1^3\theta_3 \\
&- 2\left[\frac{1}{[-1][2]} + \frac{1}{[-1][-3]}\right]\theta_1\theta_2\theta_3 = 0.
\end{aligned}
$$

From this all terms unnecessary to the desired degree of approximation have been rejected.

It appears desirable to give some details as to the treatment of the foregoing equation. First we form the various products of the Θ involved;

each is limited to the terms needed for the degree of approximation wished.

$$\theta_1^2 = \frac{225}{2^2}m^4 + \frac{855}{2^2}m^5 + \frac{5889}{2^4}m^6 + 371m^7 + \frac{697679}{2^9 . 3}m^8 + \frac{853817}{2^6 . 3^2}m^9$$
$$+ \frac{235899233}{2^{11} . 3^3}m^{10} + \frac{1733519201}{2^9 . 3^4 . 5}m^{11} + \frac{19979134939549}{2^{18} . 3^5 . 5^2}m^{12},$$
$$\theta_1^4 = \frac{50625}{2^4}m^8 + \frac{192375}{2^3}m^9 + \frac{2787075}{2^5}m^{10} + \frac{6370695}{2^5}m^{11}$$
$$+ \frac{353456169}{2^{10}}m^{12} + \frac{649258747}{2^{10}}m^{13},$$
$$\theta_1^6 = \frac{11390625}{2^6}m^{12} + \frac{129853125}{2^6}m^{13} + \frac{2868159375}{2^8}m^{14},$$
$$\theta_1^2\theta_2 = \frac{24975}{2^6}m^8 + \frac{693945}{2^8}m^9 + \frac{1188267}{2^7}m^{10} + \frac{21446525}{2^{10}}m^{11}$$
$$+ \frac{4710472379}{2^{13} . 3 . 5}m^{12},$$
$$\theta_2^2 = \frac{12321}{2^8}m^8 + \frac{155067}{2^9}m^9 + \frac{20185533}{2^{12} . 5}m^{10} + \frac{85123117}{2^9 . 3 . 5^2}m^{11},$$
$$\theta_1^4\theta_2 = \frac{5619375}{2^8}m^{12} + \frac{241552125}{2^{10}}m_{13}, \quad \theta_1^2\theta_2^2 = \frac{2772225}{2^{10}}m^{12} + \frac{55958985}{2^{11}}m^{13},$$
$$\theta_1^3\theta_3 = \frac{39382875}{2^{12}}m^{12}, \qquad \theta_1\theta_2\theta_3 = \frac{19428885}{2^{14}}m^{12}.$$

In the next place, by neglecting quantities of the 7th order, the equation may be written

$$[0] - \left[\frac{1}{[-1]} + \frac{1}{[1]}\right]\theta_1^2 - \frac{\theta_1^4}{128m^2} = 0,$$

and, if we put

$$\theta'_1 = \theta_1\left[1 - \frac{3\theta_1^2}{64m}\right],$$

it can be given the form

$$[0]^2 + 2(\theta_0 - 1)[0] + \theta'^2_1 = 0,$$

whence is derived

$$c^2 = 1 + \sqrt{(\theta_0 - 1)^2 - \theta'^2_1}.$$

By substituting the previously given developments of Θ_0 and Θ_1 we get

$$c^2 = 1 + 2m - \frac{1}{2}m^2 - \frac{225}{2^4}m^3 - \frac{3135}{2^6}m^4 - \frac{139973}{2^{10}}m^5 - \frac{4611319}{2^{12} . 3}m^6,$$
$$c = 1 + m - \frac{3}{2^2}m^2 - \frac{201}{2^5}m^3 - \frac{2367}{2^7}m^4 - \frac{111749}{2^{11}}m^5 - \frac{4095991}{2^{13} . 3}m^6.$$

These equations are correct to the last power of m set down.

This value of c may be substituted in all the terms but the two first of the equation which determines it; and the latter is thereby reduced to a

manageable form. The values of the reciprocals of the quantities denoted by the symbols [— 1], [1], [— 2], [2], [— 3], developed to the needed degree of approximation, are

$$\frac{1}{[-1]} = -\frac{1}{2^2}\mathrm{m}^{-1} - \frac{3}{2^4} - \frac{213}{2^6}\mathrm{m} - \frac{2259}{2^{10}}\mathrm{m}^2 - \frac{70973}{2^{13}}\mathrm{m}^3 - \frac{3501259}{2^{15}.3}\mathrm{m}^4,$$

$$\frac{1}{[1]} = \frac{1}{2^3} - \frac{1}{2^4}\mathrm{m} + \frac{5}{2^6}\mathrm{m}^2 + \frac{563}{2^{10}}\mathrm{m}^3 + \frac{6119}{2^{12}}\mathrm{m}^4,$$

$$\frac{1}{[-2]} = \frac{1}{2^3} + \frac{1}{2^3}\mathrm{m} + \frac{1}{2^6}\mathrm{m}^2 - \frac{643}{2^{10}}\mathrm{m}^3 - \frac{10807}{2^{12}}\mathrm{m}^4 - \frac{532047}{2^{16}}\mathrm{m}^5,$$

$$\frac{1}{[2]} = \frac{1}{2^3.3} - \frac{1}{2^3.3^2}\mathrm{m} + \frac{13}{2^6.3^3}\mathrm{m}^2 + \frac{8557}{2^{10}.3^4}\mathrm{m}^3,$$

$$\frac{1}{[-3]} = \frac{1}{2^3.3} + \frac{1}{2^4.3}\mathrm{m}.$$

The substitution being made in the eight terms of the left-hand member of the equation, the result follows, in which, for facility of verification, we write each fraction separately and in the order in which it arises from each of the eight terms.

$$\begin{aligned}
&-\frac{50625}{2^{11}}\mathrm{m}^6 + \left[-\frac{1022625}{2^{12}} + \frac{24975}{2^9}\right]\mathrm{m}^7 \\
&+\left[-\frac{90037575}{2^{16}} + \frac{49095}{2^7} - \frac{4107}{2^9}\right]\mathrm{m}^8 \\
&+\left[-\frac{1462100355}{2^{18}} + \frac{54632079}{2^{15}} - \frac{57165}{2^{10}} + \frac{11390625}{2^{18}}\right]\mathrm{m}^9 \\
&+\left[-\frac{165044625741}{2^{23}} + \frac{722443913}{2^{17}} - \frac{8198151}{2^{13}.5} + \frac{705459375}{2^{20}} - \frac{13111875}{2^{17}} - \frac{924075}{2^{15}}\right]\mathrm{m}^{10} \\
&+\left[-\frac{287970294069}{2^{22}} + \frac{87619247043}{2^{20}.5} - \frac{13794117581}{2^{17}.3^2.5^2} + \frac{92222296875}{2^{24}}\right. \\
&\qquad\qquad \left. - \frac{702232875}{2^{19}} - \frac{34533765}{2^{17}} + \frac{65638125}{2^{19}} + \frac{6476295}{2^{17}}\right]\mathrm{m}^{11}.
\end{aligned}$$

By summing the fractions the equation may be written

$$\begin{aligned}
[0] - \left[\frac{1}{[-1]} + \frac{1}{[1]}\right]\theta_1^2 = {} & \frac{50625}{2^{11}}\mathrm{m}^6 + \frac{822825}{2^{12}}\mathrm{m}^7 + \frac{65426631}{2^{16}}\mathrm{m}^8 \\
& + \frac{514143669}{2^{17}}\mathrm{m}^9 + \frac{579596224169}{2^{23}.5}\mathrm{m}^{10} + \frac{182494574380633}{2^{24}.3^2.5^2}\mathrm{m}^{11}.
\end{aligned}$$

It will be more suitable for solution if both members are multiplied by

$$-\tfrac{1}{8}[-1][1] = 4\mathrm{m} - \mathrm{m}^2 - \frac{225}{2^4}\mathrm{m}^3 - \frac{2625}{2^6}\mathrm{m}^4 - \frac{120517}{2^{10}}\mathrm{m}^5 - \frac{4587389}{2^{12}.3}\mathrm{m}^6.$$

The right member then becomes

$$\begin{aligned}
& \frac{50625}{2^9}\mathrm{m}^7 + \frac{1595025}{2^{11}}\mathrm{m}^8 + \frac{112880037}{2^{15}}\mathrm{m}^9 + \frac{1422559539}{2^{17}}\mathrm{m}^{10} \\
& \qquad + \frac{137176160137}{2^{20}.5}\mathrm{m}^{11} + \frac{477331474933393}{2^{22}.3^2.5^2}\mathrm{m}^{12}.
\end{aligned}$$

Calling this K, we have

$$\mathrm{c}^2 = 1 - \tfrac{1}{8}\theta_1^2 + \sqrt{(\theta_0 - 1 + \tfrac{1}{8}\theta_1^2)^2 - \theta_1^2 + K + \tfrac{1}{8}(\mathrm{c}^2 - \theta_0)^3}.$$

From the preceding expressions for c and Θ_0,

$$\tfrac{1}{2}(\mathrm{c}^2 - \theta_0) = -\frac{225}{2^5}\mathrm{m}^3 - \frac{3645}{2^7}\mathrm{m}^4 - \frac{159429}{2^{11}}\mathrm{m}^5 - \frac{1646333}{2^{13}}\mathrm{m}^6;$$

whence

$$\tfrac{1}{8}(\mathrm{c}^2 - \theta_0)^3 = -\frac{11390625}{2^{16}}\mathrm{m}^9 - \frac{553584375}{2^{17}}\mathrm{m}^{10} - \frac{60085546875}{2^{21}}\mathrm{m}^{11} - \frac{1228257320625}{2^{23}}\mathrm{m}^{12}.$$

The substitutions being made in the foregoing value of c^2 and the square root extracted, we get

$$\mathrm{c} = 1 + \mathrm{m} - \frac{3}{2^2}\mathrm{m}^2 - \frac{201}{2^6}\mathrm{m}^3 - \frac{2367}{2^7}\mathrm{m}^4 - \frac{111749}{2^{11}}\mathrm{m}^5 - \frac{4095991}{2^{13}.3}\mathrm{m}^6 - \frac{332532037}{2^{16}.3^2}\mathrm{m}^7 - \frac{15106211789}{2^{18}.3^3}\mathrm{m}^8 - \frac{5975332916861}{2^{23}.3^4}\mathrm{m}^9 - \frac{1547804933375567}{2^{26}.3^5.5}\mathrm{m}^{10} - \frac{818293211836767367}{2^{28}.3^6.5^2}\mathrm{m}^{11}.$$

By means of the equation

$$\frac{1}{n}\frac{d\omega}{dt} = 1 - \frac{\mathrm{c}}{1 + \mathrm{m}},$$

we derive from the last result the ratio of the motion of the perigee to the mean motion in longitude, viz.:

$$\frac{1}{n}\frac{d\omega}{dt} = \frac{3}{2^2}\mathrm{m}^2 + \frac{177}{2^6}\mathrm{m}^3 + \frac{1659}{2^7}\mathrm{m}^4 + \frac{85205}{2^{11}}\mathrm{m}^5 + \frac{3073531}{2^{13}.3}\mathrm{m}^6 + \frac{258767293}{2^{16}.3^2}\mathrm{m}^7 + \frac{12001004273}{2^{18}.3^3}\mathrm{m}^8 + \frac{4823236506653}{2^{23}.3^4}\mathrm{m}^9 + \frac{1258410742976387}{2^{26}.3^5.5}\mathrm{m}^{10} + \frac{667283922679600927}{2^{28}.3^6.5^2}\mathrm{m}^{11}.$$

In this we make the substitution $\mathrm{m} = \dfrac{m}{1 - m}$, in which m is the parameter usually employed, and prolong the resulting series only to the 9th power of m. We obtain

$$\frac{1}{n}\frac{d\omega}{dt} = \frac{3}{2^2}m^2 + \frac{225}{2^5}m^3 + \frac{4071}{2^7}m^4 + \frac{265493}{2^{11}}m^5 + \frac{12822631}{2^{13}.3}m^6 + \frac{1273925965}{2^{16}.3^2}m^7 + \frac{66702631253}{2^{18}.3^3}m^8 + \frac{29726828924189}{2^{23}.3^4}m^9.$$

The two terms ending this series are identical with M. Andoyer's and there can be no doubt as to their correctness. I do not push this series to the terms involving m^{10} and m^{11}, as I think the former in terms of the parameter m is to be preferred.

The series which has just been obtained is unsatisfactory on account of its slow convergence. It would be of great utility to transform it in such a manner that the convergence should be sensibly augmented. Here it seems no course is open but to experiment. Confining our attention to parameters of the form $\frac{\mathrm{m}}{1-\alpha \mathrm{m}}$, we may seek the value of α which brings about the greatest improvement in convergence. It is plain that the adoption of a small value for this quantity would not sensibly change the series in this respect, but as α is augmented we shall reach a value where one of the numerical coefficients vanishes; if the latter belong to a high power of m, the adjacent coefficients will be small. This is true on the assumption that the series tends to become a geometrical progression. In the present case it appears that the coefficient of m^4 is the first to vanish with augmenting α. Desiring therefore that all the coefficients may still be positive after the transformation, I adopt a value of α which is less than the value which makes the mentioned coefficient vanish. The new parameter adopted is

$$\mathsf{m} = \frac{\mathrm{m}}{1-\frac{3}{4}\mathrm{m}} = \frac{m}{1-\frac{7}{4}m}.$$

By making the denominator of α, 4, we secure the advantage that the denominators of the coefficients of the series are not augmented. With this parameter then, we have the following series:

$$\begin{aligned}\frac{1}{n}\frac{d\omega}{dt} = {} & \frac{3}{2^2}\mathsf{m}^2 + \frac{141}{2^6}\mathsf{m}^3 + \frac{57}{2^5}\mathsf{m}^4 + \frac{41213}{2^{11}}\mathsf{m}^5 + \frac{243353}{2^{12}.3}\mathsf{m}^6 + \frac{84226279}{2^{16}.3^2}\mathsf{m}^7 \\ & + \frac{1317113479}{2^{17}.3^3}\mathsf{m}^8 + \frac{1125417061277}{2^{23}.3^4}\mathsf{m}^8 + \frac{115179069708721}{2^{24}.3^5.5}\mathsf{m}^{10} \\ & + \frac{10654542330852747 7}{2^{28}.3^6.5^2}\mathsf{m}^{11}.\end{aligned}$$

The coefficients here diminish more rapidly than in the series proceeding according to powers of m.

Correspondent to the values of n and n' employed in my previous memoirs on the lunar theory we have $\mathsf{m} = 0.0860678013$. Substituting this in the series just given, we obtain the following result, exhibited term

by term; it has been assumed that the series to start from the term involving m^{11} may be regarded as a geometrical progression having the ratio $\frac{1}{3}$,

$$\begin{aligned}\frac{1}{n}\frac{d\omega}{dt} = {} & 0.0055557498 + 0.0028092554 + 0.0000977435 + 0.0000950403 \\ & + 0.0000080501 + 0.0000049959 + 0.0000011207 + 0.0000004292 \\ & + 0.0000001260 + 0.0000000418 + \underset{\text{[remainder]}}{0.0000000209} \\ = {} & 0.0085725736\,.\end{aligned}$$

The value deduced, without resorting to any expansion in powers of a parameter, is 0.0085725730. The difference of 6 units may be attributed to the uncertainty in the estimation of the remainder or to accumulated error in forming the sum of the terms of the series.

Oct. 18, 1894.

MEMOIR No. 56.

Discussion of the Observations of Jupiter with Resulting Values for the Elements of the Orbit and the Mass of Saturn.

(Astronomical Papers of the American Ephemeris, Vol. VII, pp. 5–22, 1895.)

The material employed in this discussion was derived from the published work of the following eleven observatories—the intervals of time covered by it, together with the number of observations employed in right ascension and declination, are added:

		R. A.	Dec.
Greenwich,	1750-1888	2172	2131
Palermo,	1791-1809	96	92
Paris,	1801-1883	1064	1039
Königsberg,	1814-1848	204	175
Cambridge,	1828-1865	386	233
Capetown,	1834-1860	47	75
Edinburgh,	1834-1844	240	166
Berlin,	1838-1845	49	49
Oxford,	1840-1876	131	133
Washington,	1845-1884	404	360
Brussels,	1855-1865	67	34
Whole number of observations. .		4860	4487

Only those observations were included for which the planet culminated between 16^h and 8^h of local time. An exception, however, was made in the case of the Greenwich observations in the time of Bradley.

The right ascensions were reduced to the standard of Prof. Newcomb's *Right Ascensions of the Equatorial Fundamental Stars*, and the declinations to Prof. Boss's standard.

Provisional Tables having been constructed from the theory in *Astronomical Papers*, Vol. IV, the observations of the interval 1750–1829 were compared directly with isolated places or an ephemeris computed from these tables. For the interval 1830–1888, however, it has been preferred to compare the single observations with the ephemeris contained in the *Berliner Jahrbuch* (1830–1833) or the *Nautical Almanac* (1834–1888), and thus com-

bine the material into normals. The provisional theory was then compared with these normals.

The equations of condition, formed one for each opposition, except in the time of Bradley, when sometimes additional ones were formed for quadratures, contain seven unknown quantities, the notation of which is explained as follows:

x_1 = the correction of the mean longitude for 1850.0,
x_2 = the correction of the mean motion for a century,
x_3 = the correction of the eccentricity expressed in seconds of arc,
x_4 = the correction of the longitude of the perihelion multiplied by the eccentricity,
x_5 = the correction of the inclination,
x_6 = the correction of the longitude of the ascending node multiplied by the sine of the inclination,
$1 + \frac{x_7}{1000}$ = the factor by which the mass of Saturn $\frac{1}{3501.6}$ must be multiplied.

In order to avoid a near approach to indetermination in the solution of the equations of condition the coefficients of x_7 have been derived from the perturbations of Jupiter by the action of Saturn, as they are given, not by excesses over coordinates from the mean elements, but by excesses over coordinates from elements osculating at about the middle of the period of observation. It has been found the readiest method of obtaining these coefficients to compute elliptic positions of Jupiter for the times of the equations of condition from such a set of osculating elements, and to subtract the coordinates thus obtained from the actual coordinates of the planet diminished by the small corrections arising from the action of Uranus and Neptune.

The system of osculating elements employed is the following:

$$\begin{aligned}
\text{Epoch} &= 1850,\ \text{Jan. } 0.0,\ \text{Greenwich M. T.},\\
L &= 160^\circ\ 14'\ 20''.91\,,\\
\pi &= \ \ 12\ \ \ \ 7\ \ 33.39 + 0''.146t\,,\\
\Omega &= \ \ 98\ \ 54\ \ \ 7.20 - 19''.858t - 1''.11T^2\,,\\
\log\sin i &= 8.3598404 - 114.12t + 18T^2\,,\\
\log a &= 0.7162419\,,\\
e &= 0.0485753\,,\\
n &= 109254''.93824\,.
\end{aligned}$$

In these elements the terms involving t express the secular action of Uranus and Neptune as well as the effect of the motion of the ecliptic.

The formula for passing from the correction δL_0 of an osculating element L_0 to the corresponding correction δL_m of the mean element L_m is very

simple. If $1 + \mu$ denotes the multiplier by which the mass of Saturn ought to be multiplied, the formula is

$$\delta L_m = \delta L_0 + \mu(L_m - L_0).$$

The equations which have been formed, together with the dates to which they correspond, are given below. It is to be noted that to the left members of each of the equations which are derived from the declinations must be added a term which denotes the correction to Prof. Boss's standard declinations. For convenience this term will be supposed constant through a period equivalent to about a revolution of Jupiter, and the value obtained by the solution of the equations will be attributed to the middle of the period. These corrections then will be thus denoted :

1750-1765	$-x_8$,	1826-1837	$-x_{14}$,
1766-1777	$-x_9$,	1838-1849	$-x_{15}$,
1778-1789	$-x_{10}$,	1850-1861	$-x_{16}$,
1790-1801	$-x_{11}$,	1862-1873	$-x_{17}$,
1802-1813	$-x_{12}$,	1874-1888	$-x_{18}$.
1814-1825	$-x_{13}$,		

The absolute terms of the equations which are derived from the right ascensions are $\Delta\alpha \cos \delta$, and the absolute terms of those which come from the declinations are $\Delta\delta$. For brevity the sign of equality and the zero which constitutes the right member of the equation are omitted. The number of observations on which each equation depends, together with the weight allowed to the latter in the discussion, will be given with the statement of the final residuals.

Equations from the Right Ascensions.

			x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	
1750	Nov.	19	1.234	−1.223	+0.759	−2.265	+0.404	+0.196	−0.402	+0″.56
1751	Aug.	17	0.984	0.981	1.523	1.242	0.111	0.129	0.233	−2.33
	Nov.	20	1.297	1.273	2.054	1.475	0.120	0.194	0.289	−0.46
1752	Feb.	7	1.082	1.059	1.738	1.226	+0.099	+0.209	0.239	+0.17
	Sept.	22	0.963	0.937	1.928	0.168	−0.007	−0.073	0.115	−0.77
1753	Jan.	2	1.237	1.200	2.470	0.089	+0.003	0.070	0.135	+1.52
	Mar.	29	0.994	0.962	1.993	−0.013	0.005	0.030	0.103	+1.74
	Oct.	19	0.868	0.835	1.582	+0.808	0.118	0.237	0.028	−0.06
1754	Jan.	31	1.126	1.080	1.978	1.165	0.121	0.270	−0.027	+3.85
	Dec.	5	0.841	0.800	0.870	1.515	0.329	0.199	+0.005	−2.11
1755	April	17	0.976	0.924	0.966	1.774	0.395	0.152	+0.002	+0.38
	Dec.	17	0.789	0.742	+0.014	1.643	0.368	−0.016	−0.020	+0.25
1756	April	20	1.009	0.946	−0.098	2.094	0.471	+0.057	0.033	−1.81
1757	Jan.	2	1.013	0.942	1.008	1.839	0.330	0.164	0.076	−0.74
	May	13	1.074	0.994	1.236	1.839	0.311	0.229	0.122	−1.01
	June	15	0.984	0.910	1.126	1.696	0.297	0.240	0.112	+1.02

Equations from the Right Ascensions.

			x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	
1758	Mar.	27	1.017	−0.933	−1.880	+0.870	+0.055	+0.100	−0.201	−2.93
	June	14	1.173	1.074	2.175	0.980	0.067	0.161	0.236	−1.77
	July	28	1.088	0.995	2.080	+0.729	0.062	+0.179	0.231	+0.03
1759	July	14	1.255	1.136	2.471	−0.382	0.022	−0.128	0.329	−0.61
	Sept.	20	1.072	0.968	2.116	0.369	0.022	0.078	0.276	+1.18
1760	Aug.	16	1.273	1.137	1.812	1.700	0.272	0.277	0.306	+2.16
	Nov.	25	0.986	0.879	1.312	1.416	0.243	0.179	0.227	+2.88
1761	July	13	1.079	0.955	0.438	2.045	0.414	0.116	0.158	+1.20
	Sept.	12	1.262	1.116	0.476	2.393	0.486	0.086	0.180	+1.24
	Dec.	6	1.051	0.925	−0.371	2.004	0.414	−0.017	0.145	+1.58
1762	Oct.	30	1.284	1.119	+1.011	2.272	0.377	+0.198	−0.033	+1.49
1763	Dec.	2	1.289	1.110	2.169	−1.307	0.086	+0.171	+0.046	+1.27
1765	Feb.	1	1.187	1.008	2.377	+0.081	0.013	−0.080	−0.003	+3.33
1766	Feb.	3	1.112	0.933	1.873	0.929	0.221	0.273	0.077	+2.12
1767	Mar.	8	1.035	0.857	+0.858	1.961	0.442	−0.162	0.172	+0.89
1768	April	6	1.029	0.841	−0.275	2.116	0.464	+0.079	0.232	−0.03
1769	May	9	1.091	0.880	1.396	1.762	0.275	0.231	0.237	+1.51
1770	June	10	1.192	0.949	2.277	+0.810	0.040	+0.118	−0.152	+0.62
1771	July	14	1.262	0.991	2.435	−0.587	0.044	−0.173	+0.026	−0.03
1772	Aug.	28	1.268	0.981	1.657	1.833	0.314	0.263	0.224	−0.88
1773	Sept.	28	1.269	0.968	−0.283	2.432	0.494	−0.034	0.359	+0.73
1774	Nov.	3	1.288	0.968	+1.206	2.187	0.337	+0.217	0.384	+0.35
1775	Dec.	9	1.286	0.952	2.256	−1.150	0.056	+0.140	0.281	+1.49
1777	Jan.	9	1.207	0.881	2.409	+0.297	0.032	−0.153	+0.096	+1.27
1778	Feb.	7	1.096	0.788	1.737	1.423	0.262	0.276	−0.073	+1.06
1779	Mar.	12	1.029	0.728	+0.708	2.011	0.459	−0.131	0.163	−1.87
1780	April	17	1.032	0.719	−0.422	2.098	0.448	+0.116	−0.164	+0.03
1782	June	16	1.205	0.814	2.352	+0.636	0.021	+0.081	+0.168	+1.08
1783	July	17	1.266	0.841	2.378	−0.788	0.072	−0.210	0.410	+0.50
1784	Aug.	24	1.271	0.831	1.534	1.940	0.348	−0.249	0.512	−0.72
1785	Oct.	1	1.271	0.816	−0.072	2.451	0.493	+0.009	0.432	+2.20
1786	Nov.	7	1.292	0.816	+1.392	2.085	0.295	0.228	+0.222	−1.11
1787	Dec.	12	1.279	0.794	2.347	−0.936	0.031	+0.101	−0.034	+3.13
1789	Jan.	14	1.192	0.726	2.354	+0.475	0.055	−0.186	0.210	+1.06
1790	Feb.	15	1.083	0.649	1.610	1.531	0.297	0.267	0.258	+2.16
1791	Mar.	15	1.025	0.603	+0.553	2.051	0.472	−0.099	0.194	+1.40
1792	April	28	1.035	0.597	−0.577	2.067	0.429	+0.149	−0.061	+0.91
1793	May	22	1.117	0.632	1.679	1.561	0.201	0.231	+0.127	+2.70
1794	June	18	1.218	0.676	2.412	+0.451	0.006	+0.037	0.328	+1.91
1795	July	26	1.269	0.691	2.307	−0.975	0.105	−0.236	0.450	+0.03
1796	Sept.	7	1.266	0.675	−1.317	2.074	0.392	−0.225	0.423	+0.28
1797	Oct.	6	1.273	0.665	+0.141	2.452	0.484	+0.052	0.274	+2.38
1798	Nov.	12	1.294	0.661	1.569	1.967	0.252	0.233	+0.081	+2.18
1799	Dec.	19	1.270	0.635	2.404	−0.740	0.013	+0.062	−0.066	+0.29
1801	Jan.	31	1.171	0.573	2.280	+0.643	0.082	−0.203	0.091	+1.20
1802	Feb.	28	1.071	0.512	1.479	1.630	0.331	0.250	−0.017	+2.66
1803	Mar.	28	1.023	0.478	+0.404	2.081	0.480	−0.060	+0.071	+2.84
1804	April	26	1.045	0.477	−0.745	2.031	0.403	+0.169	0.081	+1.20
1805	May	25	1.132	0.493	1.815	1.442	+0.164	+0.218	+0.022	−0.04
1806	June	23	1.229	0.535	2.455	+0.257	−0.001	−0.006	−0.138	−0.34
1807	Aug.	1	1.271	0.539	2.215	−1.160	+0.142	0.257	0.332	+1.57
1808	Sept.	14	1.265	−0.522	−1.134	−2.173	+0.424	−0.194	−0.436	+0.73

Equations from the Right Ascensions.

			x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	
1809	Oct.	17	1.274	—0.512	+0.345	—2.436	+0.468	+0.097	—0.436	+0.93
1810	Nov.	23	1.294	0.506	1.731	1.837	0.209	0.234	0.311	+2.80
1811	Dec.	25	1.264	0.481	2.447	—0.540	0.002	+0.019	—0.100	+1.73
1813	Feb.	20	1.131	0.417	2.154	+0.782	0.109	—0.209	+0.092	+2.03
1814	Feb.	23	1.060	0.380	1.334	1.729	0.364	0.241	0.203	—0.03
1815	Mar.	30	1.022	0.355	+0.245	2.106	0.483	—0.026	0.215	+0.96
1816	April	24	1.057	0.337	—0.907	1.989	0.375	+0.187	0.145	+0.49
1817	May	30	1.147	0.374	1.944	1.312	0.129	+0.202	+0.015	+3.45
1818	June	1	1.238	0.390	2.480	+0.065	0.000	—0.046	—0.174	—1.20
1819	Aug.	27	1.227	0.372	2.104	—1.267	0.170	0.238	0.278	+0.52
1820	Sept.	26	1.265	0.370	—0.950	2.258	0.463	—0.158	0.286	+1.25
1821	Nov.	5	1.258	0.354	+0.533	2.372	0.441	+0.145	—0.152	+1.03
1822	Dec.	9	1.278	0.346	1.851	1.677	0.167	+0.231	+0.003	+3.18
1824	Jan.	14	1.236	0.321	2.436	—0.350	0.000	—0.008	0.089	+1.65
1825	Feb.	3	1.144	0.285	2.109	+0.975	0.148	0.253	0.079	+1.97
1826	Mar.	16	1.043	0.248	1.199	1.785	0.389	—0.206	+0.013	+1.31
1827	April	4	1.021	0.232	+0.089	2.118	0.482	+0.010	—0.059	+0.78
1828	May	17	1.051	0.227	—1.034	1.914	0.345	0.217	0.109	+0.09
1829	June	10	1.149	0.236	2.033	+1.165	0.097	+0.186	0.114	—0.13
1830	July	24	1.230	0.239	2.458	—0.118	0.007	—0.069	—0.046	—0.51
1831	Aug.	26	1.261	0.231	1.976	1.485	0.223	0.265	+0.085	—1.48
1832	Oct.	3	1.252	0.216	—0.744	2.305	0.469	—0.114	0.221	—0.71
1833	Nov.	2	1.276	0.206	+0.754	2.347	0.414	+0.172	0.312	—0.61
1834	Dec.	11	1.283	0.193	1.997	1.521	0.127	+0.210	0.317	+0.28
1836	Jan.	20	1.221	0.170	2.434	—0.153	0.003	—0.050	0.227	+0.43
1837	Feb.	14	1.124	0.145	2.000	+1.112	0.183	0.258	+0.113	+1.14
1838	Mar.	24	1.030	0.121	+1.096	1.852	0.413	—0.179	—0.007	+0.57
1839	April	19	1.016	0.109	—0.060	2.107	0.473	+0.051	0.054	+0.50
1840	May	17	1.069	0.099	1.192	1.856	0.311	0.228	—0.025	+0.80
1841	June	10	1.175	0.101	2.160	+1.020	0.067	+0.154	+0.098	—0.04
1842	July	20	1.250	0.093	2.470	—0.327	0.021	—0.122	0.279	—0.97
1843	Aug.	28	1.264	0.080	1.843	1.646	0.267	0.270	0.399	—0.65
1844	Sept.	25	1.267	0.067	—0.540	2.387	0.487	—0.083	0.383	—1.06
1845	Oct.	23	1.286	0.054	+0.973	2.291	0.378	+0.187	0.248	—0.11
1846	Dec.	18	1.277	0.039	2.110	—1.350	0.091	+0.186	+0.057	+0.58
1848	Jan.	16	1.221	0.024	2.444	+0.048	0.013	—0.102	—0.103	+1.13
1849	Feb.	19	1.108	—0.010	1.887	1.248	0.219	0.264	0.136	+1.41
1850	Mar.	22	1.030	+0.002	+0.903	1.930	0.437	—0.157	0.130	+0.71
1851	April	28	1.012	0.013	—0.213	2.090	0.461	+0.087	—0.023	+1.40
1852	May	11	1.090	0.026	1.363	1.785	0.276	0.228	+0.104	+0.69
1853	June	13	1.189	0.041	2.256	+0.854	0.041	+0.119	0.255	+0.57
1854	Aug.	9	1.230	0.057	2.395	—0.507	0.040	—0.144	0.344	—0.20
1855	Sept.	7	1.255	0.071	1.686	1.777	0.308	0.257	0.329	—0.47
1856	Oct.	5	1.264	0.085	—0.358	2.417	0.493	—0.028	0.135	—1.03
1857	Nov.	8	1.286	0.101	+1.154	2.208	0.341	+0.216	+0.048	—0.86
1858	Dec.	24	1.271	0.014	2.210	—1.171	0.060	+0.156	—0.080	+0.05
1860	Jan.	31	1.192	0.120	2.383	+0.227	0.029	—0.128	0.130	+0.04
1861	Mar.	3	1.085	0.121	1.756	1.357	0.254	0.256	—0.020	+0.84
1862	April	12	0.998	0.123	+0.742	1.930	0.445	—0.113	+0.097	—0.12
1863	April	30	1.020	0.136	—0.372	2.083	0.446	+0.117	0.175	—0.25
1864	May	29	1.089	0.157	1.482	1.682	0.240	0.240	+0.093	—0.48
1865	July	1	1.188	+0.184	—2.308	+0.685	+0.024	+0.097	—0.027	—0.61

Equations from the Right Ascensions.

			x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	
1866	July	29	1.260	+0.209	—2.390	—0.724	+0.068	—0.196	—0.295	—1.73
1867	Sept.	8	1.260	0.223	1.512	1.929	0.352	—0.248	0.534	—2.04
1868	Oct.	10	1.266	0.238	—0.130	2.439	0.492	+0.009	0.639	—0.67
1869	Nov.	17	1.285	0.256	+1.333	2.106	0.300	0.233	0.580	—0.30
1870	Dec.	14	1.282	0.269	2.331	—0.984	0.033	+0.103	0.368	—0.15
1872	Feb.	12	1.161	0.257	2.306	+0.402	0.050	—0.154	—0.082	+0.04
1873	Mar.	18	1.051	0.244	1.602	1.441	0.284	0.240	+0.124	+0.15
1874	April	6	1.012	0.246	+0.600	2.010	0.465	—0.089	0.224	+0.08
1875	May	8	1.020	0.259	—0.525	2.050	0.425	+0.148	0.209	—0.52
1876	May	28	1.110	0.293	1.637	1.587	0.204	0.233	+0.088	—0.55
1877	June	26	1.213	0.333	2.394	+0.502	0.008	+0.047	—0.146	—1.41
1878	Aug.	9	1.256	0.359	2.312	—0.908	0.100	—0.219	0.392	—2.12
1879	Sept.	16	1.255	0.373	—1.356	2.026	0.385	—0.221	0.530	—1.53
1880	Oct.	22	1.258	0.388	+0.074	2.427	0.482	+0.058	0.493	—0.89
1881	Nov.	26	1.281	0.409	1.502	1.987	0.257	0.241	0.332	—0.90
1882	Dec.	23	1.273	0.420	2.391	—0.792	0.015	+0.067	0.155	—0.72
1884	Feb.	15	1.150	0.392	2.250	+0.578	0.077	—0.186	0.071	+0.06
1885	Mar.	5	1.067	0.375	1.506	1.593	0.326	0.248	0.041	+1.14
1886	April	11	1.008	0.366	+0.447	2.041	0.473	—0.054	0.083	+0.61
1887	April	19	1.044	0.390	—0.701	2.045	0.405	+0.167	0.152	+0.11
1888	May	25	1.134	+0.435	—1.813	+1.450	+0.158	+0.207	—0.221	—0.38

Equations from the Declinations.

			x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	
1750	Nov.	19	+0.486	—0.481	+0.296	—0.892	—1.028	—0.476	—0.162	+1″.05
1751	Aug.	17	0.178	0.176	0.270	0.225	0.623	0.725	0.040	—2.18
	Nov.	20	0.238	0.234	0.376	0.276	0.651	1.032	0.048	—4.33
1752	Feb.	7	+0.241	—0.236	+0.389	—0.270	0.446	0.941	—0.046	+0.72
	Sept.	22	—0.072	+0.070	—0.144	+0.009	—0.097	0.968	+0.018	—3.58
1753	Jan.	2	0.071	0.068	0.141	0.006	+0.061	1.231	0.023	—2.07
	Mar.	29	0.029	0.028	0.058	+0.005	0.175	0.996	0.017	+0.21
	Oct.	19	0.248	0.239	0.452	—0.235	0.412	0.825	0.021	+0.13
1754	Jan.	31	0.313	0.300	0.552	0.321	0.675	0.970	0.026	—0.91
	Dec.	5	0.348	0.331	0.359	0.628	0.795	0.481	0.011	—0.48
1755	April	17	0.384	0.364	0.385	0.696	1.000	0.385	0.012	—0.39
	Dec.	17	0.332	0.312	—0.006	0.691	0.880	—0.039	0.015	—0.47
1756	April	20	0.430	0.403	+0.034	0.892	1.105	+0.134	0.019	+0.48
1757	Jan.	2	0.335	0.311	0.336	0.606	0.998	0.495	0.024	+1.58
	May	13	0.356	0.330	0.406	0.612	0.937	0.690	0.036	+0.50
	June	15	0.349	0.323	0.399	0.603	0.840	0.676	0.036	+1.01
1758	Mar.	27	0.109	0.100	0.199	0.098	0.519	0.942	0.014	—1.71
	June	14	0.167	0.153	0.308	0.141	0.468	1.128	0.024	—0.13
	July	28	—0.184	+0.168	+0.352	0.120	+0.366	1.063	+0.032	+0.20
1759	July	14	+0.131	—0.119	—0.259	0.039	—0.212	1.219	—0.045	—0.28
	Sept.	20	0.079	0.071	0.158	0.022	0.290	1.034	0.031	—0.55
1760	Aug.	16	0.415	0.371	0.595	0.551	0.834	0.849	0.114	—0.29
	Nov.	25	0.319	0.284	0.428	0.455	0.748	0.554	0.084	+0.46
1761	July	13	0.465	0.411	0.190	0.881	0.962	0.269	0.078	+0.42
	Sept.	12	0.544	0.480	0.214	1.028	1.130	0.201	0.089	+0.29
	Dec.	6	0.447	0.394	—0.160	0.853	0.971	+0.039	0.070	+0.63
1762	Oct.	30	+0.466	—0.406	+0.360	—0.827	—1.040	—0.546	—0.012	0.00

Equations from the Declinations.

			x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	
1763	Dec.	2	+0.201	—0.173	+0.337	—0.207	—0.551	—1.095	+0.017	—1.46
1765	Feb.	1	—0.080	+0.068	—0.160	0.001	+0.198	1.179	0.014	—1.16
1766	Feb.	3	0.332	0.279	0.559	0.386	0.740	0.912	0.036	0.00
1767	Mar.	8	0.431	0.357	—0.363	0.814	1.060	—0.390	0.081	—2.27
1768	April	6	0.428	0.349	+0.109	0.881	1.115	+0.191	0.102	—1.16
1769	May	9	0.332	0.268	0.421	0.540	0.901	0.757	0.072	—2.16
1770	June	10	—0.121	+0.096	+0.230	0.084	+0.396	1.166	+0.008	—1.78
1771	July	14	+0.183	—0.143	—0.353	0.084	—0.303	1.196	—0.012	—2.29
1772	Aug.	28	0.439	0.340	0.584	0.627	0.905	0.756	+0.058	—0.67
1773	Sept.	28	0.546	0.417	—0.130	1.047	1.148	+0.079	0.145	—1.56
1774	Nov.	3	0.437	0.328	+0.403	0.745	0.995	—0.639	0.137	+4.52
1775	Dec.	9	+0.158	—0.117	+0.276	0.143	—0.453	1.144	0.051	+2.95
1777	Jan.	9	—0.155	+0.113	—0.308	0.037	+0.251	1.195	0.004	—0.12
1778	Feb.	7	0.358	0.257	0.570	0.461	0.801	0.846	0.035	—1.58
1779	Mar.	12	0.436	0.308	—0.305	0.850	1.083	—0.309	0.077	—1.96
1780	April	17	0.422	0.294	+0.166	0.858	1.098	+0.283	+0.071	—0.50
1782	June	16	—0.082	+0.055	+0.159	0.044	+0.302	1.199	—0.025	—0.09
1783	July	17	+0.229	—0.152	—0.431	0.140	—0.397	1.162	+0.052	—0.44
1784	Aug.	24	0.462	0.302	0.547	0.714	0.957	+0.686	0.168	—7.22
1785	Oct.	1	0.544	0.349	—0.039	1.048	1.152	—0.021	0.182	—2.00
1786	Nov.	7	0.405	0.256	+0.431	0.657	0.941	0.728	0.079	—0.79
1787	Dec.	12	+0.110	—0.068	+0.200	0.082	—0.360	1.180	0.007	+0.38
1789	Jan.	14	—0.190	+0.116	—0.376	0.074	+0.345	1.165	0.041	+1.32
1790	Feb.	15	0.374	0.224	0.559	0.525	0.861	0.774	0.095	—2.77
1791	Mar.	15	0.439	0.258	—0.243	0.877	1.101	—0.231	0.089	—1.90
1792	Apr.	28	0.413	0.238	+0.225	0.826	1.074	+0.372	+0.025	—0.73
1793	May	22	0.286	0.162	0.428	0.403	0.783	0.900	—0.041	+1.92
1794	June	18	—0.037	+0.020	+0.073	0.015	+0.206	1.222	—0.028	+0.83
1795	July	26	+0.268	—0.146	—0.488	0.203	—0.497	1.116	+0.073	+0.13
1796	Sept.	7	0.490	0.261	—0.517	0.800	1.011	+0.579	0.150	—0.64
1797	Oct.	6	0.536	0.280	+0.051	1.034	1.147	—0.124	0.115	+0.13
1798	Nov.	12	0.370	0.189	0.444	0.566	0.880	0.813	0.031	—0.72
1799	Dec.	19	+0.066	—0.033	+0.124	0.039	—0.256	1.207	0.006	+0.73
1801	Jan.	31	—0.213	+0.104	—0.415	0.211	+0.449	1.117	0.026	+0.84
1802	Feb.	28	0.386	0.184	0.538	0.582	0.917	0.693	+0.016	—1.56
1803	Mar.	28	0.440	0.206	—0.170	0.896	1.115	—0.139	—0.025	—0.69
1804	April	26	0.399	0.182	+0.278	0.778	1.054	+0.443	0.031	+2.76
1805	May	25	—0.257	+0.115	+0.410	0.330	0.722	0.961	0.009	+0.30
1806	June	23	+0.006	—0.003	—0.012	0.000	+0.115	1.236	0.004	+0.39
1807	Aug.	1	0.307	0.130	0.537	0.277	—0.586	1.064	0.079	+0.31
1808	Sept.	14	0.508	0.210	—0.463	0.870	1.054	+0.484	0.176	+1.02
1809	Oct.	17	0.526	0.212	+0.134	1.007	1.132	—0.236	0.185	—0.15
1810	Nov.	23	0.337	0.132	0.447	0.482	0.804	0.899	—0.085	+1.07
1811	Dec.	25	+0.019	—0.007	+0.037	0.009	—0.155	1.227	+0.002	—0.86
1813	Feb.	20	—0.227	+0.084	—0.433	0.153	+0.544	1.040	+0.005	—2.15
1814	Feb.	23	0.405	0.145	0.514	0.657	0.954	0.632	—0.067	—1.65
1815	Mar.	30	0.440	0.153	—0.112	0.906	1.123	—0.060	0.088	—1.94
1816	April	24	0.384	0.129	+0.324	0.725	1.032	+0.515	0.055	—0.53
1817	May	30	—0.227	+0.074	+0.383	—0.263	0.650	1.020	0.007	+1.69
1818	July	1	+0.046	—0.014	—0.092	+0.002	+0.009	1.242	0.009	+1.28
1819	Aug.	27	0.301	0.112	0.514	—0.315	—0.695	0.970	0.070	—0.18
1820	Sept.	26	+0.534	—0.156	—0.409	—0.950	—1.095	+0.375	—0.125	—0.10

Equations from the Declinations.

			x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	
1821	Nov.	5	+0.509	—0.143	+0.209	—0.961	—1.088	—0.357	—0.066	+0.14
1822	Dec.	9	+0.304	—0.082	+0.438	—0.400	0.702	0.972	+0.001	+0.07
1824	Jan.	14	—0.008	+0.002	—0.015	+0.004	—0.025	1.221	+0.003	—0.97
1825	Feb.	3	0.282	0.070	0.521	—0.237	+0.602	1.028	—0.015	—0.49
1826	Mar.	16	0.405	0.096	0.471	0.691	0.999	—0.530	—0.003	—0.46
1827	April	4	0.437	0.099	—0.044	0.907	1.124	+0.023	+0.027	—0.92
1828	May	17	0.373	0.081	+0.363	0.681	0.970	0.611	0.041	+0.56
1829	June	10	—0.199	+0.041	+0.351	0.205	+0.558	1.069	+0.021	0.00
1830	July	24	+0.069	—0.013	—0.139	0.004	—0.123	1.221	—0.004	+0.31
1831	Aug.	26	0.366	0.067	0.578	0.427	0.766	0.912	+0.020	+0.28
1832	Oct.	3	0.528	0.091	—0.325	0.968	1.110	+0.269	0.088	—0.28
1833	Nov.	2	0.492	0.079	+0.284	0.908	1.073	—0.446	0.122	—0.40
1834	Dec.	11	+0.259	—0.039	+0.402	—0.310	—0.626	1.036	+0.071	—0.43
1836	Jan.	20	—0.050	+0.007	—0.100	+0.008	+0.076	1.214	—0.003	—0.37
1837	Feb.	14	0.302	0.039	0.539	—0.295	0.681	0.962	—0.026	—0.17
1838	Mar.	24	0.412	0.048	—0.425	0.739	1.030	—0.446	+0.005	—0.22
1839	April	19	0.432	0.046	+0.019	0.896	1.113	+0.120	0.025	+0.02
1840	May	17	0.356	0.033	0.393	0.621	0.938	0.684	+0.009	+0.14
1841	June	10	—0.161	+0.014	+0.295	0.143	+0.488	1.123	—0.015	+0.24
1842	July	20	+0.126	—0.009	—0.248	0.031	—0.208	1.217	+0.021	—0.04
1843	Aug.	28	0.404	0.026	0.594	0.522	0.834	0.842	0.120	—0.13
1844	Sept.	25	0.544	0.029	—0.239	1.023	1.135	+0.193	0.164	—0.38
1845	Oct.	23	0.461	0.019	+0.342	0.826	1.053	—0.521	0.093	—1.41
1846	Dec.	18	+0.216	—0.007	+0.359	0.232	—0.533	1.090	0.012	—0.67
1848	Jan.	16	—0.102	+0.002	—0.205	0.003	+0.159	1.213	0.007	—0.94
1849	Feb.	19	0.325	+0.003	0.556	0.290	0.731	0.900	0.038	—0.21
1850	Mar.	22	0.425	—0.001	—0.378	0.794	1.059	—0.379	0.053	+0.10
1851	April	28	0.425	0.006	+0.084	0.878	1.096	+0.208	+0.010	—0.12
1852	May	11	0.332	0.008	0.410	0.546	0.907	0.750	—0.033	+0.25
1853	June	13	—0.122	—0.004	+0.231	0.090	+0.404	1.163	—0.032	+0.94
1854	Aug.	9	+0.151	+0.007	—0.296	0.059	—0.328	1.165	+0.035	+0.32
1855	Sept.	7	0.429	0.024	0.581	0.602	0.900	0.751	0.108	—0.16
1856	Oct.	5	0.544	0.037	—0.162	1.039	1.144	+0.081	0.059	—1.22
1857	Nov.	8	0.441	0.035	+0.389	0.760	0.996	—0.632	+0.019	—0.57
1858	Dec.	24	+0.174	+0.016	+0.302	0.162	—0.438	1.135	—0.012	—0.22
1860	Jan.	31	—0.130	—0.013	—0.260	0.022	+0.270	1.175	+0.009	—0.94
1861	Mar.	3	0.339	0.038	0.552	0.420	0.811	0.820	+0.006	—0.30
1862	April	12	0.417	0.051	—0.315	0.804	1.064	—0.269	—0.039	+0.67
1863	April	30	0.419	0.056	+0.147	0.857	1.086	+0.286	0.071	+0.21
1864	May	29	0.315	0.045	0.426	0.489	0.830	0.830	—0.025	+0.85
1865	July	1	—0.097	—0.015	+0.188	0.056	+0.288	1.190	+0.008	+1.01
1866	July	29	+0.213	+0.035	—0.406	0.120	—0.402	1.157	—0.038	—0.05
1867	Sept.	8	0.464	0.082	0.569	0.702	0.954	+0.672	0.184	—0.71
1868	Oct.	10	0.542	0.102	—0.068	1.045	1.147	—0.021	0.274	—0.15
1869	Nov.	17	0.413	0.082	+0.423	0.680	0.935	0.726	0.200	—0.68
1870	Dec.	14	+0.113	+0.024	+0.204	0.089	—0.372	1.176	—0.046	—0.54
1872	Feb.	12	—0.159	—0.035	—0.315	0.052	+0.368	1.128	+0.005	—0.13
1873	Mar.	18	0.346	0.080	0.531	0.471	0.861	0.727	—0.042	+0.57
1874	April	6	0.431	0.105	—0.261	0.854	1.092	—0.208	0.095	+0.32
1875	May	8	0.409	0.104	+0.206	0.823	1.060	+0.370	0.084	+1.20
1876	May	28	0.289	0.076	0.423	0.416	0.781	0.894	—0.021	+1.24
1877	June	26	—0.046	—0.013	+0.091	—0.021	+0.211	+1.219	+0.014	+0.34

Equations from the Declinations.

			x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	
1878	Aug.	9	+0.248	+0.071	−0.459	−0.176	−0.503	+1.105	−0.062	+0.01
1879	Sept.	16	0.480	0.143	−0.525	0.772	1.005	+0.577	0.193	−0.52
1880	Oct.	22	0.534	0.164	+0.024	1.030	1.134	−0.136	0.213	−0.33
1881	Nov.	26	0.380	0.121	0.442	0.593	0.864	0.812	0.111	−1.12
1882	Dec.	23	+0.070	+0.023	+0.132	0.045	−0.264	1.206	−0.022	−0.32
1884	Feb.	15	−0.196	−0.067	−0.384	0.095	+0.452	1.093	0.000	+0.02
1885	Mar.	5	0.382	0.134	0.543	0.566	0.912	0.693	+0.005	−0.13
1886	April	11	0.432	0.157	−0.198	0.875	1.102	−0.126	0.031	+0.28
1887	April	29	0.400	0.149	+0.263	0.786	1.057	+0.436	0.061	+0.97
1888	May	25	−0.244	−0.094	+0.383	−0.322	+0.732	+0.957	+0.058	+0.73

After the elimination of the eleven unknowns $x_8 \ldots . x_{18}$, the normal equations stand as follows:

x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.		
94.69	−2.31	− 0.14	− 9.82	+ 0.27	0.00	−2.286	− 5″.59	=0,
− 2.32	+8.31	+ 1.13	+ 0.84	− 0.09	+ 0.04	−1.194	−11.01	=0,
− 0.14	+1.13	+188.10	− 3.09	− 0.10	− 0.20	−1.790	+47.64	=0,
− 9.82	+0.84	− 3.09	+174.84	+ 1.84	+ 0.30	+4.750	+41.02	=0,
+0.27	−0.09	− 0.10	+ 1.84	+46.56	+ 0.15	+0.159	+13.48	=0,
0.00	+0.04	− 0.20	+ 0.30	+ 0.15	+45.30	+0.173	+16.89	=0,
− 2.286	−1.194	− 1.790	+ 4.750	+ 0.159	+ 0.173	+4.184	+ 3.346	=0,

After the values of the unknowns in these equations have been obtained, the values of the rest are given by the equations

$$x_8 = +0''.54 - 0.04x_1 + 0.03x_2 - 0.02x_3 + 0.41x_4 + 0.10x_5 + 0.01x_6 + 0.02x_7,$$

$x_9=$	+0.60	+0.05	−0.06	+0.12	+0.43	−0.18	+0.03	−0.05
$x_{10}=$	+0.21	+0.06	−0.05	+0.16	+0.30	−0.11	+0.30	−0.04
$x_{11}=$	+0.30	+0.01	−0.02	+0.13	+0.42	−0.09	−0.12	−0.06
$x_{12}=$	+0.03	−0.02	0.00	+0.13	+0.47	−0.02	−0.03	+0.06
$x_{13}=$	+0.27	−0.08	+0.02	+0.07	+0.54	+0.12	+0.07	+0.05
$x_{14}=$	+0.19	−0.03	0.00	+0.07	+0.47	+0.03	+0.25	−0.03
$x_{15}=$	+0.33	0.00	0.00	+0.08	+0.48	−0.06	0.00	−0.04
$x_{16}=$	+0.17	0.00	0.00	+0.07	+0.49	−0.07	0.00	−0.02
$x_{17}=$	−0.10	0.00	0.00	+0.08	+0.49	−0.06	−0.01	+0.08
$x_{18}=$	−0.13	+0.04	+0.02	+0.04	+0.52	−0.17	−0.11	+0.07

The values of the unknowns which result from the solution of these equations are

$x_1 = +0''.065$	$x_7 = -0''.172$	$x_{13} = +0''.07$
$x_2 = +1.376$	$x_8 = +0.46$	$x_{14} = -0.03$
$x_3 = -0.268$	$x_9 = +0.43$	$x_{15} = +0.21$
$x_4 = -0.234$	$x_{10} = -0.04$	$x_{16} = +0.06$
$x_5 = -0.276$	$x_{11} = +0.23$	$x_{17} = -0.23$
$x_6 = -0.371$	$x_{12} = -0.11$	$x_{18} = -0.17$

The sum of the squares of the residuals by this solution is reduced from $[nn] = 101''.42$ to $[nn.\ 18] = 49''.57$, and the probable error of a normal of the weight unity is $\pm\ 0''.293$.

The mass of Saturn which results from this investigation is, with its probable error,

$$\frac{1}{3502.20 \pm 0.53}.$$

But, as the value of this mass deduced from the measures of the satellites is somewhat larger, we will adhere to the value given by Bessel, and thus assume that $x_7 = 0$. The values of the corrections of the elements, which accord with this assumption, are

$$\begin{aligned} \delta L &= +\ 0''.07\,, \\ \delta n &= +\ 0''.01402\,, \\ \delta e &= -\ 0''.266\,, \\ e\delta\pi &= -\ 0''.239\,, \\ \delta i &= -\ 0''.277\,, \\ \sin i\delta\,\Omega &= -\ 0''.372\,. \end{aligned}$$

By applying these corrections to the elements of the provisional theory, given in *Astronomical Papers*, Vol. IV, p. 558, we obtain the following

Epoch 1850, Jan. 0.0, Greenwich M. T.

$$\begin{aligned} L &= 159^\circ\ 56'\ 25''.05 \\ \pi &= \ \ 11\ \ 54\ \ 26.72 \\ \Omega &= \ \ 98\ \ 56\ \ \ 3.54 + .35.9\delta\epsilon \\ i &= \ \ \ 1\ \ 18\ \ 41.82 + \ 0.087\delta\epsilon \\ e &= 0.04825382 \\ n &= 109256''.63954 \\ m' &= \frac{1}{3501.6} \end{aligned}$$

If we denote the correction to be applied to the declinations of equatorial stars in Prof. Boss's system by

$$x + y\,\frac{t - 1850}{100}\,,$$

the preceding values of $x_8 \ldots x_{18}$ furnish the following equations for determining x and y, to which we join their weights,

	Weight.		Weight.
$x - 0.92y + 0''.46 = 0$	0.25	$x - 0.18y - 0''.03$	0.7
$x - 0.78y + 0.43 = 0$	0.06	$x - 0.06y + 0.21$	1.0
$x - 0.67y - 0.04 = 0$	0.04	$x + 0.06y + 0.06$	1.0
$x - 0.54y + 0.23 = 0$	0.12	$x + 0.18y - 0.23$	1.0
$x - 0.42y - 0.11 = 0$	0.2	$x + 0.32y - 0.17$	1.0
$x - 0.30y + 0.07 = 0$	0.2		

The discussion of the observations of Saturn adds to these the equations

	Weight.
$x - 0.89y + 0''.57 = 0$	0.10
$x - 0.65y + 1.23 = 0$	0.04
$x - 0.36y + 0.05 = 0$	0.2
$x - 0.07y - 0.08 = 0$	1.0
$x + 0.23y - 0.02 = 0$	1.0

The normal equations derived from these are

$$7.915x - 0.167y + 0''.025 = 0,$$
$$-0.167x + 0.700y - 0.312 = 0.$$

The formula for the correction in question is then

$$+ 0''.01 + 0''.0045\ (t - 1850).$$

The residuals left by the foregoing solution in the case of each normal, together with the number of observations the latter is founded upon, and the weight it has received in the discussion, are given below:

Date.			Obs.-Cal. $\Delta\alpha \cos\delta$.	$\Delta\delta$.	No. of Obs. α.	δ.	Weight. α.	δ.
1750	Nov.	19	+0''.83	−1''.48	4	4	0.05	0.05
1751	Aug.	17	+3.78	+1.53	9	9	0.1	0.1
	Nov.	20	+2.37	+3.66	7	7	0.1	0.1
1752	Feb.	7	+1.46	−1.29	5	4	0.05	0.05
	Sept.	22	+2.43	+2.62	13	13	0.15	0.15
1753	Jan.	2	+0.64	+1.07	3	3	0.05	0.05
	Mar.	29	+0.02	−1.02	4	3	0.05	0.05
	Oct.	19	+1.70	−1.22	8	8	0.1	0.1
1754	Jan.	31	−1.71	−0.30	2	2	0.025	0.025
	Dec.	5	+3.76	−0.66	2	2	0.025	0.025
1755	April	17	+1.56	−0.63	6	5	0.1	0.1
	Dec.	17	+1.19	−0.28	4	4	0.05	0.05
1756	April	20	+3.65	−1.26	5	5	0.05	0.05
1757	Jan.	2	+2.27	−1.97	3	2	0.05	0.025
	May	13	+2.56	−0.84	14	13	0.15	0.15
	June	15	+0.43	−1.38	9	9	0.1	0.1
1758	Mar.	27	+3.88	+1.70	3	3	0.05	0.05
	June	14	+2.86	+0.13	12	12	0.15	0.15
	July	28	+0.92	−0.26	7	6	0.1	0.1
1759	July	14	+1.24	+0.33	11	10	0.15	0.15
	Sept.	20	−0.65	+0.49	11	11	0.15	0.15
1760	Aug.	16	−1.65	+0.16	3	3	0.05	0.05
	Nov.	25	−2.45	−0.73	3	1	0.05	0.025
1761	July	13	−0.52	−0.70	9	6	0.1	0.1
	Sept.	12	−0.40	−0.62	17	13	0.15	0.15
	Dec.	6	−0.86	−1.02	19	18	0.15	0.15

Date.			Obs.-Cal. $\Delta\alpha \cos\delta$.	$\Delta\delta$.	No. of Obs. α.	δ.	Weight. α.	δ.
1762	Oct.	30	—0.13	—0.43	10	10	0.1	0.1
1763	Dec.	2	+0.54	+0.78	12	11	0.15	0.15
1765	Feb.	1	—1.40	+0.28	6	5	0.1	0.1
1766	Feb.	3	—0.23	—1.10	7	7	0.1	0.1
1767	Mar.	8	+0.94	+1.32	3	3	0.05	0.05
1768	April	6	+1.67	+0.57	4	4	0.05	0.05
1769	May	9	—0.22	+1.98	5	4	0.1	0.05
1770	June	10	+0.21	+1.89	5	5	0.1	0.1
1771	July	14	+0.47	+2.40	4	4	0.05	0.05
1772	Aug.	28	+1.41	+0.50	4	3	0.05	0.05
1773	Sept.	28	+0.06	+1.21	4	4	0.1	0.05
1774	Nov.	3	+0.95	—4.98	4	4	0.05	0.05
1775	Dec.	9	+0.20	—3.60	3	2	0.05	0.025
1777	Jan.	9	+0.56	—0.82	5	5	0.1	0.1
1778	Feb.	7	+0.71	+0.59	4	4	0.1	0.05
1779	Mar.	12	+3.49	+1.17	2	2	0.025	0.025
1780	April	17	+1.40	+0.09	2	2	0.025	0.025
1782	June	16	—0.45	+0.29	4	4	0.05	0.05
1783	July	17	—0.23	+0.53	3	3	0.05	0.05
1784	Aug.	24	+1.02	+7.04	3	3	0.05	0.00
1785	Oct.	1	—1.54	+1.60	2	2	0.025	0.025
1786	Nov.	7	+2.24	+0.29	2	2	0.025	0.025
1787	Dec.	12	—1.67	—1.08	3	3	0.05	0.05
1789	Jan.	14	+0.51	—2.17	6	5	0.1	0.1
1790	Feb.	15	—0.61	+1.92	9	9	0.15	0.15
1791	Mar.	15	+0.04	+1.29	6	6	0.1	0.1
1792	April	28	+0.33	+0.49	24	24	0.25	0.2
1793	May	22	—1.81	—1.80	7	7	0.1	0.1
1794	June	18	—1.53	—0.57	12	12	0.15	0.15
1795	July	26	+0.01	—0.08	11	11	0.15	0.15
1796	Sept.	7	—0.16	+0.38	22	22	0.25	0.2
1797	Oct.	6	—1.88	—0.58	5	5	0.1	0.1
1798	Nov.	12	—1.22	+0.20	7	7	0.1	0.1
1799	Dec.	19	+0.98	—1.40	3	3	0.025	0.05
1801	Jan.	31	+0.20	—1.64	12	11	0.15	0.15
1802	Feb.	28	—1.24	+0.84	15	15	0.15	0.15
1803	Mar.	28	—1.53	+0.24	22	19	0.25	0.2
1804	April	26	—0.15	—2.84	12	12	0.15	0.15
1805	May	25	+0.62	—0.04	19	19	0.2	0.2
1806	June	23	+0.38	—0.12	22	22	0.2	0.2
1807	Aug.	1	—1.88	—0.32	26	25	0.25	0.25
1808	Sept.	14	—0.92	—1.40	25	24	0.25	0.25
1809	Oct.	17	—0.68	—0.40	22	22	0.2	0.2
1810	Nov.	23	—2.05	—1.63	8	8	0.15	0.15
1811	Dec.	25	—0.63	+0.22	12	12	0.2	0.2
1813	Feb.	20	—0.80	+1.48	25	25	0.25	0.25
1814	Feb.	23	+1.28	+1.06	17	16	0.2	0.2
1815	Mar.	30	+0.18	+1.64	18	18	0.2	0.2
1816	April	24	+0.32	+0.61	22	11	0.2	0.15

Date.			Obs. Cal. $\Delta\alpha \cos \delta$.	$\Delta\delta$.	No. of Obs. α.	δ.	Weight. α.	δ.
1817	May	30	—3.11	—1.32	17	9	0.2	0.15
1818	June	1	+0.96	—0.95	15	8	0.15	0.15
1819	Aug.	27	—1.04	+0.13	31	31	0.25	0.25
1820	Sept.	26	—1.58	—0.35	44	43	0.3	0.3
1821	Nov.	5	—0.89	—0.69	37	40	0.3	0.3
1822	Dec.	9	—2.53	—0.61	39	38	0.3	0.3
1824	Jan.	14	—0.70	+0.40	22	22	0.2	0.2
1825	Feb.	3	—0.89	—0.08	14	14	0.15	0.15
1826	Mar.	16	—0.27	+0.05	36	36	0.6	0.6
1827	April	4	+0.11	+0.81	17	19	0.4	0.4
1828	May	17	+0.47	—0.29	49	31	0.6	0.6
1829	June	10	+0.19	+0.46	38	9	0.6	0.3
1830	July	24	+0.03	+0.01	47	23	0.7	0.6
1831	Aug.	26	+0.81	—0.41	52	30	0.7	0.6
1832	Oct.	3	+0.32	—0.20	55	23	0.7	0.5
1833	Nov.	2	+0.68	—0.16	60	60	0.7	1.0
1834	Dec.	11	+0.26	—0.11	87	105	1	1
1836	Jan.	20	+0.35	—0.15	53	35	1	1
1837	Feb.	14	—0.24	—0.29	81	84	1	1
1838	Mar.	24	+0.29	—0.03	114	103	1	1
1839	April	19	+0.19	+0.06	116	103	1	1
1840	May	17	—0.44	+0.26	134	113	1	1
1841	June	10	—0.14	+0.33	79	70	1	1
1842	July	20	+0.30	+0.32	139	138	1	1
1843	Aug.	28	—0.16	—0.06	135	127	1	1
1844	Sept.	25	+0.55	—0.15	87	74	1	1
1845	Oct.	23	+0.03	+0.84	115	88	1	1
1846	Dec.	18	—0.26	+0.16	53	53	1	1
1848	Jan.	16	—0.58	+0.49	38	37	1	1
1849	Feb.	19	—0.73	—0.11	47	50	1	1
1850	Mar.	22	—0.05	—0.18	69	52	1	1
1851	April	28	—0.90	+0.36	40	38	1	1
1852	May	11	—0.55	+0.30	47	54	1	1
1853	June	13	—1.03	—0.33	51	51	1	1
1854	Aug.	9	—0.70	—0.05	66	65	1	1
1855	Sept.	7	—0.53	—0.12	102	75	1	1
1856	Oct.	5	+0.31	+0.62	77	84	1	1
1857	Nov.	8	+0.61	—0.06	59	58	1	1
1858	Dec.	24	+0.10	—0.26	58	50	1	1
1860	Jan.	31	+0.34	+0.58	83	70	1	1
1861	Mar.	3	—0.32	+0.10	62	58	1	1
1862	April	12	+0.64	—0.61	91	87	1	1
1863	April	30	+0.57	+0.21	131	143	1	1
1864	May	29	+0.37	—0.15	92	87	1	1
1865	July	1	—0.13	—0.34	77	63	1	1
1866	July	29	+0.46	+0.24	72	74	1	1
1867	Sept.	8	+0.72	+0.31	92	93	1	1
1868	Oct.	10	—0.31	—0.58	100	98	1	1
1869	Nov.	17	—0.19	+0.03	50	47	1	1

Date.			Obs.-Cal. $\Delta\alpha \cos\delta$.	$\Delta\delta$.	No. of Obs. α.	δ.	Weight. α.	δ.
1870	Dec.	14	+0.08	+0.08	19	25	1	1
1872	Feb.	12	+0.18	—0.12	59	52	1	1
1873	Mar.	18	+0.22	—0.62	37	36	1	1
1874	April	6	+0.28	—0.10	68	70	1	1
1875	May	8	+0.65	—0.62	85	88	1	1
1876	May	28	+0.18	—0.43	96	98	1	1
1877	June	26	+0.34	+0.34	63	62	1	1
1878	Aug.	9	+0.60	+0.11	58	59	1	1
1879	Sept.	16	+0.03	+0.02	71	74	1	1
1880	Oct.	22	—0.20	—0.41	74	75	1	1
1881	Nov.	26	+0.30	+0.50	77	76	1	1
1882	Dec.	23	+0.52	—0.05	35	36	0.8	0.8
1884	Feb.	15	+0.01	—0.16	49	31	0.8	0.8
1885	Mar.	5	—0.97	+0.22	35	34	0.8	0.8
1886	April	11	—0.48	+0.15	28	31	0.5	0.5
1887	April	29	—0.29	—0.21	25	26	0.5	0.5
1888	May	25	—0.36	+0.20	28	28	0.5	0.5

MEMOIR No. 57.

Discussion of the Observations of Saturn with Resulting Values for the Elements of the Orbit and the Masses of Jupiter and Uranus.

(Astronomical Papers of the American Ephemeris, Vol. VII, pp. 147–167, 1895.)

The material employed in this discussion is derived from the published work of the following eleven observatories; the intervals of time covered by it, together with the number of observations in right ascension and declination, are added:

		R. A.	Dec.
Greenwich,	1751-1888	1915	1953
Palermo,	1791-1812	58	48
Paris,	1801-1883	1035	1015
Königsberg,	1814-1847	198	185
Cambridge,	1829-1847	410	307
Capetown,	1834-1860	78	22
Edinburgh,	1835-1844	200	137
Berlin,	1838-1854	94	94
Oxford,	1840-1876	126	124
Washington,	1845-1884	438	350
Brussels,	1855-1863	33	26
Whole number of observations....		4585	4261

Only those observations were included for which the planet culminated between 16^h and 8^h of local time. An exception, however, was made in the case of the Greenwich observations in the time of Bradley, when all were included.

The right ascensions were reduced to the standard of Prof. Newcomb's *Right Ascensions of the Equatorial Fundamental Stars* (*Washington Observations*, 1870, *Appendix* III), and the declinations to Prof. Boss's standard.

In order to the combination of this material, at least an approximate estimate of the relative degree of precision of the several portions of it must be formed. The following determinations of the probable error of a single observation of Jupiter were made in the case of four observatories at the epochs of the stated intervals:

	R. A.	Dec.
	Greenwich.	
1750-1761, Oct. 16,	$\pm 0^s.104$	$\pm 1''.00$
1761, Oct. 26-1765,	± 0.159	± 1.40
1766-1811,	± 0.084	± 1.06
1812-1825,	± 0.101	± 0.90
1826-1835,	± 0.085	± 0.59
1836,	± 0.076	± 0.92
1878-1887,	± 0.052	± 0.76
	PALERMO.	
1792-1809,	$\pm 0^s.103$	$\pm 1''.15$
	PARIS.	
1801-1827,	$\pm 0^s.095$	$\pm 1''.17$
	KÖNIGSBERG.	
1814-1825,	$\pm 0^s.104$	$\pm 0''.95$

These probable errors are larger than those which correspond to a fixed star. They show that one observation of right ascension now made at Greenwich is worth as much as four made in the time of Bradley. The advance in precision of the declinations seems not to have been as great. We note that the declinations are considerably more precise than the right ascensions. Although these determinations have been made for Jupiter, it may be assumed that the probable errors for Saturn bear the same ratio to each other.

Provisional tables having been constructed from the theory in *Astronomical Papers*, Vol. IV, the observations of the interval 1751–1829 were compared directly with isolated places or an ephemeris computed from these tables. For the interval 1830–1888, however, it has been preferred to compare the single observations with the ephemeris contained in the *Berliner Jahrbuch* (1830–1833) or the *Nautical Almanac* (1834–1888), and thus combine the material into normals, three being formed about each opposition, the absolute term of which is taken to be the average of the three residuals.

These equations involve eight unknown quantities, the notation of which is explained as follows:

x_1 = the correction of the mean longitude for 1850.0,

x_2 = the correction of the mean motion for a century,

x_3 = the correction of the eccentricity expressed in seconds of arc,

x_4 = the correction of the longitude of perihelion multiplied by the eccentricity,

x_5 = the correction of the inclination,

x_6 = the correction of the longitude of the ascending node multiplied by the sine of the inclination,

$1 + \frac{x_7}{1000''}$ = the factor by which the mass of Jupiter $\frac{1}{1047.879}$ must be multiplied,

$1 + \frac{x_8}{10''}$ = the factor by which the mass of Uranus $\frac{1}{22640}$ must be multiplied.

The equations which have been formed, together with the dates to which they correspond, are given below. It is to be noted that the absolute terms of those which come from the right ascensions are $\Delta\alpha \cos\delta$, and the absolute terms of those which come from the declinations are $\Delta\delta$. For brevity the sign of equality and the zero which constitutes the right member of the equation are omitted. The number of observations on which each equation depends, together with the weight allowed to the latter in the discussion, will be given with the statement of the final residuals.

Equations from the Right Ascensions.

		x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	x_8.	
1751	May 28	0.988	−0.974	+0.658	+1.912	+0.128	+0.146	−2.182	−2.223	−2″.57
1752	Mar. 10	0.886	0.867	0.210	1.815	0.040	0.062	1.957	2.285	−2.22
1752	June 29	0.989	0.964	0.262	2.026	0.106	0.105	2.167	2.526	−0.89
	Aug. 18	0.926	0.901	+0.253	1.900	+0.060	+0.114	2.021	2.363	−1.48
1753	Apr. 9	0.926	0.896	−0.177	1.912	−0.001	−0.003	2.049	2.519	−4.13
	July 3	0.994	0.959	0.169	2.060	+0.008	+0.026	2.184	2.690	+0.24
	Aug. 26	0.930	0.896	0.148	1.927	+0.012	+0.040	2.036	2.509	+1.85
1754	July 27	0.975	0.931	0.513	1.969	−0.007	−0.056	2.154	2.611	−0.41
	Oct. 2	0.900	0.857	0.534	1.816	−0.003	0.041	1.976	2.406	+0.88
1755	July 20	0.998	0.942	0.996	1.844	+0.011	0.156	2.235	2.442	+0.19
1756	Aug. 4	0.973	0.930	1.334	1.617	0.064	0.226	2.251	2.042	−0.68
	Oct. 8	0.929	0.866	1.238	1.518	0.062	0.193	2.076	1.910	−1.06
1757	Aug. 9	0.997	0.921	1.627	1.309	0.140	0.273	2.285	1.474	+1.95
	Nov. 5	0.909	0.838	1.495	1.190	0.125	0.218	2.055	1.196	+3.06
1758	Sept. 1	1.000	0.913	1.846	0.962	0.227	0.277	2.321	0.786	+0.22
	Nov. 6	0.921	0.839	1.702	0.756	0.213	0.240	2.202	−0.728	+0.88
1759	Aug. 15	1.001	0.904	1.982	0.542	0.310	0.255	2.396	+0.032	+1.06
	Nov. 29	0.911	0.820	1.806	0.514	0.287	0.208	2.114	0.007	−1.01
1760	Sept. 5	1.024	0.914	2.062	0.121	0.379	0.191	2.503	0.858	+0.07
	Dec. 17	0.914	0.814	1.847	+0.121	0.346	0.148	2.174	0.754	−0.14
1761	Aug. 15	1.013	0.895	1.971	−0.332	0.400	0.108	2.589	1.660	+1.68
	Oct. 2	1.050	0.926	2.051	0.311	0.418	0.100	2.634	1.675	+2.64
	Dec. 7	0.972	0.856	1.909	0.269	0.393	0.078	2.387	1.515	+0.25
1762	Oct. 17	1.082	0.943	1.942	0.775	0.417	−0.002	2.808	2.453	−3.40
1763	Nov. 3	1.122	0.967	1.725	1.244	0.372	+0.087	3.010	3.160	+0.13
1764	Dec. 12	1.135	0.965	1.400	1.609	0.294	0.155	3.067	3.541	−1.31
1765	Nov. 21	1.208	1.016	0.993	2.029	0.189	0.162	3.372	4.001	+0.75
1766	Dec. 8	1.239	1.029	−0.483	2.271	0.088	0.125	3.520	4.075	+0.67
1767	Dec. 20	1.254	1.029	+0.083	2.364	+0.014	+0.037	3.635	3.887	+0.62
1769	Jan. 3	1.248	1.010	0.643	2.287	−0.009	−0.075	3.696	3.410	−0.23
1770	Jan. 19	1.222	0.977	1.144	2.061	+0.023	0.180	3.701	2.742	−1.82
1771	Feb. 2	1.182	0.933	1.549	1.700	0.099	0.256	3.676	1.930	−0.12
1772	Feb. 14	1.137	0.885	1.836	1.267	0.196	0.284	3.626	1.075	−3.65
1773	Feb. 27	1.092	0.839	2.005	0.804	0.291	0.265	3.570	+0.210	+0.03
1774	Mar. 14	1.054	0.799	2.072	−0.388	0.369	0.204	3.510	−0.586	+2.17
1775	Mar. 30	1.024	0.765	2.045	+0.123	0.413	0.119	3.446	1.320	−0.08
1776	Apr. 5	1.002	0.739	1.937	0.559	0.349	−0.023	3.388	1.972	−0.99
1777	Apr. 18	0.988	0.718	1.757	0.958	0.391	+0.057	3.326	2.507	−1.71
1778	May 2	0.982	0.704	1.512	1.319	0.332	0.123	3.259	2.911	−2.55
1779	May 10	0.981	−0.693	+1.204	+1.627	+0.252	+0.156	−3.200	−3.170	+5.69

Equations from the Right Ascensions.

		x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	x_8.	
1780	May 25	0.984	—0.685	+0.849	+1.868	+0.166	+0.157	—3.144	—3.250	+0.09
1782	June 18	0.992	0.670	+0.041	2.099	+0.026	+0.059	3.070	2.871	+1.90
1783	July 28	0.982	0.652	—0.369	2.050	—0.003	—0.014	2.984	2.394	+2.71
1784	July 13	0.996	0.652	0.801	1.955	—0.002	0.120	3.016	1.834	+6.09
1785	July 24	0.997	0.642	1.174	1.739	+0.036	0.201	2.982	1.167	+0.89
1786	Aug. 6	0.997	0.632	1.495	1.452	0.104	0.258	2.956	—0.449	+4.45
1787	Aug. 18	1.000	0.624	1.752	1.102	0.188	0.283	2.950	+0.204	+6.38
1788	Aug. 30	1.007	0.618	1.936	0.703	0.276	0.269	2.974	0.954	+1.54
1789	Sept. 11	1.021	0.616	2.045	+0.270	0.353	0.221	3.025	1.565	+0.30
1790	Sept. 22	1.042	0.618	2.069	—0.191	0.405	0.143	3.106	2.102	+0.08
1791	Oct. 8	1.072	0.624	2.007	0.656	0.423	—0.048	3.216	2.540	—0.28
1792	Oct. 24	1.108	0.634	1.848	1.114	0.398	+0.049	3.342	2.861	+0.48
1793	Nov. 7	1.150	0.646	1.584	1.553	0.333	0.125	3.481	3.066	+0.88
1794	Nov. 14	1.194	0.658	1.205	1.945	0.236	0.160	3.626	3.128	+1.17
1795	Dec. 1	1.234	0.667	0.649	2.262	0.108	0.121	3.778	3.014	—5.35
1796	Dec. 24	1.249	0.662	—0.196	2.381	+0.044	—0.086	3.779	2.755	—0.97
1797	Dec. 30	1.255	0.652	+0.381	2.365	—0.006	—0.024	3.789	2.357	+0.74
1799	Jan. 13	1.235	0.629	0.920	2.173	+0.002	0.138	3.722	1.904	—3.49
1800	Jan. 26	1.199	0.599	1.373	1.850	0.061	0.230	3.616	1.313	+2.55
1801	Feb. 28	1.150	0.562	1.697	1.439	0.151	0.269	3.467	0.815	—1.83
1802	Feb. 27	1.109	0.531	1.938	0.979	0.252	0.277	3.400	+0.352	+1.23
1803	Mar. 19	1.068	0.499	2.050	0.517	0.339	0.231	3.301	—0.019	—3.32
1804	Mar. 26	1.035	0.474	2.068	—0.064	0.398	0.157	3.201	0.309	—2.23
1805	Apr. 6	1.010	0.452	1.998	+0.368	0.393	—0.062	3.068	0.529	—2.91
1806	Apr. 24	0.991	0.433	1.845	0.764	0.408	+0.024	2.924	0.700	—1.59
1807	May 1	0.986	0.421	1.634	1.129	0.360	0.098	2.785	0.874	+0.90
1808	May 12	0.985	0.410	1.356	1.440	0.288	0.147	2.645	1.072	—2.11
1809	May 20	0.987	0.401	1.018	1.705	0.202	0.161	2.520	1.318	—1.78
1810	June 5	0.991	0.392	0.829	1.888	0.118	0.141	2.421	1.613	—0.19
1811	June 14	0.995	0.384	+0.225	2.007	0.046	0.085	2.315	1.961	—2.22
1812	June 29	0.999	0.375	—0.198	2.023	+0.003	+0.008	2.229	2.333	—2.96
1813	July 9	0.999	0.364	0.563	1.960	—0.009	—0.084	2.152	2.701	—4.62
1814	July 21	0.999	0.354	1.010	1.797	+0.016	0.170	2.089	3.027	—0.39
1815	Aug. 2	0.998	0.343	1.356	1.560	0.073	0.239	2.034	3.293	+0.73
1816	Aug. 16	0.998	0.333	1.641	1.255	0.152	0.276	2.001	3.472	—3.85
1817	Aug. 27	1.002	0.324	1.861	0.908	0.240	0.280	1.996	3.546	—1.71
1818	Sept. 6	1.012	0.317	2.004	0.505	0.323	0.246	2.027	3.518	—0.86
1819	Sept. 20	1.028	0.311	2.068	+0.086	0.387	0.179	2.041	3.371	—2.42
1820	Oct. 3	1.053	0.308	2.047	—0.357	0.420	—0.089	2.219	3.124	—2.69
1821	Oct. 18	1.086	0.306	1.932	0.814	0.413	+0.009	2.412	2.759	—2.91
1822	Nov. 7	1.123	0.305	1.717	1.254	0.365	0.097	2.612	2.276	—0.99
1823	Nov. 13	1.169	0.305	1.392	1.'81	0.280	0.150	2.871	1.669	+0.13
1824	Nov. 28	1.209	0.303	0.965	2.027	0.176	0.160	3.092	0.967	+0.24
1825	Dec. 11	1.240	0.298	—0.460	2.252	0.076	0.114	3.292	—0.190	+0.82
1827	Jan. 13	1.242	0.285	+0.100	2.316	+0.012	+0.034	3.412	+0.616	0.00
1828	Feb. 3	1.227	0.269	0.648	2.222	—0.006	—0.074	3.499	1.375	—0.12
1829	Feb. 17	1.201	0.251	1.127	1.991	+0.030	0.177	3.559	2.046	+0.41
1830	Mar. 5	1.160	0.230	1.530	1.639	0.108	0.246	3.574	2.577	—0.26
1831	Mar. 4	1.128	0.212	1.835	1.225	0.207	0.278	3.617	2.993	+0.83
1832	Mar. 22	1.082	0.192	1.993	0.773	0.301	0.251	3.577	3.178	+0.23
1833	Apr. 12	1.038	0.174	2.043	—0.309	0.372	0.186	3.520	3.221	—0.26
1834	Apr. 15	1.017	—0.160	+2.031	+0.139	+0.415	—0.105	—3.510	+3.163	+1.61

Equations from the Right Ascensions.

		x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	x_8.	
1835	May 1	0.994	—0.146	+1.919	+0.563	+0.417	—0.012	—3.444	+2.933	+0.65
1836	May 11	0.982	0.134	1.741	0.960	0.385	+0.070	3.383	2.586	—1.13
1837	May 17	0.980	0.124	1.498	1.315	0.324	0.131	3.329	2.122	—0.29
1838	May 21	0.981	0.114	1.188	1.625	0.242	0.160	3.222	1.549	—0.29
1839	June 12	0.982	0.104	0.836	1.851	0.157	0.159	3.098	0.932	+0.46
1840	June 21	0.987	0.094	0.400	2.013	0.078	0.118	2.990	+0.204	—0.16
1841	June 23	0.994	0.085	+0.008	2.086	+0.018	+0.044	2.903	—0.499	—0.31
1842	July 20	0.992	0.074	—0.405	2.043	—0.006	—0.034	2.775	1.116	—0.59
1843	Aug. 9	0.987	0.063	0.794	1.910	+0.002	0.120	2.657	1.621	—0.87
1844	Aug. 10	0.993	0.054	1.185	1.703	0.044	0.205	2.582	2.003	—1.43
1845	Aug. 21	0.994	0.043	1.502	1.414	0.114	0.259	2.511	2.194	—0.36
1846	Sept. 5	0.996	0.033	1.753	1.065	0.199	0.278	2.460	2.191	—0.13
1847	Sept. 15	1.005	0.023	1.936	0.665	0.286	0.262	2.456	1.987	—0.35
1848	Sept. 23	1.021	0.013	2.044	+0.230	0.362	0.210	2.497	1.606	—0.74
1849	Oct. 16	1.037	—0.002	2.055	—0.215	0.408	0.126	2.557	1.059	—0.66
1850	Oct. 14	1.074	+0.008	2.000	0.684	0.421	—0.035	2.701	—0.334	—1.02
1851	Nov. 5	1.106	0.020	1.827	1.139	0.391	+0.061	2.834	+0.471	—0.09
1852	Nov. 17	1.150	0.033	1.562	1.570	0.322	0.133	3.020	1.384	—0.32
1853	Nov. 9	1.197	0.046	1.169	1.967	0.218	0.156	3.225	2.326	—0.04
1854	Nov. 30	1.233	0.061	0.695	2.236	0.115	0.137	3.391	3.228	+0.62
1856	Jan. 3	1.245	0.075	—0.163	2.355	+0.035	+0.075	3.471	3.803	+0.44
1857	Jan. 29	1.232	0.087	+0.391	2.307	—0.005	—0.024	3.475	4.240	+0.06
1858	Jan. 30	1.227	0.099	0.937	2.136	+0.008	0.144	3.514	4.561	+0.87
1859	Feb. 22	1.184	0.108	1.369	1.804	0.070	0.226	3.419	4.529	+0.61
1860	Mar. 4	1.142	0.116	1.708	1.389	0.163	0.272	3.347	4.347	+0.35
1861	Mar. 27	1.088	0.122	1.909	0.930	0.260	0.262	3.233	3.960	—0.14
1862	Apr. 16	1.040	0.128	2.003	0.475	0.341	0.212	3.137	3.494	—0.15
1863	Apr. 15	1.023	0.136	2.046	—0.030	0.400	0.142	3.120	3.034	—0.42
1864	Apr. 28	0.999	0.143	1.973	+0.393	0.419	—0.051	3.009	2.487	+0.07
1865	May 6	0.987	0.152	1.832	0.803	0.403	+0.036	2.887	1.940	+0.34
1866	May 15	0.982	0.161	1.619	1.139	0.353	0.108	2.735	1.388	+0.26
1867	May 20	0.985	0.171	1.340	1.450	0.278	0.151	2.574	0.860	—0.07
1868	June 13	0.980	0.181	1.003	1.688	0.194	0.166	2.371	+0.450	—0.01
1869	June 23	0.986	0.192	0.623	1.839	0.110	0.140	2.201	—0.016	—0.36
1870	July 1	0.994	0.204	+0.210	1.973	0.041	0.081	2.036	0.321	+0.15
1871	July 27	0.986	0.213	—0.206	1.968	+0.002	+0.007	1.850	0.513	+0.33
1872	July 22	0.999	0.225	0.636	1.912	—0.007	—0.092	1.728	0.588	+0.43
1873	Aug. 1	0.999	0.235	1.027	1.742	+0.022	0.178	1.574	0.538	+0.29
1874	Aug. 25	0.991	0.244	1.356	1.499	0.082	0.237	1.440	0.373	+0.41
1875	Sept. 7	0.991	0.255	1.638	1.198	0.157	0.274	1.339	—0.018	—0.57
1876	Sept. 10	1.000	0.267	1.854	0.842	0.251	0.274	1.299	+0.234	+0.27
1877	Oct. 1	1.004	0.279	1.991	0.452	0.331	0.231	1.271	0.620	—0.51
1878	Oct. 3	1.028	0.296	2.057	+0.024	0.393	0.165	1.323	1.074	—1.08
1879	Oct. 22	1.049	0.313	2.032	—0.390	0.420	—0.072	1.405	1.546	—0.85
1880	Nov. 6	1.080	0.333	1.954	0.839	0.408	+0.025	1.558	2.007	—0.50
1881	Nov. 20	1.119	0.357	1.689	1.272	0.355	0.109	1.770	2.489	—1.31
1882	Nov. 26	1.168	0.384	1.366	1.683	0.267	0.157	2.034	2.983	—1.29
1883	Dec. 4	1.212	0.411	0.929	2.029	0.161	0.156	2.301	3.400	—1.28
1884	Dec. 19	1.240	0.434	—0.432	2.242	0.065	0.105	2.529	3.738	—1.51
1886	Jan. 1	1.251	0.450	+0.158	2.302	+0.003	+0.009	2.722	3.949	—1.26
1887	Jan. 31	1.229	0.456	0.694	2.187	—0.005	—0.094	2.827	3.965	—1.32
1888	Feb. 3	1.209	+0.460	+1.196	—1.949	+0.039	—0.202	—2.962	+3.952	—1.03

Equations from the Declinations.

		x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	x_8.	
1751	May 28	—0.175	+0.172	—0.119	—0.338	+0.723	+0.822	+0.437	+0.522	—1.″08
1752	Mar. 10	0.066	0.064	0.020	0.134	0.538	0.837	0.196	0.279	—2.03
	June 29	0.108	0.106	0.030	0.222	0.542	0.961	0.296	0.401	—1.98
	Aug. 18	—0.115	+0.112	0.029	—0.236	0.485	0.916	0.307	0.412	—0.83
1753	Apr. 9	+0.002	—0.002	—0.003	+0.004	0.370	0.969	0.053	0.107	—1.96
	July 3	—0.024	+0.023	+0.004	—0.050	0.349	1.053	0.116	0.184	—0.11
	Aug. 26	—0.037	+0.036	+0.009	—0.077	0.302	0.996	+0.142	+0.211	—1.27
1754	July 27	+0.051	—0.048	—0.025	+0.103	0.126	1.079	—0.048	—0.032	—1.86
	Oct. 2	0.036	0.034	0.017	0.074	+0.082	1.004	0.018	0.003	+1.88
1755	July 20	0.141	0.134	0.131	0.267	—0.077	1.098	0.247	0.256	—2.69
1756	Aug. 4	0.215	0.201	0.286	0.351	0.295	1.046	0.420	0.363	—2.44
1756	Oct. 8	+0.184	—0.172	—0.242	0.304	—0.310	0.970	0.449	0.311	—2.04
1757	Aug. 9	0.286	0.264	0.465	0.378	0.489	0.952	0.591	0.363	—1.42
	Nov. 5	0.232	0.214	0.377	0.311	0.489	0.848	0.466	0.267	—0.36
1758	Sept. 1	0.340	0.310	0.626	0.331	0.669	0.814	0.730	0.231	—0.60
	Nov. 6	0.302	0.275	0.556	0.301	0.647	0.731	0.636	—0.212	—1.31
1759	Aug. 15	0.389	0.352	0.771	0.213	0.797	0.655	0.881	+0.030	—0.79
	Nov. 29	0.342	0.308	0.678	0.196	0.764	0.554	0.749	0.011	+0.09
1760	Sept. 5	0.421	0.376	0.849	0.058	0.921	0.463	0.986	0.336	—0.33
	Dec. 17	0.371	0.330	0.750	+0.052	0.851	0.365	0.850	0.294	—1.40
1761	Aug. 15	0.421	0.372	0.819	—0.135	0.963	0.260	1.047	0.663	—0.04
	Oct. 2	0.437	0.386	0.856	0.123	1.004	0.240	1.066	0.662	—0.05
	Dec. 7	0.405	0.357	0.796	0.109	0.943	0.187	0.971	0.600	+0.08
1762	Oct. 17	0.434	0.379	0.783	0.305	1.038	+0.006	1.112	0.930	—0.20
1763	Nov. 3	0.406	0.350	0.633	0.440	1.025	—0.239	1.084	1.068	—4.00
1764	Dec. 12	0.360	0.306	0.447	0.509	0.925	0.489	0.992	1.053	—3.89
1765	Nov. 21	0.275	0.231	0.231	0.459	0.832	0.711	0.797	0.833	—1.07
1766	Dec. 8	0.170	0.141	—0.070	0.311	0.642	0.908	0.526	0.482	—0.79
1767	Dec. 20	+0.044	—0.036	+0.001	—0.083	0.405	1.046	—0.179	+0.064	—3.37
1769	Jan. 3	—0.084	+0.068	—0.044	+0.154	—0.136	1.111	+0.192	—0.295	+1.71
1770	Jan. 19	0.201	0.161	0.187	0.339	+0.138	1.098	0.550	0.510	—0.56
1771	Feb. 2	0.298	0.235	0.389	0.430	0.391	1.014	0.870	0.536	+0.88
1772	Feb. 14	0.369	0.287	0.593	0.414	0.606	0.876	1.123	0.388	—0.99
1773	Feb. 27	0.411	0.316	0.753	0.307	0.774	0.705	1.299	—0.106	—1.83
1774	Mar. 14	0.429	0.325	0.843	+0.143	0.907	0.500	1.396	+0.226	—0.59
1775	Mar. 30	0.427	0.319	0.854	—0.046	0.990	0.285	1.417	0.554	+1.11
1776	Apr. 5	0.339	0.250	0.657	0.185	1.031	—0.068	1.139	0.687	—3.47
1777	Apr. 18	0.375	0.272	0.669	0.358	1.031	+0.151	1.261	0.983	+1.35
1778	May 2	0.329	0.236	0.510	0.438	0.990	0.366	1.103	1.022	+1.62
1779	May 10	0.271	0.192	0.336	0.447	0.911	0.565	0.904	0.932	+0.13
1780	May 25	0.207	0.144	0.196	0.391	0.790	0.749	0.684	0.744	—4.72
1782	June 18	—0.058	+0.039	0.004	—0.122	0.449	1.015	0.219	0.237	+1.79
1783	July 28	+0.013	—0.008	0.004	+0.027	0.223	1.075	+0.008	+0.037	—6.26
1784	July 13	0.108	0.071	0.087	0.212	+0.022	1.106	—0.277	—0.131	—3.20
1785	July 24	0.186	0.120	0.219	0.325	—0.195	1.077	0.507	0.155	—1.46
1786	Aug. 6	0.257	0.163	0.384	0.376	0.402	1.001	0.713	—0.060	—5.12
1787	Aug. 18	0.320	0.199	0.559	0.356	0.588	0.884	0.898	+0.105	—1.35
1788	Aug. 30	0.371	0.228	0.712	0.263	0.749	0.731	1.057	0.386	—0.80
1789	Sept. 11	0.410	0.247	0.821	+0.114	0.879	0.549	1.186	0.650	—1.37
1790	Sept. 22	0.434	0.257	0.863	—0.074	0.972	0.344	1.277	0.882	—3.80
1791	Oct. 8	0.440	0.256	0.826	0.263	1.029	+0.116	1.315	1.033	—0.95
1792	Oct. 24	+0.425	0.243	—0.712	—0.422	—1.038	—0.127	—1.288	+1.072	—0.83

Equations from the Declinations.

		x_1.	x_2.	x_3.	x_4.	x_5.	x_6.	x_7.	x_8.	
1793	Nov. 7	+0.385	—0.216	—0.536	—0.516	—0.995	—0.373	—1.182	+0.988	—0.99
1794	Nov. 14	0.313	0.173	0.321	0.508	0.897	0.609	0.977	0.771	—1.86
1795	Dec. 1	0.181	0.098	0.109	0.327	0.739	0.829	0.582	0.383	—2.20
1796	Dec. 24	+0.108	—0.057	0.019	—0.205	0.509	0.993	—0.363	+0.168	—7.40
1797	Dec. 30	—0.028	+0.014	0.010	+0.052	—0.267	1.092	+0.043	—0.125	—0.35
1799	Jan. 13	0.153	0.078	0.114	0.269	+0.018	1.113	0.418	0.309	+0.70
1800	Jan. 26	0.260	0.130	0.297	0.402	0.281	1.059	0.742	0.361	+0.73
1801	Feb. 28	0.331	0.162	0.486	0.418	0.525	0.933	0.956	0.310	+0.07
1802	Feb. 27	0.394	0.189	0.686	0.352	0.710	0.780	1.170	0.196	+0.18
1803	Mar. 19	0.422	0.197	0.807	0.210	0.859	0.584	1.270	—0.056	+2.36
1804	Mar. 26	0.430	0.197	0.858	+0.032	0.960	0.379	1.304	+0.075	+3.39
1805	Apr. 6	0.391	0.175	0.773	—0.136	1.018	—0.162	1.169	0.160	+1.63
1806	Apr. 24	0.392	0.171	0.732	0.296	1.032	+0.062	1.150	0.242	+2.07
1807	May 1	0.351	0.150	0.585	0.397	1.012	0.276	0.994	0.283	+1.39
1808	May 12	0.299	0.125	0.415	0.434	0.949	0.485	0.814	0.302	+2.09
1809	May 20	0.235	0.096	0.246	0.404	0.847	0.674	0.619	0.293	+0.35
1810	June 5	0.166	0.066	0.142	0.315	0.704	0.841	0.430	0.252	—0.34
1811	June 14	0.087	0.034	0.022	0.175	0.532	0.972	0.230	0.155	—0.27
1812	June 29	—0.008	+0.003	0.000	—0.016	0.329	1.062	+0.046	+0.003	+0.13
1813	July 9	+0.076	—0.028	0.043	+0.148	+0.116	1.103	—0.133	—0.218	+0.02
1814	July 21	0.155	0.055	0.156	0.279	—0.104	1.095	0.293	0.479	—1.47
1815	Aug. 2	0.230	0.079	0.311	0.360	0.316	1.038	0.438	0.764	—1.18
1816	Aug. 16	0.294	0.098	0.472	0.376	0.514	0.937	0.561	1.025	+0.32
1817	Aug. 27	0.351	0.113	0.650	0.321	0.685	0.799	0.671	1.240	—2.19
1818	Sept. 6	0.395	0.124	0.782	0.202	0.827	0.630	0.765	1.368	—2.68
1819	Sept. 20	0.424	0.128	0.854	+0.041	0.937	0.433	0.819	1.382	—1.01
1820	Oct. 3	0.437	0.128	0.852	—0.142	1.010	+0.214	0.906	1.287	—1.40
1821	Oct. 18	0.431	0.122	0.770	0.318	1.040	—0.022	0.944	1.080	—1.58
1822	Nov. 7	0.403	0.109	0.620	0.446	1.018	0.270	0.932	0.801	—1.67
1823	Nov. 13	0.345	0.090	0.416	0.493	0.947	0.509	0.846	0.478	—1.93
1824	Nov. 28	0.262	0.066	0.214	0.439	0.811	0.737	0.676	—0.196	—1.26
1825	Dec. 11	0.152	0.037	—0.047	0.278	0.617	0.927	0.425	+0.010	—0.98
1827	Jan. 13	+0.040	—0.009	+0.003	—0.075	0.358	1.054	—0.129	+0.031	—0.75
1828	Feb. 3	—0.083	+0.018	—0.042	+0.150	—0.085	1.101	+0.214	—0.085	+1.11
1829	Feb. 17	0.198	0.041	0.183	0.330	+0.184	1.075	0.563	0.334	+0.09
1830	Mar. 5	0.292	0.058	0.381	0.415	0.429	0.980	0.876	0.652	+0.19
1831	Mar. 4	0.370	0.070	0.599	0.405	0.633	0.850	1.165	0.983	+0.62
1832	Mar. 22	0.408	0.073	0.750	0.297	0.798	0.665	1.333	1.220	+1.24
1833	Apr. 12	0.422	0.071	0.830	+0.132	0.914	0.457	1.419	1.340	+1.79
1834	Apr. 15	0.424	0.067	0.847	—0.052	0.994	0.251	1.455	1.356	+1.12
1835	May 1	0.404	0.059	0.782	0.222	1.025	—0.031	1.394	1.235	+0.87
1836	May 11	0.371	0.051	0.660	0.358	1.020	+0.186	1.274	1.019	+1.07
1837	May 17	0.325	0.041	0.500	0.432	0.977	0.395	1.097	0.745	+0.77
1838	May 21	0.265	0.031	0.325	0.437	0.895	0.590	0.873	0.456	+0.73
1839	June 12	0.202	0.021	0.174	0.380	0.765	0.771	0.641	0.221	+0.04
1840	June 21	0.127	0.012	0.054	0.259	0.607	0.918	0.392	—0.049	—0.04
1841	June 23	—0.043	+0.004	0.002	—0.089	0.424	1.027	+0.133	+0.008	—0.17
1842	July 20	+0.031	—0.002	0.013	+0.065	+0.205	1.087	—0.077	—0.037	—0.38
1843	Aug. 9	0.108	0.007	0.085	0.210	—0.018	1.095	0.279	0.166	—0.08
1844	Aug. 10	0.191	0.010	0.226	0.328	0.229	1.065	0.482	0.363	—0.58
1845	Aug. 21	0.262	0.011	0.394	0.375	0.432	0.983	0.647	0.546	—0.34
1846	Sept. 5	+0.322	—0.011	—0.565	+0.348	—0.615	+0.858	—0.782	—0.666	—0.42

Equations from the Declinations.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
1847	Sept. 15	+0.374	−0.009	−0.719	+0.252	−0.770	+0.703	−0.902	−0.689	−0.47
1848	Sept. 23	0.412	0.005	0.826	+0.099	0.894	0.519	1.001	0.592	−0.99
1849	Oct. 16	0.431	−0.001	0.857	−0.084	0.979	0.302	1.060	0.376	−1.52
1850	Oct. 14	0.438	+0.003	0.818	0.273	1.032	+0.085	1.101	−0.071	−1.35
1851	Nov. 5	0.420	0.008	0.697	0.427	1.029	−0.161	1.076	+0.248	−0.95
1852	Nov. 17	0.377	0.011	0.516	0.511	0.982	0.406	0.995	0.526	−0.62
1853	Nov. 9	0.295	0.011	0.294	0.482	0.884	0.631	0.800	0.641	−0.63
1854	Nov. 30	0.199	0.010	0.117	0.360	0.715	0.846	0.552	0.584	−0.97
1856	Jan. 3	+0.092	+0.006	0.017	−0.174	0.476	1.007	−0.260	+0.334	−0.28
1857	Jan. 29	−0.027	−0.002	0.008	+0.051	−0.203	1.086	+0.071	−0.046	+0.21
1858	Jan. 30	0.159	0.013	0.120	0.278	+0.059	1.106	0.451	0.559	+0.51
1859	Feb. 22	0.258	0.024	0.296	0.396	0.323	1.034	0.738	0.970	+0.88
1860	Mar. 4	0.340	0.035	0.505	0.417	0.548	0.913	0.987	1.297	+0.10
1861	Mar. 27	0.387	0.043	0.676	0.336	0.730	0.736	1.139	1.434	+0.91
1862	Apr. 16	0.411	0.050	0.790	0.193	0.863	0.536	1.229	1.426	+0.90
1863	Apr. 15	0.425	0.057	0.850	+0.018	0.963	0.341	1.286	1.326	+0.87
1864	Apr. 28	0.413	0.059	0.817	−0.157	1.014	−0.123	1.234	1.107	+1.19
1865	May 6	0.387	0.059	0.720	0.310	1.028	+0.093	1.121	0.845	+1.01
1866	May 15	0.346	0.057	0.573	0.397	1.001	0.307	0.958	0.580	+0.54
1867	May 20	0.292	0.051	0.401	0.427	0.936	0.510	0.756	0.345	+0.72
1868	June 13	0.232	0.043	0.240	0.399	0.817	0.699	0.556	0.190	+1.13
1869	June 23	0.161	0.031	0.104	0.304	0.673	0.860	0.354	0.072	−0.04
1870	July 1	0.082	0.017	−0.019	0.162	0.499	0.986	0.162	0.035	+0.53
1871	July 27	−0.007	−0.001	+0.002	−0.013	0.286	1.060	+0.006	0.040	+0.60
1872	July 22	+0.083	+0.019	−0.055	+0.159	+0.081	1.103	−0.151	0.081	+0.03
1873	Aug. 1	0.164	0.039	0.167	0.286	−0.137	1.089	0.265	0.103	+0.13
1874	Aug. 25	0.231	0.057	0.314	0.351	0.353	1.016	0.343	−0.084	−0.54
1875	Sept. 7	0.295	0.076	0.484	0.360	0.527	0.918	0.405	+0.017	−1.10
1876	Sept. 10	0.354	0.094	0.659	0.303	0.708	0.771	0.463	0.125	−1.03
1877	Oct. 1	0.393	0.109	0.779	0.182	0.845	0.590	0.496	0.301	−1.09
1878	Oct. 3	0.425	0.122	0.852	+0.016	0.950	0.398	0.541	0.521	−1.61
1879	Oct. 22	0.435	0.130	0.845	−0.156	1.013	+0.173	0.572	0.729	−1.45
1880	Nov. 6	0.427	0.132	0.775	0.326	1.032	−0.064	0.600	0.889	−1.82
1881	Nov. 20	0.396	0.126	0.601	0.446	1.002	0.307	0.608	0.979	−0.85
1882	Nov. 26	0.337	0.111	0.398	0.482	0.926	0.545	0.566	0.955	−0.75
1883	Dec. 4	0.247	0.084	0.194	0.412	0.788	0.765	0.447	0.781	−1.10
1884	Dec. 19	0.137	0.048	0.051	0.247	0.584	0.949	0.258	0.484	−0.17
1886	Jan. 1	+0.010	+0.004	0.001	−0.019	0.338	1.069	−0.001	+0.082	+0.36
1887	Jan. 31	−0.105	−0.039	0.058	+0.187	−0.054	1.106	+0.258	−0.311	+0.96
1888	Feb. 3	−0.226	−0.086	−0.223	+0.366	+0.206	−1.080	+0.572	−0.748	+0.87

The normal equations which result from these equations of condition follow:

$$
\begin{array}{rrrrrrrrl}
x_1. & x_2. & x_3. & x_4. & x_5. & x_6. & x_7. & x_8. & \\
77.80 & +\ 0.29 & -\ 7.30 & -\ 1.43 & +\ 0.11 & -\ 0.15 & -196.19 & +\ 88.23 & -34''.26=0, \\
+\ 0.29 & +\ 5.91 & -\ 2.19 & -\ 3.44 & +\ 0.01 & +\ 0.03 & +\ 5.21 & +\ 11.81 & -\ 1.09=0, \\
-\ 7.30 & -\ 2.19 & +160.45 & +\ 0.91 & +\ 0.08 & +\ 0.01 & -\ 30.23 & +107.75 & +26.50=0, \\
-\ 1.43 & -\ 3.44 & +\ 0.91 & +144.45 & +\ 0.05 & +\ 0.08 & +\ 9.07 & -131.00 & +\ 0.18=0, \\
+\ 0.11 & +\ 0.01 & +\ 0.08 & +\ 0.05 & +40.16 & +\ 0.09 & -\ 0.46 & -\ 2.02 & +34.84=0, \\
-\ 0.15 & +\ 0.03 & +\ 0.01 & +\ 0.08 & +\ 0.09 & +37.29 & +\ 0.58 & -\ 0.21 & -\ 8.82=0, \\
-196.19 & +\ 5.21 & -\ 30.23 & +\ 9.07 & -\ 0.46 & +\ 0.58 & +522.99 & -252.71 & +74.80=0, \\
+\ 88.23 & +11.81 & +107.75 & -131.00 & -\ 2.02 & -\ 0.21 & -252.71 & +392.54 & -\ 5.82=0,
\end{array}
$$

Their solution gives

$$
\begin{array}{ll}
x_1 = +\,1''.955 & x_5 = -\,0''.881 \\
x_2 = +\,0.110 & x_6 = +\,0.238 \\
x_3 = +\,0.190 & x_7 = +\,0.479 \pm 0''.116 \\
x_4 = -\,0.244 & x_8 = -\,0.258 \pm 0.0384
\end{array}
$$

By this solution the sum of the squares of the absolute terms is reduced from [nn] = 124.67 to [nn.8] = 67.12. From which it results that the probable error of a normal of the weight unity is $\pm\ 0''.33$. The corresponding probable errors of the corrections of the masses of Jupiter and Uranus are given above. The values of these masses, given by this discussion, are then

$$\text{Mass of Jupiter} = \frac{1}{1047.378 \pm 0.121},$$

$$\text{Mass of Uranus} = \frac{1}{23239 \pm 89}.$$

As the values of these masses are also derivable from other sources, it is desirable to have the corrections expressed in terms of the two indeterminates x_7 and x_8; they are as follow:

$$
\begin{aligned}
x_1 &= +\,0''.428 + 2.551x_7 - 1.183x_8, \\
x_2 &= +\,0.115 - 0.933x_7 - 1.718x_8, \\
x_3 &= -\,0.144 + 0.292x_7 - 0.754x_8, \\
x_4 &= +\,0.007 - 0.062x_7 + 0.859x_8, \\
x_5 &= -\,0.869 + 0.004x_7 + 0.054x_8, \\
x_6 &= +\,0.240 - 0.005x_7 + 0.001x_8.
\end{aligned}
$$

The elements on which the provisional tables were founded are the following:

Epoch 1850, Jan. 0.0, Greenwich M. T.

$$
\begin{aligned}
L &= 14^\circ\ 49'\ 38''.13 \\
\pi &= 90\ \ \ 6\ 41.50 \\
\Omega &= 112\ \ 20\ 49.05 \\
i &= 2\ \ 29\ 40.19 \\
e &= 0.05606038 \\
n &= 43996''.20594
\end{aligned}
$$

The constant of the mean obliquity of the ecliptic for 1850 used in the discussion was adopted from Leverrier and is $23^\circ 27' 31''.83$. If we suppose this ought to receive the correction $\delta\varepsilon$, the inclination and the longitude of

the ascending node of Saturn will receive proportionate corrections. The corrected elements then become

$$
\begin{aligned}
L &= 14^\circ\, 49'\, 38''.558 + 2.551x_7 - 1.183x_8\,,\\
\pi &= 90\ \ 6\ 41.62\ \ -1.11\ x_7 + 15.32\ x_8\,,\\
\Omega &= 112\ \ 20\ 54.56\ \ -0.11\ x_7 + 0.02\ x_8 + 21.23\delta\epsilon\,,\\
i &= 2\ \ 29\ 39.321 + 0.004x_7 + 0.054x_8 + 0.380\delta\epsilon\,,\\
e &= 0.05605968 + 142x_7 - 366x_8\,,\\
n &= 43996''.20709 - 0.00933x_7 - 0.01718x_8\,.
\end{aligned}
$$

In the case of e the coefficients of the indeterminates are in units of the 8th decimal.

In order to see what the material we have used was capable of giving, the masses of Jupiter and Uranus were derived from the equations of conditions belonging to the last century; the results were

$$\text{Mass of Jupiter} = \frac{1}{1046.117 \pm 0.919},\quad \text{Mass of Uranus} = \frac{1}{20927 \pm 943}.$$

The values of Bouvard $\frac{1}{1070.5}$ and $\frac{1}{17918}$, which were obtained from the observations down to 1814 inclusive, can then only have resulted from the too rude reduction and the too imperfect theory.

The residuals left by the above solution in the case of each normal, together with the number of observations the latter is founded upon and the weight it has received in the discussion, are as follows:

Date.			Obs.-Cal.		No. of Obs.		Weight.	
			$\Delta\alpha \cos\delta$.	$\Delta\delta$.	α.	δ.	α.	δ.
1751	May	28	+1''.62	+1''.25	8	8	0.1	0.1
1752	Mar.	10	+1.34	+1.94	3	3	0.05	0.05
	June	29	—0.05	+1.91	5	5	0.05	0.05
	Aug.	18	+0.54	+0.74	2	2	0.03	0.03
1753	April	9	+3.24	+1.64	2	2	0.03	0.03
	July	3	—1.19	—0.25	6	6	0.1	0.1
	Aug.	26	—2.74	+0.92	3	2	0.04	0.03
1754	July	27	—0.48	+1.23	4	4	0.05	0.05
	Oct.	2	—1.68	—2.52	3	3	0.04	0.04
1755	July	20	—0.93	+1.83	3	3	0.04	0.04
1756	Aug.	4	+0.11	+1.36	2	2	0.03	0.03
	Oct.	8	+0.54	+1.05	3	3	0.04	0.04
1757	Aug.	9	—2.30	+0.18	14	13	0.2	0.2
	Nov.	5	—3.37	—0.80	9	9	0.15	0.15
1758	Sept.	1	—0.34	—0.71	5	5	0.06	0.06
	Nov.	6	—1.04	+0.04	6	6	0.1	0.1
1759	Aug.	15	—0.96	—0.57	7	7	0.1	0.1
	Nov.	29	+1.07	—1.37	4	4	0.06	0.06
1760	Sept.	5	+0.22	—1.02	5	5	0.06	0.06
	Dec.	17	+0.38	+0.11	7	4	0.1	0.05

Date.			Obs.-Cal. $\Delta\alpha \cos\delta$.	$\Delta\delta$.	No. of Obs. α.	δ.	Weight. α.	δ.
1761	Aug.	15	—1.26	—1.25	10	9	0.15	0.15
	Oct.	2	—2.22	—1.29	12	10	0.15	0.15
	Dec.	7	+0.10	—1.35	17	15	0.2	0.2
1762	Oct.	17	+3.89	—1.06	13	11	0.1	0.1
1763	Nov.	3	+0.32	+2.82	24	23	0.2	0.2
1764	Dec.	12	+1.64	+2.85	5	4	0.06	0.06
1765	Nov.	21	—0.57	+0.16	5	5	0.06	0.06
1766	Dec.	8	—0.68	+0.08	1	1	0.02	0.02
1767	Dec.	20	—0.83	+2.90	3	3	0.04	0.04
1769	Jan.	3	—0.14	—1.89	3	3	0.04	0.04
1770	Jan.	19	+1.34	+0.68	2	2	0.03	0.03
1771	Feb.	2	—0.39	—0.46	1	1	0.02	0.02
1772	Feb.	14	+3.11	+1.65	3	3	0.04	0.04
1773	Feb.	27	—0.57	+2.66	2	2	0.03	0.03
1774	Mar.	14	—2.71	+1.55	1	1	0.02	0.02
1775	Mar.	30	—0.51	—0.09	1	1	0.02	0.02
1776	April	5	+0.30	+4.42	1	1	0.02	0.02
1777	April	18	+1.04	—0.43	2	2	0.03	0.03
1778	May	2	+1.82	—0.82	1	1	0.02	0.02
1779	May	10	—6.47	+0.48	1	1	0.02	0.02
1780	May	25	—0.88	+5.14	2	2	0.03	0.03
1782	June	18	—2.52	—1.90	1	1	0.02	0.02
1783	July	28	—3.19	+5.90	1	1	0.02	0.02
1784	July	13	—6.36	+2.63	1	1	0.02	0.02
1785	July	24	—0.95	+0.72	2	1	0.03	0.02
1786	Aug.	6	—4.29	+4.25	1	1	0.02	0.02
1787	Aug.	18	—6.01	+0.39	1	1	0.02	0.02
1788	Aug.	30	—0.98	—0.21	4	4	0.05	0.05
1789	Sept.	11	+0.39	+0.34	5	5	0.06	0.06
1790	Sept.	22	+0.69	+2.77	3	3	0.04	0.04
1791	Oct.	8	+1.02	—0.09	7	8	0.1	0.1
1792	Oct.	24	+0.13	—0.18	12	12	0.15	0.15
1793	Nov.	7	—0.45	—0.06	2	2	0.03	0.03
1794	Nov.	14	—0.98	+1.00	6	5	0.06	0.06
1795	Dec.	1	+5.23	+1.50	2	2	0.03	0.03
1796	Dec.	24	+0.57	+6.93	1	1	0.02	0.02
1797	Dec.	30	—1.36	+0.18	3	2	0.04	0.03
1799	Jan.	13	+2.75	—0.54	1	1	0.02	0.02
1800	Jan.	26	—3.35	—0.24	3	3	0.04	0.04
1801	Feb.	28	+1.04	+0.67	7	7	0.1	0.1
1802	Feb.	27	—1.94	+0.76	14	14	0.15	0.2
1803	Mar.	19	+2.71	—1.29	13	7	0.15	0.1
1804	Mar.	26	+1.69	—2.27	20	18	0.2	0.2
1805	April	6	+2.38	—0.56	21	21	0.2	0.2
1806	April	24	+1.12	—1.06	12	10	0.15	0.15
1807	May	1	—1.44	—0.47	15	15	0.2	0.2
1808	May	12	+1.52	—1.32	19	19	0.2	0.2
1809	May	20	+1.10	+0.25	19	17	0.2	0.2
1810	June	5	—0.58	+0.71	15	15	0.2	0.2
1811	June	14	+1.37	+0.39	23	23	0.25	0.25
1812	June	29	+2.04	—0.25	20	18	0.25	0.2
1813	July	9	+3.62	—0.44	8	8	0.1	0.1
1814	July	21	—0.60	+0.78	24	25	0.3	0.3

Date.			Obs.-Cal. $\Delta\alpha \cos\delta$.	$\Delta\delta$.	No. of Obs. α.	δ.	Weight. α.	δ.
1815	Aug.	2	—1.76	+0.22	18	7	0.2	0.1
1816	Aug.	16	+2.82	—1.48	18	10	0.25	0.15
1817	Aug.	27	+0.70	+0.80	12	10	0.15	0.15
1818	Sept.	6	—0.14	+1.13	14	8	0.2	0.15
1819	Sept.	20	+1.36	—0.62	14	14	0.2	0.2
1820	Oct.	3	+1.67	—0.25	14	13	0.2	0.2
1821	Oct.	18	+1.82	—0.03	12	12	0.2	0.2
1822	Nov.	7	—0.16	+0.22	27	30	0.4	0.5
1823	Nov.	13	—1.34	+0.69	12	13	0.2	0.2
1824	Nov.	28	—1.50	+0.34	8	9	0.15	0.15
1825	Dec.	11	—2.09	+0.41	8	8	0.15	0.15
1827	Jan.	13	—1.17	+0.57	10	7	0.2	0.15
1828	Feb.	3	—0.86	—0.95	9	8	0.2	0.2
1829	Feb.	17	—1.12	+0.40	23	15	0.5	0.4
1830	Mar.	5	—0.14	+0.47	46	20	0.6	0.5
1831	Mar.	4	—0.93	+0.17	63	39	0.7	0.6
1832	Mar.	22	—0.06	—0.38	95	55	1	1
1833	April	12	+0.65	—0.98	64	66	1	1
1834	April	15	—1.10	—0.33	121	78	1	1
1835	May	1	—0.06	—0.13	72	62	1	1
1836	May	11	+1.08	—0.37	102	92	1	1
1837	May	17	+0.77	—0.16	95	99	1	1
1838	May	21	+0.62	—0.16	119	104	1	1
1839	June	12	—0.28	+0.37	114	104	1	1
1840	June	21	+0.14	+0.33	98	84	1	1
1841	June	23	+0.14	+0.26	60	58	1	1
1842	July	20	+0.25	+0.25	134	130	1	1
1843	Aug.	9	+0.44	—0.27	101	96	1	1
1844	Aug.	10	+0.92	+0.01	78	76	1	1
1845	Aug.	21	—0.14	—0.46	101	83	1	1
1846	Sept.	5	—0.38	—0.57	84	81	1	1
1847	Sept.	15	—0.09	—0.64	108	115	1	1
1848	Sept.	23	+0.33	—0.20	72	65	1	1
1849	Oct.	16	+0.32	+0.33	33	29	1	1
1850	Oct.	14	+0.74	+0.18	52	51	1	1
1851	Nov.	5	—0.21	—0.10	27	24	1	1
1852	Nov.	17	+0.05	—0.27	38	41	1	1
1853	Nov.	9	—0.26	—0.06	38	39	1	1
1854	Nov.	30	—0.93	+0.53	36	48	1	1
1856	Jan.	3	—0.80	+0.13	38	34	1	1
1857	Jan.	29	—0.37	—0.05	56	51	1	1
1858	Jan.	30	—1.10	—0.10	63	71	1	1
1859	Feb.	22	—0.72	—0.23	74	70	1	1
1860	Mar.	4	—0.34	+0.72	74	58	1	1
1861	Mar.	27	+0.25	+0.04	56	59	1	1
1862	April	16	+0.35	+0.09	77	74	1	1
1863	April	15	+0.64	+0.18	105	108	1	1
1864	April	28	+0.13	—0.16	68	66	1	1
1865	May	6	—0.24	+0.03	67	62	1	1
1866	May	15	—0.32	+0.45	75	77	1	1
1867	May	20	—0.14	+0.16	64	62	1	1
1868	June	13	—0.34	—0.40	70	63	1	1
1869	June	23	—0.14	+0.59	63	55	1	1

Date.			Obs.-Cal.		No. of Obs.		Weight.	
			$\Delta\alpha \cos \delta$.	$\Delta\delta$.	α.	δ.	α.	δ.
1870	July	1	—0.77	—0.20	96	90	1	1
1871	July	27	—1.02	—0.48	42	41	1	1
1872	July	22	—1.14	—0.16	60	59	1	1
1873	Aug.	1	—0.97	—0.50	73	79	1	1
1874	Aug.	25	—1.01	—0.05	74	73	1	1
1875	Sept.	7	+0.06	+0.32	86	91	1	1
1876	Sept.	10	—0.68	+0.11	72	75	1	1
1877	Oct.	1	+0.13	+0.08	57	58	1	1
1878	Oct.	3	+0.72	+0.53	57	53	1	1
1879	Oct.	22	+0.52	+0.37	72	73	1	1
1880	Nov.	6	+0.15	+0.81	66	65	1	1
1881	Nov.	20	+0.87	—0.04	72	72	1	1
1882	Nov.	26	+0.76	+0.03	35	36	0.8	0.8
1883	Dec.	4	+0.63	+0.60	44	37	0.8	0.8
1884	Dec.	19	+0.79	—0.02	30	26	0.8	0.8
1886	Jan.	1	+0.51	—0.23	17	17	0.5	0.5
1887	Jan.	31	+0.60	—0.51	23	23	0.5	0.5
1888	Feb.	3	+0.44	—0.12	12	12	0.5	0.5

From the consideration of the declinations of both Jupiter and Saturn it has been concluded that Prof. Boss's system of declinations, in the region neighboring the equator, needs, in the average, a correction which, for different epochs, is given by the formula

$$+ 0''.01 + 0''.0045\,(t - 1850).$$

Accordingly the residuals in declination, just given, have been thus corrected.

It will be noticed that the declinations are much better represented by the theory than the right ascensions, as the sum of the squares of the residuals is about 20 for the former against 47 for the latter. The residuals of the right ascensions frequently show a systematic character, especially in the latter half of the period. However, all the efforts I have made to detect periodicity in them have led to no result. They may be attributed to one of three causes. Either some error has been committed in the theory, or some force acts on Saturn of which we know nothing, or the observations are affected with systematic errors which their combination has not completely eliminated. The last seems to have the greatest degree of probability in its favor.

The mass of Jupiter given by the foregoing discussion is in fair agreement with the values which have been obtained from other sources, but the mass obtained for Uranus is to a considerable degree smaller than the values given by observations of the satellites. It seems to me possible that its determination from the observation of Saturn is unfavorably influenced by the presence of systematic errors in the latter.

MEMOIR No. 58.

The Periodic Solution as a First Approximation in the Lunar Theory.

(Astronomical Journal, Vol. XV, pp. 137-143, 1895.)

The lunar theory may be developed in either of two ways. In the first, the coefficients of the periodic terms in the expressions for the coordinates are exhibited as functions of the elements on which they depend, the latter being left indeterminate. This may be called a literal lunar theory. In the second certain numerical values are attributed to these elements at the beginning even of the investigation, and all the following computations are performed on numerical quantities so that the final results present the mentioned coefficients in the form of numbers. This may be called a numerical lunar theory. Both these methods of treatment are to be desired; the former having the greater attraction for the mathematician; while the latter recommends itself to the practical astronomer who wishes merely to have the power of predicting the position of the moon. The latter is also preferable on account of the far less labor required for its elaboration. A literal theory probably demands from four to six times as much labor as suffices for a numerical theory. The latter indeed has the disadvantage of depending on values of the elements which afterwards may be discovered to need correction; however, this objection is of very slight weight since the literal theories we already possess are amply sufficient for assigning the corresponding corrections of the coefficients.

Nearly all the lunar theories in existence have been elaborated by successive steps of approximation in which the elliptic theory has been taken as the starting point. It may be asked whether some labor may not be saved by adopting as the first approximation the best values attainable for the coordinates.

It is well known that the problem of three bodies admits a periodic solution in which the motions of the two planets about the body, considered as central, take place in the same plane, and where the eccentricities peculiar to each planet vanish. It is the object of the present article to elaborate this solution in the case of the moon, employing the numerical method. It is very desirable that the approximation in this solution should be pushed

to a high degree of precision, much more than at first sight would seem at all necessary, because, in the further elaboration of the lunar theory by this method, the motions of the perigee and node depend on the values here assigned to the coefficients of the periodic terms.

In the particular solution just mentioned there are only four independent arbitrary constants introduced by integration. Two of these simply define the phases of the planets at the origin of time, while the remaining two may be taken to be the average rates of increase of their longitudes. These four constants may be denoted in their order as ε, ε', n, n'. It is expedient to eliminate the masses of the bodies from the equations by introducing two constants to take their places; these we denote by the symbols a, a'. The masses of the sun, earth and moon being severally denoted by m_1, m_2 and m_3, the mentioned constants are connected by the equations

$$m_1 + m_2 + m_3 = M = a'^3 n'^2, \quad m_2 + m_3 = a^3 n^2$$

For convenience we adopt the notations

$$\mu = \frac{m_3}{m_2 + m_3}, \quad \mu' = \frac{m_2 + m_3}{m_1 + m_2 + m_3}$$

For properly exhibiting the effect of the moon's mass on its motion relative to the earth, the system of coordinates, where the moon is referred to the earth, but the sun to the center of gravity of the earth and moon, is well adapted. Denoting such rectangular coordinates together with the radii as x, y, r for the moon, and x', y', r' for the sun, it is well known that the differential equations of motion are

$$m_2\mu \frac{d^2x}{dt^2} = \frac{\partial \Omega}{\partial x} \qquad m_2\mu \frac{d^2y}{dt^2} = \frac{\partial \Omega}{\partial y}$$
$$m_1\mu' \frac{d^2x'}{dt^2} = \frac{\partial \Omega}{\partial x'} \qquad m_1\mu' \frac{d^2y'}{dt^2} = \frac{\partial \Omega}{\partial y'}$$

where Ω, the potential function, is equal to the sum of the products of every two masses into the reciprocal of their distance, or

$$\Omega = \frac{m_1 m_2}{\Delta_{1,2}} + \frac{m_1 m_3}{\Delta_{1,3}} + \frac{m_2 m_3}{\Delta_{2,3}}$$

The distances are given in terms of the coordinates by the equations

$$\Delta_{1,2}^2 = (x' + \mu x)^2 + (y' + \mu y)^2$$
$$\Delta_{1,3}^2 = [x' - (1 - \mu)x]^2 + [y' - (1 - \mu)y]^2$$
$$\Delta_{2,3}^2 = x^2 + y^2 = r^2$$

The ratio $\left(\frac{r}{r'}\right)$ is so small that it is advisable to develop Ω in a series

of its powers. Thus is obtained as a suitable potential for the determination of the moon's coordinates

$$\begin{aligned}\frac{1}{m_2\mu}\,\Omega = \frac{m_1+m_2}{r} + m_1 \Big\{ & [(1-\mu)^{-1}+\mu^{-1}]\frac{1}{r'} \\ & + [(1-\mu)+\mu]\frac{r^2}{r'^3}(\tfrac{3}{2}S^2-\tfrac{1}{2}) \\ & + [(1-\mu)^2-\mu^2]\frac{r^3}{r'^4}(\tfrac{5}{2}S^3-\tfrac{3}{2}S) \\ & + [(1-\mu)^3+\mu^3]\frac{r^4}{r'^5}(\tfrac{35}{8}S^4-\tfrac{15}{4}S^2+\tfrac{3}{8}) \\ & + [(1-\mu)^4-\mu^4]\frac{r^5}{r'^6}(\tfrac{63}{8}S^5-\tfrac{35}{4}S^3+\tfrac{15}{8}S) \\ & + \ldots\ldots\ldots\ldots \Big\}\end{aligned}$$

S being determined by the equation

$$rr'S = xx' + yy'$$

To get the similar function for the relative motion of the sun about the center of gravity of the earth and moon it is only necessary to multiply the preceding expression by

$$\frac{m_2\mu}{m_1\mu'} = \frac{M}{m_1}\mu(1-\mu)$$

Here, neglecting the first term as independent of solar coordinates, it will suffice to take the two next following and write

$$\frac{1}{m_1\mu'}\,\Omega = \frac{M}{r'} + M\mu(1-\mu)\frac{r^2}{r'^3}(\tfrac{3}{2}S^2-\tfrac{1}{2})$$

For our purpose it is sufficiently accurate to suppose that the motion of the sun about the center of gravity of the earth and moon is circular and uniform. The various inequalities have coefficients less than 0''.0002, and it is easy to see that the effect of neglecting them on the position of the moon is much less than this quantity. Therefore, in the last expression, we can substitute for $\frac{r^2}{r'^3}(\frac{3}{2}S^2-\frac{1}{2})$ its non-periodic term. Here a rude approximation suffices, and, turning to Pontécoulant, *Théorie Analytique du Système du Monde*, Tom. IV, p. 100, we find that this non-periodic term is

$$\frac{a^2}{a'^3}[\tfrac{1}{4}-\tfrac{179}{96}m^2-\tfrac{87}{16}m^3-\tfrac{7681}{576}m^4-\tfrac{7103}{288}m^5-\tfrac{14028053}{331776}m^6]$$

m denoting the ratio of the month to the year. Thus, putting

$$K = \mu(1-\mu)\frac{a}{a'^2}[\tfrac{1}{4}-\tfrac{179}{96}m^2-\tfrac{87}{16}m^3-\ldots]$$

we may take for the motion of the sun

$$\frac{1}{m_1\mu'}\,\Omega = \frac{M}{r'}\left[1 + K\frac{a'^2}{r'^2}\right]$$

Employing λ' to denote the longitude, the solar differential equations of motion are

$$\frac{d^2r'}{dt^2} - r'\frac{d\lambda'^2}{dt^2} + \frac{M}{r'^2} + 3MK\frac{a'^2}{r'^4} = 0$$

$$\frac{d\lambda'}{dt} = n'$$

As K is an excessively small quantity its square may be neglected, and the differential equations are satisfied by the values

$$r' = a'(1 + K), \quad \lambda' = \varepsilon' + n't$$

Certain modifications can now be made in the differential equations for the coordinates of the moon. Denoting the longitude by λ, let $\phi = \lambda - \lambda'$, and let τ denote the mean value of the same, so that $\tau = \varepsilon - \varepsilon' + (n - n')\,t$. Making τ the independent variable instead of t, and putting

$$\mathrm{m} = \frac{n'}{n - n'}, \quad R = \frac{1}{m_2\mu(n - n')^2}\,\Omega$$

and adopting r and ϕ as the variables for expressing the position of the moon, the differential equations become

$$\frac{d\,r}{d\tau^2} - r\left(\frac{d\phi}{d\tau} + \mathrm{m}\right)^2 = \frac{\partial R}{\partial r}$$

$$\frac{d}{d\tau}\left[r^2\left(\frac{d\phi}{d\tau} + \mathrm{m}\right)\right] = \frac{\partial R}{\partial \phi}$$

As R now contains no variables but r and ϕ, r' being constant, if we multiply these equations severally by dr and $d\phi$, and add the resulting equations we have an exact differential, which, being integrated, gives

$$\frac{dr^2 + r^2d\phi^2}{d\tau^2} = 2R + \mathrm{m}^2r^2 + 2C$$

C being the arbitrary constant. This equation can be substituted for the second of the former, and, if we put

$$a^2V = 2R + \mathrm{m}^2r^2 + 2C$$

the two equations of the problem can take the form

$$\frac{d^2r^2}{a^2d\tau^2} - 4\mathrm{m}\,\frac{r^2}{a^2}\frac{d\phi}{d\tau} = 2V + r\frac{\partial V}{\partial r}$$

$$\frac{dr^2 + r^2d\phi^2}{a^2d\tau^2} = V$$

By making ϕ the independent variable we shall obtain the advantage of having but one equation to integrate. We adopt r^2 as the unknown to

be determined, and, as $\frac{r^2}{a^2}$ differs from unity by a small quantity, we put $\frac{r^2}{a^2} = 1 + u$. Also, for the same reason, we put $\frac{d\tau}{d\phi} = 1 + \nu$. Then our equations may be written

$$\frac{d}{d\phi}\left[\frac{1}{1+\nu}\frac{du}{d\phi}\right] - 4\mathrm{m}(1+u) = (1+\nu)\left(2V + r\frac{\partial V}{\partial r}\right)$$

$$1 + u + \frac{1}{4(1+u)}\frac{du^2}{d\phi^2} = (1+\nu)^2 V$$

In order to have small quantities to deal with we put

$$V = 1 + N, \quad 2V + r\frac{\partial V}{\partial r} = U - 4\mathrm{m}$$

N and U are then of the same order of smallness as m^2. Then the equations which solve the problem are

$$(1+\nu)^2 = \frac{1 + u + \frac{1}{4(1+u)}\frac{du^2}{d\phi^2}}{1+N}$$

$$\frac{d}{d\phi}\left[\frac{1}{1+\nu}\frac{du}{d\phi}\right] - 4\mathrm{m}(u - \nu) = (1+\nu)\, U$$

$$\tau = \phi + \int \nu d\phi$$

It is plain that the arbitrary constant C, which enters into the expression for V and thus into those for N and U, is not independent of those which have already been noted, viz.: a, a', n, n'; but, in the numerical method, we do not need to know its expression in terms of the latter, we simply assign to it such a numerical value as will make the non-periodic term of ν vanish in the periodic development of the latter as a function of ϕ. As $\frac{2C}{a^2}$ is nearly equivalent to -1, the expressions for N and U will be simplified if we replace it by

$$1 - \mathrm{m}^2 - 2(1+\mathrm{m})^2 + C$$

where the new C is a quantity of the order of m^2.

With these modifications, if we adopt the following notation for certain constants,

$$a_1 = (1-\mu')\mathrm{m}^2\frac{a'^3}{r'^3}$$

$$a_2 = (1-\mu')(1-2\mu)\,\mathrm{m}^2\frac{a}{a'}\frac{a'^4}{r'^4}$$

$$a_3 = (1-\mu')(1-3\mu+3\mu^2)\,\mathrm{m}^2\frac{a^2}{a'^2}\frac{a'^5}{r'^5}$$

$$a_4 = (1-\mu')(1-4\mu+6\mu^2-4\mu^3)\,\mathrm{m}^2\frac{a^3}{a'^3}\frac{a'^6}{r'^6}$$

the expressions of the functions N and U, in terms of u and ϕ, are

$$\begin{aligned}
N = C &+ 2(1+\mathrm{m})^2[(1+u)^{-\frac{1}{2}} - 1] + \mathrm{m}^2 u \\
&+ a_1(1+u)[\tfrac{3}{2}\cos 2\phi + \tfrac{1}{2}] \\
&+ a_2(1+u)^{\frac{3}{2}}[\tfrac{5}{4}\cos 3\phi + \tfrac{3}{4}\cos\phi] \\
&+ a_3(1+u)^2[\tfrac{35}{32}\cos 4\phi + \tfrac{5}{8}\cos 2\phi + \tfrac{9}{32}] \\
&+ a_4(1+u)^{\frac{5}{2}}[\tfrac{63}{64}\cos 5\phi + \tfrac{35}{64}\cos 3\phi + \tfrac{15}{32}\cos\phi] \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots \\
U = 2C &+ 2(1+\mathrm{m})^2[(1+u)^{-\frac{1}{2}} - 1] + 4\mathrm{m}^2 u \\
&+ 4a_1(1+u)[\tfrac{3}{2}\cos 2\phi + \tfrac{1}{2}] \\
&+ 5a_2(1+u)^{\frac{3}{2}}[\tfrac{5}{4}\cos 3\phi + \tfrac{3}{4}\cos\phi] \\
&+ 6a_3(1+u)^2[\tfrac{35}{32}\cos 4\phi + \tfrac{5}{8}\cos 2\phi + \tfrac{9}{32}] \\
&+ 7a_4(1+u)^{\frac{5}{2}}[\tfrac{63}{64}\cos 5\phi + \tfrac{35}{64}\cos 3\phi + \tfrac{15}{32}\cos\phi] \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

From these expressions it will be seen that if the numerical value of N has been obtained through the first, that of U immediately results by summing the several terms of N multiplied by simple integers.

The coefficients in the periodic solution here treated are functions of three independent constants, which may be taken to be m, $\frac{a}{a'}$ and μ. The values we assign to them are

$$\mathrm{m} = 0.0808489338\,, \quad \frac{a}{a'} = 0.002573603\,, \quad \mu = \frac{1}{82.5}$$

The first is so approximate that it will scarcely need correction. The second corresponds to the value $8''.8$ of the constant of the solar parallax which is adopted simply because it is a round number. The third, which results from the principal constants of nutation and precession, is about as close a value as at present can be assigned to this constant and at the same time be expressed in few figures. The connected constants, which are functions of these, have the values:

$$\mu' = \frac{1}{328242.3}\,, \quad K = 0.0000000188$$

$$\begin{aligned}
a_1 &= 0.00653\ 65298\ 16 \\
a_2 &= 0.00001\ 64146\ 16 \\
a_3 &= 0.00000\ 00417\ 39 \\
a_4 &= 0.00000\ 00001\ 06
\end{aligned}$$

The approximate value of u, to be substituted in the differential equations, and afterwards corrected, will be obtained from the results given by Mr. Ernest W. Brown and myself (*Amer. Jour. Math.*, Vols. I and XIV).

After correcting the numbers to make them correspond to the adopted moon's mass and the constant of solar parallax, we have

$$\frac{a_0}{a} = 0.9990930780$$

$$\frac{r}{a_0}\cos(\phi - \tau) = \left\{\begin{array}{l} 1 + 0.00028\,81665 \cos \tau \\ - 0.00718\,00404 \cos 2\tau \\ - 0.00000\,75187 \cos 3\tau \\ + 0.00000\,60337 \cos 4\tau \\ - 0.00000\,00034 \cos 5\tau \\ + 0.00000\,00326 \cos 6\tau \\ 0.00000\,00000 \cos 7\tau \\ + 0.00000\,00002 \cos 8\tau \end{array}\right\}$$

$$\frac{r}{a_0}\sin(\phi - \tau) = \left\{\begin{array}{l} - 0.00061\,02619 \sin \tau \\ + 0.01021\,15492 \sin 2\tau \\ + 0.00000\,72231 \sin 3\tau \\ + 0.00000\,57245 \sin 4\tau \\ + 0.00000\,00058 \sin 5\tau \\ + 0.00000\,00275 \sin 6\tau \\ 0.00000\,00000 \sin 7\tau \\ + 0.00000\,00002 \sin 8\tau \end{array}\right\}$$

By computing the special values of these functions and thence those of $\frac{r^2}{a^2}$ and $\phi - \tau$ for the thirteen values of τ evenly distributed from 0° to 180°, and then applying the formulas for deriving periodic series from special values to the latter, we get

$$u = \frac{r^2}{a^2} - 1 = \left\{\begin{array}{l} 0.00173\,50214 \\ + 0.00056\,71290 \cos \tau \\ - 0.01433\,41815 \cos 2\tau \\ - 0.00001\,08564 \cos 3\tau \\ - 0.00001\,42655 \cos 4\tau \\ - 0.00000\,00213 \cos 5\tau \\ - 0.00000\,00366 \cos 6\tau \\ - 0.00000\,00004 \cos 7\tau \\ 0.00000\,00000 \cos 8\tau \end{array}\right\}$$

$$\phi - \tau = \left\{\begin{array}{ll} - & 125''.72648 \sin \tau \\ + & 2106.27394 \sin 2\tau \\ + & 0.73663 \sin 3\tau \\ + & 8.74177 \sin 4\tau \\ + & 0.00765 \sin 5\tau \\ + & 0.04898 \sin 6\tau \\ + & 0.00008 \sin 7\tau \\ + & 0.00031 \sin 8\tau \end{array}\right\}$$

The second of these equations must now be reversed so as to exhibit τ as a function of ϕ, and the right member of the first changed into a function of the same variable. This is accomplished through the method of special values, or by an application of Lagrange's Theorem. For convenience, exhibiting the first in terms of the radian, we get

$$\tau = \phi + \left\{ \begin{array}{l} +\,0.00061\,26173 \sin \phi \\ -\,0.01021\,12121 \sin 2\phi \\ -\,0.00001\,29397 \sin 3\phi \\ +\,0.00006\,18797 \sin 4\phi \\ +\,0.00000\,01885 \sin 5\phi \\ -\,0.00000\,05352 \sin 6\phi \\ -\,0.00000\,00028 \sin 7\phi \\ +\,0.00000\,00055 \sin 8\phi \end{array} \right\}$$

$$u = \left\{ \begin{array}{l} -\,0.00188\,15691 \\ +\,0.00057\,84401 \cos \phi \\ -\,0.01433\,26460 \cos 2\phi \\ -\,0.00002\,26099 \cos 3\phi \\ +\,0.00013\,20733 \cos 4\phi \\ +\,0.00000\,04284 \cos 5\phi \\ -\,0.00000\,13789 \cos 6\phi \\ -\,0.00000\,00070 \cos 7\phi \\ +\,0.00000\,00156 \cos 8\phi \end{array} \right\}$$

From the latter equation we derive the following special values of u and its differential, the first to 11 decimals, the second to 10.

ϕ	u	$\frac{du}{d\phi}$
0°	— 1552 72535 0	0
15	— 1368 51204 8	+1377 95258
30	— 861 19787 7	+ 2414 49781
45	— 158 89281 4	+ 2829 74425
60	+ 552 93712 0	+ 2478 32319
75	+ 1076 30147 8	+ 1419 13080
90	+ 1258 45447 0	— 64 84608
105	+ 1043 07758 0	— 1540 57776
120	+ 490 52899 0	— 2578 13247
135	— 243 83254 6	— 2901 65947
150	— 961 31365 3	— 2458 99497
165	— 1477 08311 0	— 1398 70662
180	— 1663 97567 0	0

In order to make ν have 0 for its non-periodic term, we discover that the proper value for our second C is — 0.00691 797355. With this value

we determine the values of N with 11 decimals and of ν with 10, as follow:

ϕ	N	ν
0°	+ 2423 62364 7	— 1960 34362
15	+ 2077 04404 1	— 1699 81270
30	+ 1126 28190 9	— 980 20757
45	— 181 26557 4	+ 21 24791
60	— 1497 24415 9	+ 1042 98441
75	— 2461 13925 6	+ 1799 70812
90	— 2801 88479 9	+ 2067 32850
105	— 2420 67794 5	+ 1762 32100
120	— 1422 31459 5	+ 973 86650
135	— 81 41117 5	— 70 74302
150	+ 1241 28206 1	— 1086 15603
165	+ 2199 48222 6	— 1812 72120
180	+ 2548 20012 6	— 2075 28823

The values of U with 11 decimals are:

ϕ	U
0°	+ 5575 26677 1
15	+ 4848 34346 5
30	+ 2851 89637 7
45	+ 100 41000 3
60	— 2676 95407 9
75	— 4720 17206 3
90	— 5454 62497 2
105	— 4670 63959 3
120	— 2591 36210 6
135	+ 202 39483 6
150	+ 2954 06634 9
165	+ 4944 15126 9
180	+ 5667 66586 5

When these values are substituted in the differential equation

$$\frac{d}{d\phi}\left[\frac{1}{1+\nu}\frac{du}{d\phi}\right] - 4\mathrm{m}(u-\nu) - (1+\nu)\,U = 0$$

the left member, instead of becoming 0, takes the value, in the periodic form,

$$438 + 171.0 \cos\phi + 1270 \cos 2\phi - 214 \cos 3\phi + 100 \cos 4\phi - 23 \cos 5\phi - 44 \cos 6\phi$$
$$- 3 \cos 7\phi - 62 \cos 8\phi,$$

where the coefficients are noted in units of the 10^{th} decimal.

A second trial has been made with this differential equation by substituting in it the value

$$u = \begin{cases} -\,0.00188\,16351 \\ +\,0.00057\,82699 \cos\phi \\ -\,0.01433\,26793 \cos 2\phi \\ -\,0.00002\,26122 \cos 3\phi \\ +\,0.00013\,20725 \cos 4\phi \\ +\,0.00000\,04277 \cos 5\phi \\ -\,0.00000\,13727 \cos 6\phi \\ -\,0.00000\,00070 \cos 7\phi \\ +\,0.00000\,00153 \cos 8\phi \\ +\,0.00000\,00001 \cos 9\phi \\ -\,0.00000\,00002 \cos 10\phi \end{cases}$$

Precisely as in the first trial it is found that $C = -0.00691811469$, and that ν, in units of the 11th decimal, has the special values

ϕ	ν
0°	— 1960 36519 9
15	— 1699 83366 4
30	— 980 22578 1
45	+ 21 23523 8
60	+ 1042 97800 8
75	+ 1799 70665 7
90	+ 2067 33150 0
105	+ 1762 32890 1
120	+ 973 87813 5
135	— 70 73027 3
150	— 1086 14268 3
165	— 1812 70686 1
180	— 2075 27315 3

The value of the left member of the differential equation, instead of becoming 0, is, in the developed periodic form (the coefficients in units of the 11th decimal)

$$+\,27426 + 37\cos\phi + 21763\cos 2\phi + 1818\cos 3\phi + 2707\cos 4\phi + 1501\cos 5\phi$$
$$-\,21995\cos 6\phi - 27\cos 7\phi + 969\cos 8\phi + 63\cos 9\phi + 182\cos 10\phi$$

In order to discover the correction of the substituted value, it is necessary to form the equations to *variation* of the two differential equations of the problem. Here we must attribute a variation δC to the constant C as well

as the variation δu to u. Differentiating the expressions for N and U with respect to u we get as the special values

ϕ	$\frac{dN}{du}$	$\frac{dU}{du}$
0°	—1.17634	—1.11732
15	1.17433	1.11928
30	1.16878	1.12460
45	1.16126	1.13188
60	1.15375	1.13914
75	1.14836	1.14449
90	1.14654	1.14654
105	1.14888	1.14487
120	1.15478	1.13999
135	1.16273	1.13329
150	1.17062	1.12656
165	1.17636	1.12163
180	—1.17848	—1.11984

From these are readily determined the special values of $\delta \nu$, and, if we put

$$\delta \nu = F\delta C + G\delta u + H\frac{d.\delta u}{d\phi}$$

ϕ	F	G	H
0°	—0.47860	+1.06092	0
15	0.48150	1.06374	+0.00348
30	0.48959	1.07154	0.00608
45	0.50101	1.08251	0.00710
60	0.51290	1.09403	0.00619
75	0.52184	1.10281	+0.00353
90	0.52505	1.10599	—0.00016
105	0.52144	1.10257	0.00384
120	0.51216	1.09367	0.00644
135	0.50006	1.08209	0.00728
150	0.48851	1.07108	0.00620
165	0.48038	1.06334	—0.00354
180	—0.47746	+1.06057	0

By writing the variation of the differential equation thus

$$\frac{d}{d\phi}\left[\frac{1}{1+\nu}\frac{d.\delta u}{d\phi} - \frac{1}{(1+\nu)^2}\frac{du}{d\phi}\delta\nu\right] = L\delta C + M\delta u + P\frac{d.\delta u}{d\phi}$$

we get

ϕ	L	M	P
0°	+2.0884	—1.05596	0
15	2.0984	1.06930	—0.00096
30	2.1248	1.10615	0.00179
45	2.1620	1.15771	0.00229
60	2.2005	1.21071	0.00217
75	2.2294	1.25039	—0.00131
90	2.2398	1.26483	+0.00006
105	2.2282	1.24971	0.00142
120	2.1984	1.20971	0.00225
135	2.1593	1.15685	0.00234
150	2.1218	1.10567	0.00182
165	2.0953	1.06921	+0.00097
180	+2.0858	—1.05608	0

By substituting the periodic developments which correspond to the special values, this equation may be reduced to the form

$$\begin{aligned}
&[1.00010 - 62 \cos\phi + 2043 \cos 2\phi + 3 \cos 3\phi + 6 \cos 4\phi] \frac{d^2\delta u}{d\phi^2} \\
&+ [121 \sin\phi - 6957 \sin 2\phi - 12 \sin 3\phi - 20 \sin 4\phi] \frac{d.\delta u}{d\phi} \\
&+ [1.15885 + 93 \cos\phi - 16644 \cos 2\phi - 45 \cos 3\phi + 266 \cos 4\phi]\, \delta u \\
&= [2.16208 + 137 \cos\phi - 10493 \cos 2\phi - 35 \cos 3\phi + 268 \cos 4\phi]\, \delta C \\
&+ \text{ residual,}
\end{aligned}$$

in which the numerical coefficients are all to five places of decimals. In this we substitute the periodic series with indeterminate coefficients

$$\delta u = a_0 + a_1 \cos\phi + a_2 \cos 2\phi + a_3 \cos 3\phi + \ldots$$

and equate to zero each coefficient of the cosine of the multiple of ϕ, and thus obtain a group of equations determining the a. As the non-periodic term of $\delta\nu$ should vanish, this furnishes the equation determining δC,

$$-0.50104\delta C + 1.08284 a_0 + 20 a_1 - 1849 a_2 - 6 a_3 + 47 a_4 = 0$$

where again the coefficients have all five decimals.

After the elimination of δC the group of equations determining the a in the first trial are

$$\begin{aligned}
-3.514 a_0 + 0.02529 a_2 \qquad\qquad\qquad\qquad &+ 438 = 0 \\
-0.002 a_0 + 0.10010 a_1 + 0.00038 a_2 - 0.069 a_3 &+ 171.0 = 0 \\
+0.056 a_0 - 2.844\ a_2 - 0.108\ a_4 \qquad &+ 1270 = 0 \\
-0.1268 a_1 + 0.003 a_2 - 7.84 a_3 &- 214 = 0 \\
-0.003\ a_0 - 0.194 a_2 - 14.84 a_4 &+ 100 = 0 \\
-0.28\ a_3 - 23.84 a_5 &- 23 = 0 \\
-34.84 a_6 &- 44 = 0 \\
\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots
\end{aligned}$$

The final terms of these equations are in units of the tenth decimal. Their solution leads to the values

$$a_0 = +128, a_1 = -1707, a_2 = +449, a_3 = 0, \quad a_4 = +1$$
$$a_5 = -\quad 1, a_6 = -\quad 1, a_7 = \quad 0, a_8 = -1, \delta C = +258.5$$

The equations determining the a for the second trial differ only from those for the first trial in having different absolute terms, and their solution gives the values

$$a_0 = +786.1, a_1 = -8.0, a_2 = +780.3, \quad a_3 = +\quad 23.4$$
$$a_4 = +\quad 8.7, a_5 = +6.0, a_6 = -\quad 63.2, \quad a_7 = -\quad 0.1$$
$$a_8 = +\quad 1.5, a_9 = +0.1, a_{10} = +\quad 0.2, \delta C = +1670.2$$

When these corrections are applied severally to the coefficients of the approximate values of u employed in the two trials, the results are two values of u which differ only in their three first coefficients by 2 and 3 units in the tenth decimal, the following coinciding exactly. We give then only the result of the second trial, which as it has been made with more care, is to be preferred.

$$u = \left\{ \begin{array}{l} -0.00188\,15565 \\ +0.00057\,82691 \cos \phi \\ -0.01433\,26013 \cos 2\phi \\ -0.00002\,26099 \cos 3\phi \\ +0.00013\,20734 \cos 4\phi \\ +0.00000\,04283 \cos 5\phi \\ -0.00000\,13790 \cos 6\phi \\ -0.00000\,00070 \cos 7\phi \\ +0.00000\,00155 \cos 8\phi \\ +0.00000\,00001 \cos 9\phi \\ -0.00000\,00002 \cos 10\phi \end{array} \right\}$$

The values of C from the two trials also agree well, they are

$$C = -0.00691\,749770, \quad C = -0.00691\,749767.$$

To derive from the just given expression the coordinates of the moon as functions of the time it is necessary to have the correct value of ν. This is obtained by computing the correction $\delta\nu$ to be added to the values of ν in the second trial. We employ the equation

$$\delta\nu = F\delta C + G\delta u + H\frac{d.\delta u}{d\phi}$$

The principal results obtained are, in units of the tenth decimal.

ϕ	δu	$\frac{d.\delta u}{d\phi}$	$\delta \nu$
0°	+ 1535.0	0	+ 829.3
15	1475.5	— 523	763.7
30	1220.5	1448	+ 481.4
45	752.8	1960	— 35.6
60	305.0	1301	530.8
75	99.8	— 308	762.5
90	80.0	+ 37	788.3
105	128.2	406	731.0
120	350.0	1367	— 481.3
135	805.1	1916	+ 22.1
150	1246.5	1288	511.3
165	1454.5	+ 371	743.1
180	+ 1493.0	0	+ 786.1

When the corrections in the last column are applied to the mentioned set of values of ν and the periodic development of this quantity derived, we get

$$\nu = \left\{\begin{array}{l} +\ 0.00061\,24345\,1 \cos \phi \\ -\ 0.02042\,24560\,4 \cos 2\phi \\ -\ 0.00003\,88175\,1 \cos 3\phi \\ +\ 0.00024\,75192\,1 \cos 4\phi \\ +\ 0.00000\,09433\,5 \cos 5\phi \\ -\ 0.00000\,32169\,3 \cos 6\phi \\ -\ 0.00000\,00186\,3 \cos 7\phi \\ +\ 0.00000\,00433\,7 \cos 8\phi \\ +\ 0.00000\,00002\,3 \cos 9\phi \\ -\ 0.00000\,00006\,2 \cos 10\phi \end{array}\right\}$$

The integration of this expression gives

$$\tau = \phi + \left\{\begin{array}{l} +\ 126.32369'' \sin \phi \\ -2106.21697 \sin 2\phi \\ -\ 2.66890 \sin 3\phi \\ +\ 12.76362 \sin 4\phi \\ +\ 0.03892 \sin 5\phi \\ -\ 0.11059 \sin 6\phi \\ -\ 0.00055 \sin 7\phi \\ +\ 0.00112 \sin 8\phi \\ +\ 0.00001 \sin 9\phi \\ -\ 0.00001 \sin 10\phi \end{array}\right\}$$

From this formula are derived the following corresponding values of τ and ϕ:

τ	ϕ		
°	°	′	″
15	15	17	8.75640
30	30	29	29.55616
45	45	33	37.86824
60	60	28	27.66264
75	75	15	23.69115
90	89	57	53.58146
105	104	40	32.45790
120	119	27	54.62453
135	134	23	25.41221
150	149	28	26.23660
165	164	41	47.23948

From these special values we derive the periodic development of ϕ in terms of τ:

$$\phi = \tau + \left\{ \begin{array}{rl} & '' \\ - & 125.68900 \sin \tau \\ + & 2106.27704 \sin 2\tau \\ + & 0.73711 \sin 3\tau \\ + & 8.74182 \sin 4\tau \\ + & 0.00763 \sin 5\tau \\ + & 0.04902 \sin 6\tau \\ + & 0.00006 \sin 7\tau \\ + & 0.00032 \sin 8\tau \end{array} \right\}$$

A comparison of this formula with the corresponding one employed at the outset shows that the coefficient of the parallactic inequality in the latter needs the correction $+ 0''.0375$.*

Astronomers have been accustomed to employ the function $\frac{a}{r}$ of the moon's radius. In order to obtain this we substitute the just given special values of ϕ in the expression for $\frac{r^2}{a^2}$ in terms of ϕ; then, by extracting the

* It should be explained that this rather large correction is due to no imperfection in Prof. E. W. Brown's coefficient, but arises from a mistake committed in transforming his series.

square root and taking the reciprocal, we have the following special values:

τ	$\frac{a}{r}$
°	
0	1.00785 52819
15	1.00687 82756
30	1.00422 86149
45	1.00065 67204
60	0.99714 55263
75	0.99463 05661
90	0.99376 63268
105	0.99478 25258
120	0.99743 74708
135	1.00106 63735
150	1.00472 62613
165	1.00742 99059
180	1.00842 50531

Thence we have the following periodic development:

$$\frac{a}{r} = \left\{ \begin{array}{l} \quad 1.00090\,73946 \\ -0.00028\,72775 \cos \tau \\ +0.00718\,65934 \cos 2\tau \\ +0.00000\,23529 \cos 3\tau \\ +0.00004\,58501 \cos 4\tau \\ +0.00000\,00385 \cos 5\tau \\ +0.00000\,03269 \cos 6\tau \\ +0.00000\,00004 \cos 7\tau \\ +0.00000\,00025 \cos 8\tau \end{array} \right\}$$

The coefficients of the expressions for ϕ and $\frac{a}{r}$ may be trusted as correctly corresponding to the assumed values of the three elements, m, $\frac{a}{a'}$ and μ, to within a very few units of the last decimal noted.

MEMOIR No. 59.

On the Convergence of the Series Used in the Subject of Perturbations.

(Bulletin of the American Mathematical Society. Vol. II, pp. 93-97, 1896.)

The perturbations of the planets and the coördinates of the moon have been developed by astronomers in infinite series of terms involving sines or cosines of linear functions of two or more arguments with positive or negative integral multipliers. These arguments vary proportionally with the time, and their periods, in accordance with notions derived from the theory of probabilities, are supposed to be incommensurable with each other. Recently M. Poincaré has much insisted that, under the latter condition, these series, in the rigorous mathematical sense, are divergent (*Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. II, pp. 277-280). However, the reasons brought forward to sustain this opinion are scarcely convincing, and I think there has been some scepticism among astronomers in reference to the matter. Without attempting to find any flaw in M. Poincaré's logic, I simply wish to point out a class of cases where the convergency of the series can be shown in spite of the incommensurability of the component arguments.

In many problems of dynamics, where the integral of conservation of areas has place, we shall often have the longitude λ of the moving point given by a quadrature. We choose as our example the equation

$$\frac{1}{\mathrm{n}}\frac{d\lambda}{dt} = \sum_{i=0}^{i=\infty}\sum_{i'=-\infty}^{i'=+\infty} a^{i+|i'|}\cos(il + i'l'), \tag{1}$$

in which $l = nt + c$ and $l' = n't + c'$, and a is a positive constant less than unity. Here λ corresponds to M. Poincaré's log x (p. 279 of the above-quoted volume). Under the condition named, the series of (1) is convergent. Now let both members of the equation be integrated; putting μ for $\frac{n'}{n}$, we have

$$\lambda = \varepsilon + \mathrm{n}t + \frac{\mathrm{n}}{n}\Sigma\frac{1}{i + i'\mu}a^{i+|i'|}\sin(il + i'l'), \tag{2}$$

ε being the added arbitrary constant, and the sign of summation Σ having the same extension as that of the double sign in (1), except that the com-

bination $i = i' = 0$ is omitted. When μ is an irrational quantity, the summation of this equation constitutes a divergent series according to M. Poincaré.

We prefer to write (2) in the more expanded form

$$\lambda = \varepsilon + \text{n}t + \frac{\text{n}}{n}\left\{ \Sigma \frac{1}{i} a^i \left[\sin il + \frac{1}{\mu} \sin il' \right] \right.$$
$$+ 2\Sigma\Sigma \frac{i}{i^2 - i'^2\mu^2} a^{i+i'} \sin il \cos i'l'$$
$$\left. - 2\mu\Sigma\Sigma \frac{i'}{i^2 - i'^2\mu^2} a^{i+i'} \cos il \sin i'l' \right\}, \tag{3}$$

where the extent of the summation, in all cases, is from 1 to $+\infty$. This will be the signification of the sign Σ hereafter.

Before we proceed to consider the question of convergence in reference to (3), it may be of interest to point out that the series of (1) admits of summation. For, by arranging it according to cosines of multiples of l', we have

$$\frac{1}{\text{n}} \frac{d\lambda}{dt} = [1 + a \cos l + a^2 \cos 2l + a^3 \cos 3l + \ldots]$$
$$+ [1 + 2a \cos l + 2a^2 \cos 2l + 2a^3 \cos 3l + \ldots]\, a \cos l'$$
$$+ [1 + 2a \cos l + 2a^2 \cos 2l + 2a^3 \cos 3l + \ldots]\, a^2 \cos 2l'$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$= [1 + a \cos l + a^2 \cos 2l + \ldots]$$
$$+ [1 + 2a \cos l + 2a^2 \cos 2l + \ldots][a \cos l' + a^2 \cos 2l' + \ldots]$$
$$= \frac{1 + a^4 - a(1 + a^2)(\cos l + \cos l') + 2a^2 \cos l \cos l'}{(1 - 2a \cos l + a^2)(1 - 2a \cos l' + a^2)}.$$

Then λ can be expressed by the following quadrature:

$$\lambda = \varepsilon + \text{n}t +$$
$$\frac{\text{n}}{2}\int_0^t \left[\frac{(1 - a^2)^2}{[1 - 2a \cos (nt + c) + a^2][1 - 2a \cos (n't + c') + a^2]} - 1 \right] dt. \tag{4}$$

Our supply of functions in the integral calculus is inadequate to the expression of this quadrature in finite terms; but there is no bar to our finding the amount of motion of λ between any two given times t_0 and t_1 by the process of mechanical quadratures.

Quadratures may also be invoked to aid in the expression of (3). For, by putting

$$X_i = \frac{1}{1 - i^2\mu^2} a \cos l + \frac{1}{4 - i^2\mu^2} a^2 \cos 2l + \frac{1}{9 - i^2\mu^2} a^3 \cos 3l + \ldots, \tag{5}$$

as well as
$$P_i = -\frac{dX_i}{dl}, \quad Q_i = iX_i,$$

(3) takes the form

$$\lambda = \varepsilon + \mathrm{n}t + \frac{\mathrm{n}}{n}\left\{\Sigma\frac{1}{i}\,a^i\left[\sin il + \frac{1}{\mu}\sin il'\right]\right.$$
$$+ 2\,(P_1 a\cos l' + P_2 a^2\cos 2l' + P_3 a^3\cos 3l' + \ldots)$$
$$\left. - 2\mu\,(Q_1 a\sin l' + Q_2 a^2\sin 2l' + Q_3 a^3\sin 3l' + \ldots)\right\} \qquad (6)$$

But it will be perceived that, putting

$$L = \frac{a\cos l - a^2}{1 - 2a\cos l + a^2},$$

X_i satisfies the linear differential equation of the second order

$$\frac{d^2X_i}{dl^2} + i^2\mu^2 X_i + L = 0. \qquad (7)$$

By the integration of this, we have

$$X_i = \cos(i\mu l)\int L\sin(i\mu l)\,dl - \sin(i\mu l)\int L\cos(i\mu l)\,dl. \qquad (8)$$

The first summation of (6) may be obtained through the use of the well-known equation

$$\Sigma\frac{1}{i}\,a^i\sin il = -\tfrac{1}{2}l + \operatorname{arc\,tan}\left[\frac{1+a}{1-a}\tan\frac{l}{2}\right]. \qquad (9)$$

We come now to the consideration of the question of convergence of the two double summations in (3). In these we may put

$$\sin il = \cos il = \sin i'l' = \cos i'l' = 1;$$

the matter at issue is not thereby changed. Hence it suffices to determine the convergence or divergence of the two series

$$\Sigma\Sigma\frac{i}{i^2 - i'^2\mu^2}\,a^{i+i'}, \qquad \Sigma\Sigma\frac{i'}{i^2 - i'^2\mu^2}\,a^{i+i'}.$$

It is necessary now to specify the precise nature of the quantity μ. As an example, we assume that $\mu = \sqrt{h}$, h being a non-square integer. The divisor in the two series is then $i^2 - hi'^2$. From the theory of indeterminate equations of the second degree, we learn that the least absolute value of this expression is unity; that is, we may write

$$|i^2 - hi'^2| \geqq 1.$$

Unity may therefore be substituted for this divisor in the summations just given without thereby modifying the question of their convergence, which is thus narrowed to the convergence or divergence of the single expression

$$\Sigma\Sigma i a^{i+i'}.$$

But this series is convergent, being equivalent to the product

$$[\Sigma i a^i][\Sigma a^i] = [a + 2a^2 + 3a^3 + \ldots][a + a^2 + a^3 + \ldots].$$

If we agree to take j terms of each factor of this as an approximation, the error committed will be

$$\frac{j(1-a)+2}{(1-a)^3} a^{j+2},$$

which may be made as small as we please by taking j sufficiently large. This expression is also a superior limit to the error committed in either of the double summations of (3) when the series is pushed to the same extent in reference to the varying integers i and i'.

As a more general example, including the former, we will take

$$\mu^2 = \frac{p}{q} + \sqrt{\frac{p'}{q'}},$$

where p, q, p', and q' are integers, and $\frac{p'}{q'}$ is not an exact square. If we substitute this value of μ^2 in the expression $i^2 - i'^2\mu^2$, and rationalize and render integral this denominator, multiplying it by the proper factor, we find that it becomes

$$q'(q^2i^2 - p^2i'^2)^2 - p'q^4i'^4.$$

Now this expression, which, is integral, cannot vanish, for this would make $\sqrt{\frac{p'}{q'}}$ rational; consequently, its absolute value is at least unity. We may then substitute unity for it in the summations we consider without affecting the question of their convergence. Thus, the latter is seen to depend on the convergence of

$$\Sigma\Sigma i^3 a^{i+i'} \text{ and } \Sigma\Sigma i^2 i' a^{i+i'}.$$

As these summations are quite plainly convergent, there is nothing further to be said.

As a still greater generalization, let us suppose that μ is an irrational root of an algebraic equation with rational coefficients. Then, in a similar way as before, $i^2 - i'^2\mu^2$ may be rationalized and rendered integral by multiplying by the proper factor. The absolute value of the thus modified denominator is at least unity. On consulting the form of the numerator, it is gathered that the convergence of our series depends on that of various summations whose general type is

$$\Sigma\Sigma i^{\nu} i'^{\nu'} a^{i+i'},$$

where ν and ν' are finite positive integers. The convergence is therefore established.

When μ is rational, we would simply transfer the terms which, in the integration, become proportional to t to the term nt of (3). Then the remainder would constitute a periodic and convergent series. Thus, in all cases where μ is a root of an algebraic equation with rational coefficients, (3) may be affirmed to be a convergent series.

Our method of treatment cannot be applied in the case where μ is a root of a transcendental equation. But it may be remarked that the higher the degree of the equation which has μ for a root, the larger become the exponents ν and ν'. Thus, one is led to think that, when μ is a root of a transcendental equation, these exponents become infinite. Should this be correct, the summations, whose general type has just been given, become divergent. But we would not be warranted in concluding thence the divergence of (3). The whole question turns on the properties of the integral

$$\int\left[\frac{(1-a^2)^2}{[1-2a\cos(nt+c)+a^2][1-2a\cos(n't+c')+a^2]}-1\right]dt.$$

It is possible that the ratio $\frac{n'}{n}$ may have values which would make this expression tend towards infinity as the limits of integration were removed farther from each other. But I am not aware that this has been proved. But it is something gained to have established that, when $\frac{n'}{n}$ is an irrational root of an algebraic equation with rational coefficients, the expression is always contained between finite limits, whatever may be the limits of integration.

Our conclusions still hold when in (1) we substitute the general coefficient $C_{i,i'}$ for $\alpha^{i+|i'|}$, provided we have the condition

$$|C_{i,i'}| \lesseqgtr a^{i+|i'|},$$

α being positive and less than unity. Also, we might assume a different rate of decrement in the coefficients with augmenting multiples of l' from that which belongs to l. Calling this α', for $\alpha^{i+|i'|}$ we should have $\alpha^i\alpha'^{|i'|}$, and the course of reasoning would be scarcely changed by this modification. In case there are more than two elementary arguments, the mode of proceeding is quite similar. The ratios $\frac{n'}{n}, \frac{n''}{n}, \frac{n'''}{n}$, etc., being irrational roots of algebraical equations, the divisors introduced by integration must be rationalized and rendered integral by multiplying both terms of the fraction by the proper factor. The convergence of the series is made out as before.

MEMOIR No. 60.

Remarks on the Progress of Celestial Mechanics Since the Middle of the Century.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 27, 1895.

(Bulletin of the American Mathematical Society, Second Series, Vol. II, pp. 125–136, 1896.)

The application of mathematics to the solution of the problems presented by the motion of the heavenly bodies has had a larger degree of success than the same application in the case of the other departments of physics. This is probably due to two causes. The principal objects to be treated in the former case are visible every clear night, consequently the questions connected with them received earlier attention; while, in the latter case, the phenomena to be discussed must ofttimes be produced by artificial means in the laboratory; and the discovery of certain classes of them, as, for instance, the property of magnetism, may justly be attributed to accident. A second cause is undoubtedly to be found in the fact that the application of quantitative reasoning to what is usually denominated as physics generally leads to a more difficult department of mathematics than in the case of the motion of the heavenly bodies. In the latter we have but one independent variable, the time; while in the former generally several are present, which makes the difference of having to integrate ordinary differential equations or those which are partial. Thus it happens that, while the science of astro-mechanics is started by Newton, that of thermal conductivity receives its first treatment, at the hands of Fourier, more than a century later. In addition to these two causes, ever since the discovery of the telescope the application of optical means to the discovery of whatever might be found in the heavens has always had a fascination for mankind. And, as the ability to co-ordinate and correlate the facts observed much enhances the enjoyment of scientific occupation, it has resulted that many who began as observers ended as mathematical astronomers. Thus our science has had relatively a large number of cultivators.

A thoroughly satisfactory history of our subject is yet to be written. We have only either slight sketches of the whole, or elaborate treatments of special divisions of the science, and none of them coming down to recent

times. Among the former may be mentioned Gautier's *Essai historique sur le problème des trois corps*, which appeared in 1817. Also Laplace's historical chapters in the last volume of the *Mécanique Céleste*. Todhunter's History of the theories of attraction and the figure of the earth is an example of the latter class. Such books as Todhunter's—of which Delambre has given an earlier example in his *Histoire de l'Astronomie*—can hardly be regarded as history; they resemble rather extensive tables of contents of the literature examined, accompanied by short comments. However, in many cases, they are more useful to the student than formal histories would be, as, when judiciously compiled, they may, as epitomes in our libraries, take the place of a large mass of scientific literature. The History of Physical Astronomy by Robert Grant, is a book that comes down to 1850, and professedly covers the whole of our subject. But only one-third of this book is devoted to astro-mechanics, the rest dealing with what is really observational and descriptive astronomy. Moreover, the author indulges so much in diffusive veins of writing, that but a small fraction of the 200 pages is really given to purely historic statement. As far as the Lunar Theory is concerned, the third volume of M. Tisserand's *Traité de Mécanique Céleste* constitutes a fair history. But it must be borne in mind that the author's plan is to notice only the disquisitions having a first-class importance; hence his history is incomplete in this respect.

In America we are not well situated for investigations of this character, on account of the meagreness of our libraries. Of no inconsiderable number of memoirs and even books, having at least some importance in our subject, there exist no copies in the United States. Hence, should an American be inclined to undertake the task of writing the history of our subject, he must at least perform some of the work abroad.

In the present discourse it is proposed to touch very lightly the more important steps made since the middle of the century, the time at our disposal not admitting fuller treatment.

And first we will take up Delaunay's method, proposed for employment in the lunar theory, but quite readily extended to all classes of problems in dynamics. The first sketch of this method, given of course by the author himself, appeared in the *Comptes Rendus* of the Paris Academy of Sciences, in 1846. It professes to be merely an extract from a memoir offered for publication in the collections of the Academy, which must, however, have been afterwards withdrawn to make place for the two volumes of the *Théorie du Mouvement de la Lune*. When this extract is compared with the earlier chapters of the latter work, it is perceived that Delaunay has, to some

extent, modified and improved his method in the interim between 1846 and 1860. In this long period nothing appeared from the author on this subject. He must have been profoundly engaged in applying his method to the motion of the moon. Tisserand's exposition of this method is somewhat more brief than the author's own. But when the necessary modifications are introduced into Delaunay's procedures, to make them applicable to the more general case of the motion of a system of bodies, the establishment of the formulas can be rendered still more brief.

There is one point in reference to Delaunay's method which, as far as I am aware, has escaped notice. This method consists in a series of operations or transformations, in each of which the position of the moon in space is defined by six variables, the number three being doubled in order that the velocities, as well as the co-ordinates, may be expressed without differentials. The aim of the transformations is to make one-half of these, which Poincaré has called the linear variables, continually approach constancy, while the other half, named the angular variables, continually approach a linear function of the time. But at any stage of the process the position of the moon, as well as its velocity, is definitely fixed by the six variables produced by the last transformation, provided that the proper degree of variability is attributed to them, just as, before any transformation was made, the six elements of elliptic motion, usually denominated osculating, defined them; the point of difference to be noticed being that the more the transformations are multiplied, the more complex becomes the character of the expression of the former quantities in terms of the latter. But, however great may be the number of transformations, the series evolved have always one consistent trait, viz., that the angular variables are involved in them only through cosines or sines of linear functions of these variables, the linear functions being formed with integral coefficients. Now, as in all this work we are obliged to employ infinite series, the question of their convergence is an extremely important one. The inquiry in this respect may be divided into two parts, mainly independent of each other. These are, convergence as respects the angular variables, and convergence as respects the linear variables. The first part is much the more simple. Regarding each of the coefficients of the series we employ as a whole, that is, representing it by a definite integral, it is quite easily perceived that the said series are both legitimate and convergent when, giving the angular variables the utmost range of values, still no two of the bodies can occupy the same point of space. In the contrary case the series are evidently divergent. This condition affords certain limiting conditions for the values

of the linear variables. Could we trace these limiting conditions through all the transformations, and obtain by comparison the formulas to which these tend when the number of transformations is made infinite, we should be in possession of the conditions of stability of motion of the system of bodies. The second part of the inquiry relates to the expression of the mentioned coefficients by infinite series proceeding according to powers and products of certain parameters which are functions of the linear variables. It is well known that, in the case of elliptic elements, Laplace and Cauchy almost simultaneously showed that the series are convergent when the eccentricity does not exceed a fraction which is about two-thirds. The determination of the conditions of convergence, after certain transformations have been made in the signification of the elements, is undoubtedly a more complex problem; nevertheless, it seems to be within the competency of analysis as it exists at present.

The discovery of the criterion for the convergence of series proceeding according to powers and products of parameters is due to Cauchy, and is a most remarkable contribution to the science of mathematics. Supposing that the parameters begin from zero values, this criterion amounts to saying that the moment the function, which the series is to represent, ceases to be holomorphic, or becomes infinite, that moment the series ceases to be convergent. Consequently, if a space, having as many dimensions as there are parameters in the case, be conceived, and a surface be constructed in it formed by the consensus of all the points where the considered function ceases to be holomorphic, then, provided the values of the parameters define a point within this surface, that is, on the same side where lies the origin, the series will be convergent. Generally this surface will be closed, and, within it, the function will not take infinity as its value.

Without any mathematical reasoning the propriety of the principle just enunciated may be perceived. Since it is possible for the series in powers and products to give only one value for the function, the moment the latter may have any one of several values, the series fails to give them all; and, as there is no reason why any particular value should be selected, the conclusion must be that it does not represent any of them. Also, it is easy to see that, when the function takes infinity as its value, the series fails to represent it.

In applying this principle to the series involved in the treatment of the problem of many bodies by Delaunay's method, it appears, at first sight, as if we must have some finite representation of the coefficients in question in order to discover the particular points at which they cease to be holomorphic,

such, for instance, as is given by an algebraic or transcendental equation. But this is not imperative, as it is often possible to make this discovery from certain recognized properties of the function considered, without being in possession of its form explicitly or implicitly. It appears probable that, in the class of cases considered, the mentioned coefficients can be represented by multiple definite integrals, all taken between the limits 0 and π, the independent variables being those which have been denominated angular. Such functions are always holomorphic, provided that the expressions under the signs of integration are themselves holomorphic between the mentioned limits. If the statement just made be admitted, although it may be impossible to write explicitly the mentioned expressions, we may, nevertheless, be certain that they remain holomorphic, provided that the linear variables, which may be the same as the parameters considered, are so restricted in their range of values that no matter what values the angular variables receive, no distance between any two bodies of the system can vanish. Or, in other words, that the R of Delaunay must never become infinite. Thus it seems probable that the conditions of convergence for Delaunay's series are precisely identical with those for the stability of motion of the system.

The series arising in Delaunay's method, as applied to the moon, contain five parameters; the number would be six were the moon's mass not neglected. We should also have six in the application of the method to two planets moving about the sun; however, should we employ the well-known functions $b_s^{(i)}$ of Laplace, the number would be reduced to five. It ought to be possible, therefore, after the performance of a limited number of operations, to assign limiting values to these parameters, below which the series would certainly be convergent. This also involves the possibility of finding limits to the errors committed by truncating the series at a certain order of terms. Again, provided the time is limited to a certain interval, the capacity of these truncated series for representing the co-ordinates of the planets could be shown by giving superior limits to the errors necessarily involved.

One more remark may be made before we leave Delaunay's method. In every operation or transformation half the integrals are obtained without the intervention of the time, and from these solely are obtained the ranges of values for all the linear variables. As no integrating divisors appear in their expressions, it follows that the question of stability is not affected in any way by the vanishing of these. Moreover, the presence of a libration in the angle of operation does not necessitate any change in the procedure. The integrating divisors which appear in the expressions for

the angular variables, obtained through quadratures, may cause difficulty, but this can generally be removed by a modification of the parameters employed in the development of the coefficients in series. Beyond this it does not seem necessary to attend particularly to the terms which Professor Gyldén has designated as critical.

To give a succinct idea of the scope of this method, it may be said that it is applicable whenever, in the system, the planets maintain their order of succession from the sun. In systems where that undergoes change, as is the case with the group of minor planets, supposing their action on each other is sensible, it is not applicable.

Delaunay's method has not yet received all the developments and applications it is susceptible of.

The treatise of Hansen on the shortest and most ready method of deriving the perturbations of the small planets was published in the interval 1857-1861. But as the principles on which it is founded had been elaborated and communicated to the public some years earlier, it is, perhaps, more properly to be assigned to the first half of the century. In consequence, I pass it over with this slight mention.

Perhaps the most conspicuous labors in our subject, during the period of time we consider, are those of Professor Gyldén and M. Poincaré. We will limit our attention, for the remainder of this discourse, to the consideration of these investigations.

Professor Gyldén began work with the methods of Hansen and was gradually led to modifications of them looking towards their use for indefinite lengths of time. This quality has latterly become imperative with him, and he has recently published the first volume of what is evidently intended to be a lengthy work entitled *Traité Analytique des Orbites Absolues des Huit Planètes Principales.* To show the drift of Professor Gyldén's investigations, we cannot do better than give an analysis of this volume. At the outset the author introduces a class of curves he names periplegmatic. The definition of this sort of curve is that it describes continually the space between two concentric spheres, and, at every point, turns its concavity towards the intersection of the radius vector with the inner sphere. In an application to the solar system, the sun is supposed to occupy the common centre of the spheres. The investigation is at first limited to the case where this curve is plane. A differential equation of the second order is derived which the radius vector of this curve satisfies, the independent variable being the angle described. The perpendicular distance between the spheres is called the diastem. The spheres are sup-

posed to be drawn so that they touch the curve at the points where the radius becomes a maximum or minimum. Thus, in some cases, the spheres are regarded as fixed, in others, as movable. In the latter case, however, the sum of their radii is supposed to remain constant. Thence we have two groups of periplegmatic curves; those with constant and those with variable diastems. The author gives examples of both these groups, in most cases of which the line of apsides is variable, and considers the situation and density of the points of intersection of these curves with themselves.

The idea of an absolute orbit of a planetary body is this: an oval symmetrical with regard to an axis movable in space. While the axis remains constant in length (the half of it is called the protometre), the velocity of its motion may vary, and the diastem may also vary. Professor Gyldén, however, admits into the expressions of these variations only terms whose period would become infinite did the planetary masses vanish. These terms he calls elementary. But elementary terms in the diastem and the longitude of the perihelion can produce terms in the co-ordinates having periods which differ but little from the time of revolution of the planet These are also called elementary terms. But the two classes are distinguished, the first as being of the type (A), and the second as of the type (B). In all the formulas relative to this matter the author insists on keeping the arc described by the radius as the independent variable.

The co-ordinates are only approximately given by the preceding apparatus of expressions. They must then have certain complements added to them; these, however, are all composed of terms which would vanish with the planetary masses.

In deriving the elementary terms in the radius of a planet through the integration of a linear differential equation of the second order, Professor Gyldén attaches much price to his method of establishing the convergence of the series formed by the successive terms. As the latter are obtained through division by divisors of the order of the planetary masses, it might be feared that some of them would turn out to be very large. But the author prevents this by retaining in the coefficient of the dependent variable in the differential equation a quantity equivalent to the sum of the squares of all the coefficients in the integral. This is named the horistic or limiting function. It is plain such an expression could be introduced in the mentioned coefficient, provided that the linear equation is the truncated form of an equation containing the cube of the variable. And in the problem of planetary motion the approximations may always be so ordered that this shall be the case.

With regard to the co-ordinate which exhibits the departure of the planet from a fixed plane, Professor Gyldén does not greatly deviate from the procedure of Hansen in following the displacement of the instantaneous plane of the orbit. Only here, as in the preceding treatment of the radius, he would sharply distinguish the elementary and non-elementary terms.

At this point is introduced certain new nomenclature. As before we had diastem now we have anastem to denote the product of the radius and the sine of the inclination; and what has generally been called the true argument of the latitude is here called the anastematic argument. Any angular magnitudes which are constantly moving through the circumference are astronomic arguments; and when they have the same mean velocity of rotation they are isokinetic; and isokinetic arguments are homorhythmic when, in each revolution through the circumference, they always retake together the same corresponding points. In like manner, the true anomaly is the diastematic argument, and we have diastematic and anastematic coefficients and moduli. It will be seen from this that Professor Gyldén does not shrink from imposing on us the labor of learning new terms.

Thus far we have been engaged in deriving the equations of the path followed by a heavenly body; it remains to show how we may find the point on that path occupied by the body at a given moment. There is then necessary an equation between the time and the variable assumed as independent, that is, the orbit longitude, or, more properly, the amount of angle described by the radius vector. If we suppose the absolute orbit to be described by the planet so that equal areas are passed over by the radius in equal times, it is plain that, on the attainment of a given longitude, a definite amount of time must have elapsed since the epoch. This is what Professor Gyldén calls the *reduced* time; and he computes the difference between it and the actual time required by the theory of gravity for the planet to arrive at the stated direction. This mode of proceeding does not differ from Hansen's except in the point that the absolute orbit is substituted for a fixed ellipse.

But this gives us correctly only the orbit longitude; for the radius and the latitude, which correspond in the absolute orbit to this reduced time, are not quite those which the planet has at the actual time. Consequently, Professor Gyldén proposes to compute two corrections, the one to be applied to the product of the eccentricity into the cosine of the true anomaly, the other to the sine of the latitude. Also the reduction of the orbit longitude to the plane of reference must be manipulated so that it comes out correctly.

The employment of the orbit longitude as independent variable throughout all the integrations necessitates a mass of very intricate transformations of terms from one shape into another. Also the integrations which bear on elementary terms must be kept distinct from those which bear on non-elementary terms. A degree of complexity is thus imparted to the subject, which makes it difficult to see when one has really gathered up all the warp and woof of it. Professor Gyldén has nowhere removed the scaffolding from the front of his building and allowed us to see what architectural beauty it may possess; it is necessary to compare a large number of equations scattered through the volume before one can opine how the author means to proceed.

The advantages claimed for the method are that it prevents the time from appearing outside the trigonometrical functions, and that it escapes all criticism on the score of convergence. The first is readily conceded, but many simpler methods possessing this advantage are already elaborated, and it is not so clear that the second ought to be granted.

No completely worked out example of the application of this method has yet been published. The great labor involved will naturally deter investigators from employing it.

In 1890 was published the memoir of M. H. Poincaré, entitled *Sur le problème des trois corps et les équations de la dynamique,* and which obtained the prize of the King of Sweden. Most of the results of this memoir were worked over and presented anew with greater elaboration and clearness by their author in *Les Méthodes Nouvelles de la Mécanique Céleste.* Here we find a large number of new and very interesting theorems.

First is to be noted the class of particular solutions in the problem of the motion of a system of material points which are now named *periodic solutions.* The initial relative positions and velocities of the several points are so adjusted that, after the lapse of a definite time, the latter retake them. Hence is evident a method which may be employed to elaborate this special case of motion, viz., by the tentative process with mechanical quadratures. M. Poincaré has divided this sort of solutions into three classes, of which, however, the second and third are not essentially different. He has shown that, in the latter classes, the values of the arbitrary constants of the problem must be so adjusted that no secular inequalities, or, as Professor Gyldén calls them, elementary terms, may arise. The number and variety of these particular solutions is far greater than one would at first sight imagine.

We come now to a second class of particular solutions named by the author *asymptotic*. It arises from the consideration of solutions differing very little from periodic solutions. Here we have to deal with linear differential equations having periodic coefficients. The integrals of these contain in their terms exponential factors, and on the nature of the exponents of these factors depends the quality of the resulting solutions. M. Poincaré has named these exponents *characteristic*. They are roots of an algebraic equation of a degree equal to the number of dependent variables involved in the question. If any of these roots are imaginary with real portions or wholly real, we are in presence of asymptotic solutions. The algebraic equation mentioned contains the unknown only in even powers; hence the characteristic exponents are in pairs having the same absolute value, but with contrary signs. In all the cases presented by astronomy, where, on account of the near approach to circular motion, a periodic solution can be taken as a first approximation, it appears that the squares of the characteristic exponents are all real and negative. Thus, there is no call here to consider this sort of solution, and this fact must much diminish the interest of the astronomer in it. M. Poincaré has, however, elaborated it with great pains, showing how the effect of higher powers of the deviations from the periodic solution may be taken into account. The series resulting are, nevertheless, divergent, as in other cases.

The second volume of the *Méthodes Nouvelles* is devoted to the elaboration and consideration of various processes for developing the integrals of planetary motion according to the powers of a small parameter. The chief of these are due to Professor Newcomb and MM. Lindstedt and Bohlin; but M. Poincaré has augmented the number of them by introducing modifications of his own. All involve the principle of recurrence; that is, the first step is the only one which is independent, the following depend on all that precede. These methods, in their general aspect, do not differ from the old developments in powers of the disturbing force, except the operations are so adjusted that the time never escapes from the trigonometric functions. This is accomplished by greatly augmenting the number of the elementary arguments, and by supposing that the rate of motion of each of these is developable according to integral powers of the before-mentioned parameter, or, in some cases, of its square root.

When there is more than one elementary argument, the series obtained in all these ways are pronounced to be generally divergent in the rigorous sense of the word. M. Poincaré brings forward several methods of proof of this. The first depends on the presence of small divisors in the ex-

pressions of the coefficients. However, when we do not insist on developments in powers of a parameter, this method of proof has no application. Another method is derived from the principle that two characteristic exponents vanish for every uniform integral that exists. But the integrals which necessitate this conclusion must not only be uniform, they must be valid for every possible case of the problem. Now the integrals known as those of the conservation of living forces and of areas are of this nature; but the integrals derivable from the series of Delaunay, Newcomb, and Lindstedt are valid only for a limited range in the values of the linear variables. For instance, in the problem of the three bodies, if the deformation of the triangle formed by these bodies is such that we cannot find any two sides, one of which sustains to the other an invariable relation of greater to less, we cannot apply the mentioned series. And here it is well to note that the defect of convergence does not arise from the application of the processes of integration, but already exists in the development of the perturbative function before integration commences. Thus Delaunay's development of this function at the beginning of his lunar theory is divergent and illusory, unless we have the lunar radius in apogee always less than the solar radius in perigee, and that without regard to the mode of expressing the coefficients. Some of the particular integrals, relied upon by M Poincaré to establish the vanishing of all the characteristic exponents in case we accept M. Lindstedt's series as valid, lie, so to speak, on the boundary of the domain in which these series are convergent.

In the third place an appeal is made to the alleged non-existence of analytic and uniform integrals beyond those already known. Were this non-existence clearly established it would decide the question on the side where M. Poincaré has placed himself. But, at least as far as the non-existence of integrals of this nature in a limited domain for the linear variables is concerned, the proof given for it is quite defective. This proof consists in ascertaining how these integrals, supposing them to exist, would behave, should we attempt to derive periodic solutions from them. It is difficult to present this matter without the assistance of algebraic formulæ; nevertheless, it may be attempted. Let there be a number of equations whose left members are formed by the product of two factors. When we pass to a periodic solution, one of these factors becomes zero. What conclusion can we draw from each of the thus modified equations? Evidently one of two things: either the remaining factor of the left member is infinite and the right member indeterminate, or it is finite and the right member a vanishing quantity. Now in case we are obliged to accept the first con-

clusion, were it only but once, M. Poincaré has demonstrated the non-existence of integrals; but, granting that it is proper in every case to accept the latter conclusion, the demonstration fails. Now he declines to consider the latter alternative, saying that he does not believe that any problem of dynamics, presenting itself naturally, occurs where the right members of the mentioned equations would all vanish. But it should be borne in mind that, while they do not vanish in the general equations, the adjustment of the values of the linear parameters required by the passage to a periodic solution may bring about their vanishing. Thus, in the lunar theory, a periodic solution is brought about by making $e = 0$, $e' = 0$, and $\gamma = 0$, the result is the vanishing of every coefficient having any of these quantities as a factor.

M. Poincaré appeals in another place to the fact that the Lindstedt series, if convergent, would establish the non-existence of asymptotic solutions. But this observation is irrelevant for the reason that the domains of the two things are quite distinct. In any case where Lindstedt series are applicable there are no asymptotic solutions, and, where there are asymptotic solutions, Lindstedt's series would be illusory.

We owe much to M. Poincaré for having commenced the attack on this class of questions. But the mist which overhangs them is not altogether dispelled; there is room for further investigation.

MEMOIR No. 61.

Jupiter-Perturbations of Ceres, of the First Order, and the Derivation of the Mean Elements.

(Astronomical Journal, Vol. XVI, pp. 57-62, 1896.)

Although Ceres was the first discovered of the minor planets, no thoroughly adequate expressions for its absolute perturbations have ever been published, and the ephemerides of its position have been derived by the method of special perturbations. The fullest account of the discovery of this planet is contained in a paper by Lalande (*Conn. des Temps de l'année XIII*, pp. 453-465). It is there stated that Burckhardt computed the perturbations in the incredibly short space of two days, and that he afterwards computed tables founded upon them. I have not been able to find this work in print, and perhaps it was never published. For some time after its discovery Gauss busied himself with this planet (*Werke*, Band VI, pp. 199-313). He computed no less than thirteen sets of elements, and gave analytical expressions for the perturbations with tables for facilitating their computation, but did not go beyond terms of the first order with respect to the eccentricities and mutual inclination. Damoiseau (*Conn. des Temps*, 1846, *Additions*, p. 32), has also given expressions for the perturbations. But, although a large number of terms are computed, some even of the fifth order with respect to the eccentricities and mutual inclination, the individual coefficients do not seem very exact, probably because only the term of the lowest order was taken into account; and the secular motions assigned to the eccentricity and longitude of the perihelion are much too rapid. These are the only disquisitions on this subject I am aware of.

In 1874, at the time of publishing a modified form of computing absolute perturbations (*Astr. Nachr.* No. 1982), I entered into an engagement to illustrate it by computing the first-order perturbations of Ceres by Jupiter. Considerable work was done shortly after, but other occupations hindered its completion at that time. However, I have now brought it to a conclusion, and the results are here presented.

The elements of Ceres and Jupiter employed as the basis of the work are the following:

EPOCH, 1854 JANUARY $0^d.0$.

$$
\begin{array}{rlrl}
 & \;\;\circ\quad '\quad '' & & \;\;\circ\quad '\quad '' \\
\pi = & 148\ 55\ 23.41 & \pi' = & 11\ 58\ 42.2 \\
\Omega = & 80\ 50\ 50.79 & \Omega' = & 98\ 56\ 38.3 \\
i = & 10\ 37\ \ 8.54 & i' = & 1\ 18\ 39.6 \\
\phi = & 4\ 24\ 38.41 & \phi' = & 2\ 45\ 55.24 \\
\mu = & 769''.63875 & \mu' = & 299''.1286 \\
\log a = & 0.0000000 & \log a' = & 0.2737570
\end{array}
$$

$$m' = \frac{1}{1050}$$

Those of Ceres are on the authority of Ernest Schubert (*A.J.* III, 153–159, 162–165). The computations were carried through with the stated mass of Jupiter, a value now antiquated; but this imperfection will be removed at the end, when the final results will be changed to suit a more modern value of this constant.

From the given elements result the following values of the mutual inclination of orbits and angular distances of the perihelia from the ascending node of Jupiter's orbit on the orbit of Ceres:

$$I = 9°22'53''.85, \quad \Pi = 250°34'28''.01, \quad \Pi' = 113°35'30''.69$$

The circumference being divided into 16 equal parts with reference to v, the true anomaly of Ceres, we compute the quantities K, K', A and A' for each of the points of division from the equations

$$
\begin{aligned}
K \cos(\Pi' - A) &= \cos(v + \Pi) \\
K \sin(\Pi' - A) &= \cos I \sin(v + \Pi) \\
K' \cos(\Pi' - A') &= \cos I \cos(v + \Pi) \\
K' \sin(\Pi' - A') &= \sin(v + \Pi)
\end{aligned}
$$

Let ζ and ζ' denote the mean anomalies severally of Ceres and Jupiter, and w' an angle such that

$$\zeta' = v - \frac{\mu'}{\mu}(v - \zeta) + w'$$

Then the circumference is divided into 24 equal parts with reference to w'. Let r and r' denote the radii of Ceres and Jupiter; h, double the areal velocity of Ceres, and v' the true anomaly of Jupiter. We compute for all

combinations of the 16 values of v, with the 24 values of w', the four following quantities:

$$\Delta^2 = 1 + \left(\frac{r'}{r}\right)^2 - 2K\frac{r'}{r}\cos(v' + A)$$

$$X = \frac{m'}{h^2} Kr\left[\frac{1}{\Delta^3} - \left(\frac{r}{r'}\right)^3\right] r' \cos(v' + A) - \frac{m'}{h^2}\frac{r^2}{\Delta^3}$$

$$Y = \frac{m'}{h^2} K'\left[\frac{1}{\Delta^3} - \left(\frac{r}{r'}\right)^3\right] r' \sin(v' + A')$$

$$Z = \frac{m'}{h^2} \sin I\left[\frac{1}{\Delta^3} - \left(\frac{r}{r'}\right)^3\right] r' \sin(v' + \Pi')$$

For convenience the three last are expressed in seconds of arc. For about two-thirds of the values of w' where Ceres and Jupiter are quite distant from each other, differencing with reference to v will serve as a test of their correctness. For the remaining values of w' they must be checked by a duplicate computation.

Mechanical quadratures are now applied to these 384 values of X, Y and Z, first with reference to the variable v, and then with reference to the variable w'. After these periodic series are obtained we replace w' by $\vartheta' - v$, so that the signification of ϑ' is given by the equation

$$\vartheta' = \zeta' + \frac{\mu'}{\mu}(v - \zeta)$$

whence $\frac{d\vartheta'}{dv} = \frac{\mu'}{\mu}$. The general argument of these series is $iv + i'\vartheta'$, i and i' being integers. The following table gives the periodic developments of X, Y, Z.

Arg. i	i'	X cos	X sin	Y cos	Y sin	Z cos	Z sin
		″	″	″	″	″	″
0	0	+20.57678	. . .	+ 0.03455	. . .	+ 3.256	. . .
1	0	— 9.34693	— 0.64788	— 0.13276	+ 0.06894	—13.41794	— 4.48970
2	0	— 0.1197	— 0.9805	— 0.6521	+ 0.8212	+ 2.9068	+ 1.4363
3	0	+ 0.285	+ 0.439	+ 0.348	— 0.180	— 0.311	— 0.112
4	0	— 0.058	— 0.081	— 0.027	+ 0.019	+ 0.018	— 0.012
5	0	+ 0.01	+ 0.01	0.00	0.00	. . .	. . .
— 4	— 1	+ 0.016	+ 0.003	0.000	— 0.001	. . .	. . .
— 3	— 1	+ 0.124	— 0.046	+ 0.014	+ 0.029	. . .	. . .
— 2	— 1	— 0.491	+ 0.270	— 0.045	— 0.201	+ 0.49	+ 0.07
— 1	— 1	+ 0.003	— 0.540	— 0.182	+ 0.561	— 3.652	— 0.597
0	— 1	+ 7.813	+ 4.269	+ 1.056	— 1.641	+ 9.640	+ 4.226
1	— 1	—17.4732	—15.5278	— 4.3858	+ 4.8851	— 4.254	— 3.363
2	— 1	+ 3.476	+ 2.216	— 0.429	— 0.244	+ 4.42	+ 7.85
3	— 1	— 0.227	+ 1.202	+ 1.080	— 0.082	— 0.79	— 1.77
4	— 1	+ 0.022	— 0.524	— 0.250	— 0.006	+ 0.11	+ 0.12
5	— 1	0.000	+ 0.070	+ 0.025	— 0.001	. . .	. . .

Arg. i	i'	X cos "	X sin "	Y cos "	Y sin "	Z cos "	Z sin "
— 3	— 2	+ 0.024	+ 0.007	— 0.004	+ 0.007	. . .	. . .
— 2	— 2	— 0.158	— 0.035	+ 0.011	+ 0.022	+ 0.01	+ 0.06
— 1	— 2	+ 0.498	+ 0.182	— 0.193	+ 0.111	— 0.55	— 0.55
0	— 2	— 0.0673	+ 1.8213	+ 1.6813	— 0.3220	+ 2.928	+ 3.946
1	— 2	— 3.5564	—17.5296	—11.6928	+ 1.5478	— 5.833	—13.083
2	— 2	+ 4.7181	+57.1832	+49.4221	— 3.7845	+ 1.106	+ 4.584
3	— 2	— 1.713	— 7.047	— 3.678	+ 1.367	+ 1.15	— 5.39
4	— 2	+ 0.849	— 0.345	— 0.518	— 0.615	— 0.30	+ 1.07
5	— 2	— 0.21	+ 0.09	+ 0.14	+ 0.13	. . .	. . .
6	— 2	+ 0.02	— 0.04	— 0.01	— 0.01	. . .	. . .
— 2	— 3	— 0.025	— 0.024	0.000	+ 0.001	. . .	. . .
— 1	— 3	+ 0.133	+ 0.110	— 0.036	— 0.002	— 0.01	— 0.11
0	— 3	— 0.3421	+ 0.0131	+ 0.4117	+ 0.0765	+ 0.238	+ 1.096
1	— 3	+ 0.9744	— 3.6564	— 3.11454	— 0.76372	— 0.151	— 5.149
2	— 3	— 6.7575	+19.4914	+15.9989	+ 6.4363	— 2.869	+ 8.918
3	— 3	+22.268	—29.430	—26.060	—20.161	+ 1.52	— 2.94
4	— 3	— 3.486	+ 4.038	+ 2.478	— 0.526	— 2.61	+ 1.89
5	— 3	— 0.71	— 0.16	+ 0.08	+ 0.65	+ 0.51	— 0.36
6	— 3	+ 0.21	— 0.03	— 0.02	— 0.13	. .	. . .
7	— 3	— 0.03	+ 0.02	0.00.	+ 0.01	. . .	. . .
— 1	— 4	+ 0.012	+ 0.038	0.000	+ 0.003	. . .	. .
0	— 4	— 0.053	— 0.116	+ 0.050	+ 0.039	— 0.07	+ 0.16
1	— 4	+ 0.3645	+ 0.2471	— 0.4406	— 0.3832	+ 0.454	— 1.149
2	— 4	— 3.1579	+ 3.2819	+ 2.9694	+ 2.7147	— 2.568	+ 3.174
3	— 4	+14.084	— 8.344	— 7.349	—12.076	+ 5.07	— 3.00
4	— 4	—21.886	+ 4.199	+ 3.532	+19.926	— 2.05	+ 0.81
5	— 4	+ 2.15	— 1.28	— 1.04	— 1.15	+ 1.86	+ 0.15
6	— 4	+ 0.35	+ 0.40	+ 0.31	— 0.36	— 0.34	— 0.02
7	— 4	— 0.13	— 0.07	+ 0.01	— 0.08	. . .	. . .
— 1	— 5	— 0.002	+ 0.007	+ 0.001	0.000	. . .	. . .
0	— 5	+ 0.0092	— 0.0365	+ 0.0001	+ 0.0052	— 0.015	+ 0.009
1	— 5	+ 0.0166	+ 0.0640	— 0.0406	— 0.0846	+ 0.163	— 0.154
2	— 5	— 0.6539	+ 0.1132	+ 0.29957	+ 0.71839	— 0.994	+ 0.565
3	— 5	+ 4.3150	— 0.6108	— 0.6302	— 3.8565	+ 2.941	— 0.506
4	— 5	—11.700	— 2.142	— 1.833	+ 8.855	— 3.51	— 0.69
5	— 5	+11.155	+ 6.713	+ 6.319	—10.118	+ 1.33	+ 0.48
6	— 5	— 1.40	— 0.13	+ 0.16	+ 2.47	— 0.71	— 0.77
7	— 5	— 0.03	— 0.37	— 0.34	+ 0.05	+ 0.13	+ 0.12
8	— 5	+ 0.04	+ 0.17	+ 0.06	0.00	. . .	. . .
0	— 6	+ 0.006	— 0.005	— 0.001	— 0.001	. . .	. . .
1	— 6	— 0.022	+ 0.020	— 0.006	— 0.009	+ 0.028	— 0.007
2	— 6	— 0.041	— 0.035	— 0.009	+ 0.125	— 0.239	+ 0.028
3	— 6	+ 0.7851	— 0.0235	+ 0.2033	— 0.8253	+ 0.943	+ 0.179
4	— 6	— 3.166	— 1.956	— 1.730	+ 2.910	— 1.76	— 1.09
5	— 6	+ 5.01	+ 6.41	+ 5.80	— 4.58	+ 1.35	+ 1.71
6	— 6	— 2.20	—10.44	— 6.57	+ 1.90	— 0.42	— 0.78
7	— 6	+ 0.59	+ 0.49	+ 0.21	— 0.51	+ 0.05	+ 0.58
8	— 6	— 0.13	+ 0.21	+ 0.19	+ 0.14	. . .	. . .
9	— 6	0.00	— 0.19	— 0.03	— 0.03	. .	. . .

Arg. i	i'	X cos ″	X sin ″	Y cos ″	Y sin ″	Z cos ″	Z sin ″
1	− 7	− 0.008	+ 0.001	+ 0.001	+ 0.001	. . .	. . .
2	− 7	+ 0.017	− 0.004	− 0.008	+ 0.012	− 0.037	− 0.012
3	− 7	+ 0.0646	+ 0.0679	+ 0.0936	− 0.1189	+ 0.196	+ 0.123
4	− 7	− 0.495	− 0.585	− 0.654	+ 0.513	− 0.466	− 0.594
5	− 7	+ 0.890	+ 2.798	+ 2.558	− 0.857	+ 0.39	+ 1.32
6	− 7	+ 0.35	− 5.31	− 4.90	+ 1.26	+ 0.09	− 1.28
7	− 7	− 1.73	+ 3.80	+ 3.47	+ 1.67	− 0.13	+ 0.53
8	− 7	− 0.10	− 0.42	− 0.27	− 1.36	+ 0.21	− 0.24
9	− 7	+ 0.13	− 0.07	− 0.02	− 0.16	. . .	. . .
10	− 7	− 0.02	− 0.03	− 0.02	+ 0.02	. . .	. . .
1	− 8	− 0.001	− 0.001	0.000	0.000	. . .	. . .
2	− 8	+ 0.0062	+ 0.0029	− 0.0008	+ 0.0003	− 0.002	− 0.003
3	− 8	− 0.0062	+ 0.0028	+ 0.0190	− 0.0150	+ 0.025	+ 0.035
4	− 8	− 0.0267	− 0.1307	− 0.1593	+ 0.0433	− 0.082	− 0.184
5	− 8	− 0.042	+ 0.759	+ 0.673	+ 0.025	− 0.03	+ 0.56
6	− 8	+ 0.87	− 2.00	− 1.87	− 0.79	+ 0.38	− 0.81
7	− 8	− 2.42	+ 2.38	+ 2.20	+ 2.25	− 0.52	+ 0.57
8	− 8	+ 2.12	− 0.93	− 0.84	− 1.96	+ 0.27	− 0.18
9	− 8	− 0.08	+ 0.19	+ 0.23	− 0.03	. . .	. . .
10	− 8	− 0.10	+ 0.03	− 0.10	+ 0.08	. . .	. . .
2	− 9	+ 0.001	+ 0.001	0.000	0.000	. . .	. . .
3	− 9	− 0.0032	− 0.0045	+ 0.0018	+ 0.0004	+ 0.001	+ 0.006
4	− 9	+ 0.0050	− 0.0103	− 0.0263	− 0.0040	+ 0.002	− 0.044
5	− 9	− 0.068	+ 0.133	+ 0.151	+ 0.065	− 0.06	+ 0.16
6	− 9	+ 0.44	− 0.46	− 0.46	− 0.41	+ 0.28	− 0.28
7	− 9	− 1.41	+ 0.66	+ 0.67	+ 1.31	− 0.52	+ 0.22
8	− 9	+ 2.14	− 0.10	− 0.08	− 3.11	+ 0.44	− 0.03
9	− 9	− 1.20	− 0.40	− 0.35	+ 1.10	− 0.19	− 0.03
10	− 9	+ 0.08	− 0.01	− 0.18	+ 1.06	. . .	. . .
11	− 9	+ 0.11	− 0.02	+ 0.17	+ 0.05	. . .	. . .
3	−10	0.0000	− 0.0006	− 0.0002	+ 0.0001	. . .	. . .
4	−10	+ 0.0001	+ 0.0022	− 0.0025	+ 0.0018	+ 0.003	− 0.007
5	−10	− 0.0149	+ 0.0122	+ 0.0173	+ 0.0146	− 0.028	+ 0.030
6	−10	+ 0.129	− 0.062	− 0.018	− 0.135	+ 0.12	− 0.06
7	−10	− 0.51	+ 0.18	+ 0.05	+ 0.48	− 0.28	+ 0.02
8	−10	+ 1.02	+ 0.30	+ 0.28	− 0.96	+ 0.34	+ 0.11
9	−10	− 0.98	− 0.80	− 0.77	+ 0.93	− 0.22	− 0.13
10	−10	+ 0.32	+ 0.62	+ 0.52	− 0.35	. . .	. . .
11	−10	+ 0.06	− 0.05	+ 0.10	+ 0.19	. .	. . .
4	−11	− 0.0008	+ 0.0008	0.0000	− 0.0001	. . .	. . .
5	−11	− 0.0006	− 0.0004	+ 0.0015	+ 0.0039	− 0.007	+ 0.003
6	−11	+ 0.024	− 0.023	+ 0.001	− 0.031	+ 0.04	0.00
7	−11	− 0.13	− 0.03	+ 0.09	+ 0.13	− 0.10	− 0.03
8	−11	+ 0.31	+ 0.14	+ 0.21	− 0.31	+ 0.15	+ 0.11
9	−11	− 0.38	− 0.60	− 0.58	+ 0.35	− 0.11	− 0.19
10	−11	+ 0.13	+ 0.79	+ 0.73	+ 0.32	. . .	. . .
11	−11	+ 0.16	− 0.36	− 0.32	+ 0.03	. . .	. . .
12	−11	− 0.16	− 0.10	+ 0.07	− 0.70	. . .	. . .
13	−11	+ 0.22	+ 0.20	− 0.16	+ 0.24		. .

In order to pass from the values of these three functions to the expressions of the perturbations of the three coordinates, it is necessary to derive first the function

$$T = X + 2fr^3 + 2r^3 \int r^{-2} \left(\frac{e \sin v}{p} X + Y \right) dv,$$

where p denotes the semi-parameter and f is the arbitrary constant which completes the integral of the expression. To discover the value of the latter, let X contain the terms

$$B_0 + B_1 \cos v$$

and T, without the addition of $2fr^3$, the terms

$$A_0 + A_1 \cos v$$

Then

$$f = -\frac{1}{6ap^2} \left[3A_0 + B_0 + \frac{e}{2}(3A_1 + B_1) + \tan \frac{\varphi}{2} \cos^2 \varphi . A_1 \right]$$

In the present case we have

$$A_0 = +20''.96790, \quad B_0 = +20''.57678, \quad A_1 = -12''.75542, \quad B_1 = -9''.34693,$$

consequently

$$f = -13''.68833$$

The perturbations of the radius, of the orbit-longitude, and a latitude referred to the plane of the primitive orbit, are then given by the equations

$$\delta r = \int T \sin (\bar{v} - v)\, dv$$

$$\delta \lambda = \int \left[\int Y dv - 2 \frac{\delta r}{r} \right] dv$$

$$\delta \beta = \int Z \sin (\bar{v} - v)\, dv$$

where $\bar{v}$ denotes a v invariable in the integration which afterwards is to be made equal to v.

In integrating the perturbation of the radius δr contains the portion

$$K^{(c)} \cos v + K^{(s)} \sin v$$

depending on arbitrary constants. In order to make the terms involving the argument v disappear from the perturbation of the orbit longitude $\delta\lambda$, we must put

$$K^{(c)} = +3''.6988, \quad K^{(s)} = +0''.4188$$

After the integration is accomplished we substitute for the v, which appears as a multiplier of coefficients, its value given by the infinite series

$$v = nt + E_1 \sin v - \tfrac{1}{2} E_2 \sin 2v + \tfrac{1}{3} E_3 \sin 3v - \ldots$$

where

$$E_i = 2\left(1 + i\sqrt{1-e^2}\right)\left(\frac{e}{1+\sqrt{1-e^2}}\right)^i$$

Before the integration was performed it was discovered that the μ of the osculating elements for 1854 Jan. 1 differed widely from the mean value. In integrating, therefore, the value $\mu = 770''.75$ was assumed, and the mass of Jupiter was put at $\frac{1}{1047.355}$. The following table gives the expressions for the perturbations, those of the common logarithm of the radius being in units of the seventh decimal:

Arg.		$\delta\lambda$		$\delta(\log r)$		$\delta\beta$	
i	i'	sin	cos	cos	sin	sin	cos
		$''$	$''$			$''$	$''$
0	0	. . .	. . .	− 138.3	. . .	. . .	+ 2.75
. .	.	. . .	. . .	+ 0.4127nt	. . .	. . .	. . .
1	0	. . .	. . .	+ 68.0	+ 9.3	. . .	. . .
. .	.	− 1.01940nt	− 6.51277nt	+ 10.7318nt	− 68.5638nt	− 6.72591nt	+ 2.25052nt
2	0	− 1.05	+ 0.77	+ 13.8	+ 12.1	− 0.31	− 0.45
. .	.	− 0.01960nt	− 0.12522nt	+ 0.4127nt	− 2.6364nt	. . .	. . .
3	0	+ 0.06	− 0.09	− 0.9	− 1.5	+ 0.01	+ 0.02
−3	−1	− 0.01	0.00	− 0.2	+ 0.1	. . .	. . .
−2	−1	+ 0.14	+ 0.05	+ 2.7	− 0.9	− 0.01	− 0.10
−1	−1	− 0.54	− 1.18	− 3.6	+ 18.6	+ 0.65	+ 3.95
0	−1	+ 41.31	− 29.74	+ 124.2	+ 92.8	+ 4.99	+11.38
1	−1	+162.88	−146.19	−1133.5	−1017.4	− 5.39	− 6.82
2	−1	+ 7.35	− 5.09	− 123.1	− 89.2	− 4.92	− 2.77
3	−1	+ 0.07	− 0.47	− 1.5	− 8.6	+ 0.30	+ 0.14
4	−1	0.00	+ 0.05	0.0	+ 1.0	− 0.01	− 0.01
−2	−2	+ 0.01	− 0.01	+ 0.3	+ 0.1	− 0.01	0.00
−1	−2	− 0.22	− 0.85	− 3.4	+ 17.1	+ 0.26	+ 0.25
0	−2	+ 5.10	− 44.13	+ 37.3	+ 337.8	+ 9.95	+ 7.38
1	−2	+107.73	−600.51	− 326.8	−1966.3	−13.81	− 6.16
2	−2	+ 43.06	−514.81	− 522.1	−6206.4	− 9.23	− 2.23
3	−2	− 0.26	− 5.05	− 0.4	− 135.8	+ 1.37	− 0.29
4	−2	+ 0.13	+ 0.15	− 2.4	+ 3.3	− 0.11	+ 0.03
5	−2	− 0.01	− 0.01	+ 0.3	− 0.2	. . .	. . .
−1	−3	− 0.01	− 0.09	− 0.2	+ 2.2	+ 0.03	0.00
0	−3	+ 1.27	− 8.76	+ 16.3	+ 103.6	− 3.09	− 0.67
1	−3	+ 10.62	− 65.53	− 30.7	+ 313.4	− 5.30	− 0.16
2	−3	+143.46	+396.34	−1343.5	+3689.0	+29.64	− 9.54
3	−3	+ 29.57	+ 43.22	− 455.6	+ 685.5	+ 1.24	− 0.64
4	−3	+ 0.06	− 0.48	+ 0.3	− 5.0	− 0.27	+ 0.37
5	−3	− 0.09	− 0.01	+ 1.9	− 0.3	+ 0.03	− 0.04
6	−3	+ 0.01	0.00	− 0.2	0.0	. . .	. . .

Arg. i	i'	$\delta\lambda$ sin	$\delta\lambda$ cos	δ (log r) cos	δ (log r) sin	$\delta\beta$ sin	$\delta\beta$ cos
		$''$	$''$			$''$	$''$
0	—4	+ 0.09	— 0.10	+ 1.8	+ 1.3	— 0.11	+ 0.05
1	—4	+ 3.45	+ 1.61	+ 12.7	— 0.9	— 1.66	+ 0.65
2	—4	+ 51.89	+ 65.22	— 308.5	+ 377.4	+ 3.98	— 3.22
3	—4	+ 44.38	+ 28.87	— 588.3	+ 386.4	+ 2.75	— 4.64
4	—4	— 8.99	— 1.36	+ 145.6	— 19.8	— 0.16	+ 0.41
5	—4	+ 0.15	+ 0.17	— 2.1	+ 3.0	— 0.01	— 0.17
6	—4	+ 0.03	— 0.03	— 0.7	— 0.5	0.00	+ 0.02
0	—5	+ 0.44	+ 0.27	+ 9.0	— 5.5	0.00	+ 0.01
1	—5	+ 21.22	+ 13.45	+ 209.2	— 132.7	— 1.34	+ 1.41
2	—5	+297.15	+193.91	— 313.6	+ 174.6	+ 0.57	— 1.00
3	—5	+184.80	+ 45.15	—2022.8	+ 497.3	+ 4.14	—24.06
4	—5	— 4.64	+ 2.30	+ 55.2	+ 40.5	+ 0.21	+ 1.09
5	—5	+ 2.38	— 1.51	— 41.8	— 26.9	— 0.06	— 0.16
6	—5	— 0.21	— 0.02	+ 2.6	— 0.5	+ 0.05	+ 0.05
7	—5	0.00	+ 0.02	+ 0.1	+ 0.4	0.00	— 0.01
1	—6	+ 0.08	+ 0.07	+ 1.1	— 0.9	+ 0.01	— 0.04
2	—6	+ 0.15	+ 0.24	+ 4.5	— 0.5	+ 0.03	— 0.27
3	—6	— 12.82	+ 1.02	+ 103.6	+ 10.4	+ 0.33	+ 1.72
4	—6	— 5.61	+ 3.05	+ 80.4	+ 42.8	+ 0.61	+ 0.98
5	—6	+ 1.60	— 2.03	— 26.8	— 34.2	— 0.28	— 0.22
6	—6	— 0.27	+ 1.08	+ 4.8	+ 23.0	+ 0.06	+ 0.03
7	—6	+ 0.04	— 0.01	— 0.7	— 0.1	— 0.03	0.00
8	—6	— 0.01	— 0.01	+ 0.1	— 0.2	. . .	. . .
2	—7	— 0.15	+ 0.17	— 1.0	— 1.1	— 0.02	— 0.08
3	—7	— 3.63	+ 2.37	+ 15.3	+ 10.6	+ 0.13	+ 0.21
4	—7	— 3.43	+ 3.80	+ 42.1	+ 46.0	+ 0.92	+ 0.72
5	—7	+ 0.46	— 1.42	— 7.0	— 22.4	— 0.31	— 0.09
6	—7	— 0.14	+ 0.95	+ 0.9	+ 17.0	+ 0.13	— 0.01
7	—7	— 0.16	— 0.33	+ 3.1	— 6.2	— 0.03	+ 0.01
8	—7	+ 0.05	+ 0.01	— 0.3	+ 0.3	+ 0.01	+ 0.01
9	—7	+ 0.01	0.00	— 0.1	+ 0.1	. . .	. . .
2	—8	— 0.24	+ 0.10	— 2.8	— 1.1	+ 0.01	+ 0.01
3	—8	— 2.90	+ 0.92	— 4.7	— 2.9	+ 0.04	+ 0.03
4	—8	+ 1.11	— 4.17	— 11.0	— 41.2	— 0.93	— 0.41
5	—8	+ 0.01	— 0.83	— 0.4	— 12.7	— 0.22	+ 0.01
6	—8	+ 0.22	+ 0.51	— 3.7	+ 8.8	+ 0.11	— 0.05
7	—8	— 0.27	— 0.27	+ 5.1	— 5.0	— 0.04	+ 0.04
8	—8	+ 0.13	+ 0.06	— 2.6	+ 1.1	+ 0.01	— 0.01
9	—8	0.00	— 0.01	0.0	— 0.1	. . .	. . .
3	—9	— 0.01	— 0.01	— 0.1	0.0	+ 0.01	0.00
4	—9	— 0.02	— 0.33	+ 0.2	— 2.3	— 0.06	0.00
5	—9	— 0.16	— 0.39	+ 2.0	— 5.1	— 0.12	+ 0.05
6	—9	+ 0.17	+ 0.19	— 2.7	+ 3.0	+ 0.05	— 0.05
7	—9	— 0.21	— 0.11	+ 3.8	— 1.9	— 0.02	+ 0.05
8	—9	+ 0.23	+ 0.01	— 3.7	+ 0.1	0.00	— 0.02
9	—9	— 0.06	+ 0.02	+ 1.1	+ 0.4	0.00	+ 0.01
10	—9	— 0.03	0.00	+ 0.2	0.0	. . .	. . .

Arg. i	i'	$\delta\lambda$ sin	$\delta\lambda$ cos	$\delta (\log r)$ cos	$\delta (\log r)$ sin	$\delta\beta$ sin	$\delta\beta$ cos
		$''$	$''$			$''$	$''$
3	—10	+ 0.06	— 0.02	+ 0.6	+ 0.2	...	...
4	—10	+ 0.61	— 0.16	— 0.9	— 0.4	— 0.01	0.00
5	—10	— 0.17	— 0.30	+ 1.9	— 3.3	— 0.12	+ 0.11
6	—10	+ 0.09	+ 0.02	— 1.3	+ 0.3	+ 0.02	— 0.03
7	—10	— 0.11	— 0.02	+ 1.8	— 0.5	0.00	+ 0.03
8	—10	+ 0.10	— 0.03	— 1.9	— 0.5	— 0.01	— 0.02
9	—10	— 0.06	+ 0.05	+ 1.1	+ 0.9	0.00	+ 0.01
10	—10	+ 0.01	— 0.02	— 0.2	— 0.4	...	...
4	—11	0.00	— 0.01	...	...	...	...
5	—11	+ 0.04	+ 0.01	— 0.3	+ 0.1	+ 0.01	— 0.02
6	—11	+ 0.04	+ 0.01	— 0.6	+ 0.3	0.00	— 0.02
7	—11	— 0.04	— 0.02	+ 0.7	— 0.1	0.00	+ 0.02
8	—11	+ 0.04	— 0.02	— 0.7	— 0.4	— 0.01	— 0.01
9	—11	— 0.03	+ 0.04	+ 0.5	+ 0.8	+ 0.01	+ 0.01
10	—11	— 0.01	— 0.03	0.0	— 0.7	...	...
11	—11	0.00	+ 0.01	— 0.1	+ 0.2	...	...
12	—11	+ 0.01	0.00	...	...	...	...

In order to arrive at mean elements for Ceres, as well as to see how closely the preceding perturbations were capable of representing the heliocentric coordinates of the planet, 10 normal positions were formed for the times of as many oppositions from material as follows:

Opposition of 1802 — Derived from 4 observations at Greenwich on Mar. 6, 14, 18, 25.

" " 1807 — Derived from Gauss's *Werke*, VI, 299.

" " 1830 — Derived from 5 observations at Greenwich on Apr. 27, 28, 30; May 1, 3.

" " 1857 — Derived from 4 observations at Greenwich on Feb. 16, 24, 26, 28.

" " 1863 — Derived from 5 observations at Greenwich on July 6, 9, 10, 16, 18.

" " 1866 — Derived from 3 observations at Greenwich, 1 at Paris on Jan. 22, 23, 24; Feb. 2.

" " 1873 — Derived from 4 observations at Greenwich, 3 at Paris on Sept. 19, 20, 23, 26, 27; Oct. 2, 6.

" " 1883 — Derived from 8 observations at Greenwich on Nov. 23, 26, 27, 29; Dec. 4, 5, 14, 15.

" " 1885 — Derived from eleven observations at Greenwich on Mar. 27, 28, 30, 31; Apr. 2, 4, 6, 11, 16, 17, 18.

" " 1890 — Derived from 7 observations at Greenwich on May 14, 15, 21, 22, 23, 24, 26.

The normals, with certain additional data, are

Greenwich M. T.		Hel. Long. M. E. of Date	Geoc. Lat.	Log. Earth's Radius	v
	d	° ′ ″	° ′ ″		° ′ ″
1802 Mar.	17.15072	176 21 32.3	+17 7 57.2	9.9982161	27 22 46.0
1807 May	3.14789	222 13 53.8	+10 40 15.9	0.0038415	72 47 3.6
1830 Apr.	30.32003	219 55 21.2	+11 11 24.4	0.0035208	70 37 38.6
1857 Feb.	13.77639	145 34 58.1	+15 36 20.0	9.9947561	357 15 27.9
1863 July	16.16898	293 33 38.9	− 8 49 22.5	0.0070483	144 41 38.8
1866 Jan.	20.79681	121 5 35.9	+11 9 43.7	9.9931177	332 20 44.7
1873 Sept.	23.34662	0 51 22.7	−15 45 57.5	0.0011276	211 24 17.1
1883 Dec.	6.86562	74 57 7.3	− 1 44 29.6	9.9933823	285 41 43.1
1885 Apr.	2.34757	193 15 32.4	+15 51 21.8	0.0001720	43 6 40.2
1890 May	17.37448	236 51 47.4	+ 6 53 49.4	0.0051454	86 39 46.0

	ϑ'	$\delta\lambda$	$\delta \log r$	$\delta\beta$
	° ′ ″	′ ″		″
1802	139 25 1.0	+22 36.9	− 5035	− 3.3
1807	296 45 40.8	+24 33.2	+ 7507	+59.9
1830	274 31 6.4	− 1 7.6	+ 8938	+58.6
1857	4 20 30.5	−17 5.9	− 6064	−34.8
1863	201 16 52.6	+11 56.4	+ 6398	+23.9
1866	274 6 12.9	+12 23.9	−10719	−43.4
1873	146 35 49.7	+11 13.2	− 6087	−10.4
1883	94 51 25.5	+ 26.1	− 8011	−69.5
1885	140 26 1.5	+22 49.0	+ 1114	+22.6
1890	297 3 7.8	+15 32.9	+ 9161	+77.9

The Earth's radius is corrected for its latitude according to the precept of Gauss (*Theoria Motus*, Art. 72). To ϑ' are added the long-period inequalities of the mean longitude of Jupiter. The values of the perturbations given in the last columns contain only the periodic terms; that of $\log r$ is in units of the seventh decimal. The terms due to the action of Mars and Saturn were computed from the formulas of Damoiseau.

The secular perturbations through the action of Mars, Jupiter and Saturn were computed by the method of Gauss, the values of the attracting masses being severally $\frac{1}{3093500}$, $\frac{1}{1047.355}$, $\frac{1}{3501.6}$. The results obtained are annexed to the following system of elements, which is that compared with the foregoing normals. The secular motions of π, Ω and i include the effect of precession and motion of the ecliptic. The adopted value of general precession for 1850 is 50″.25787.

Epoch, 1850 Jan. 0.0 G. M. T.

$$
\begin{aligned}
L &= 309^\circ\ 30'\ 36''.7 \\
\pi &= 148\ 23\ 18.1 + 106''.6658\,t \\
\Omega &= 80\ 50\ 45.0 - 4.9064\,t \\
i &= 10\ 37\ 10.1 - 0.5676\,t \\
e &= 0.0784500 - 0.0000033347\,t \\
\mu &= 770''.72275 \\
\log a &= 0.4420721
\end{aligned}
$$

The residuals left by the comparison of these elements with the normals are (Obs. — Cal.).

	Hel. Long. ″	Geoc. Lat. ″		Hel. Long. ″	Geoc. Lat. ″
1802	+ 50.0	— 6.2	1866	— 41.9	+ 24.7
1807	+ 64.5	— 36.5	1873	+ 1.4	+ 21.2
1830	— 8.7	— 30.6	1883	— 86.1	+ 29.3
1857	— 112.3	— 3.6	1885	— 132.2	— 9.7
1863	— 3.1	— 31.2	1890	— 43.4	— 41.6

The equations of condition for correcting the assumed elements follow. The final terms of those which arise from the longitudes are made to be residuals in geocentric longitude multiplied by the cosine of the geocentric latitude. For convenience, 40 Julian years is the unit of time for δn; and logarithms are written in place of the coefficients and final terms.

	δL	δn	δe	$e\delta\pi$	δi	$\sin i\delta\Omega$		
1802	0.2672	0.3445n	0.2115	0.4953n	8.4476	9.1657n	1.8969n	= 0
1807	0.2183	0.2463n	0.4881	0.0219n	9.1591	8.4948	2.0072n	= 0
1830	0.2218	9.9136n	0.4832	0.0777n	9.1653	8.2980	1.1371	= 0
1857	0.2752	9.5258	9.1437n	0.5506n	9.0604n	8.9829n	2.2512	= 0
1863	0.1233	9.6528	0.2255	0.3470	9.1107n	8.7608	0.6714	= 0
1866	0.2651	9.8687	0.2068n	0.4965n	9.1698n	8.3681	1.8261	= 0
1873	0.1189	9.8921	0.1796n	0.3641	8.6749	9.1163n	0.3141n	= 0
1883	0.2123	0.1409	0.4831n	0.0063n	8.4786	9.1571	2.1329	= 0
1885	0.2557	0.2008	0.3524	0.4166n	9.0183	9.0255n	2.3191	= 0
1890	0.2006	0.2047	0.4968	9.5788n	9.0397	8.9905	1.8349	= 0
1802	8.6001n	8.6774	9.1570	9.1793	0.2006	9.1811	0.7924	= 0
1807	9.3875n	9.4155	9.6191n	9.4182	9.9952	0.0921	1.5623	= 0
1830	9.3848n	9.0766	9.5902n	9.4489	0.0162	0.0778	1.4857	= 0
1857	9.1716	8.4222	9.2357	9.4569n	0.1608	9.8347n	0.5563	= 0
1863	9.3113n	8.8408n	9.3040n	9.5934n	9.9144n	0.1066	1.4942	= 0
1866	9.4239	9.0275	9.0576n	9.7023n	0.0161	0.0884n	1.3927n	= 0
1873	8.5646	8.3378	8.7992	9.1512	0.1667n	9.4141n	1.3263n	= 0
1883	9.4808	9.4094	9.7573n	9.2342n	9.2080n	0.1956n	1.4669n	= 0
1885	9.1333n	0.0784n	8.4282n	9.4843	0.1678	9.7837	0.9868	= 0
1890	9.4417n	9.4458n	9.7255n	9.1310	9.8069	0.1592	1.6191	= 0

With equal weights, these 20 equations produce the normal equations

δL	δn	δe	$e\delta\pi$	δi	$\sin i\Omega$	$''$	
23.232	+ 3.028	+14.472	—23.316	— 0.031	— 2.473	+ 826.0	= 0
+ 3.028	+17.398	— 8.914	+ 2.845	+ 0.014	— 0.301	+1068.6	= 0
+14.472	— 8.914	+54.309	—10.240	+ 0.716	— 1.276	— 286.3	= 0
—23.316	+ 2.845	—10.240	+53.884	+ 0.465	+ 2.198	—1179.9	= 0
— 0.031	+ 0.014	+ 0.716	+ 0.465	+13.300	+ 1.850	+ 95.5	= 0
— 2.473	— 0.301	— 1.276	+ 2.198	+ 1.850	+11.657	+ 265.5	= 0

Their solution gives the corrections and corrected elements:

	Epoch 1850 Jan. 0^d0 G. M. T.
$\delta L = -4''3$	$L = 309°\ 30'\ 32''4$
$\delta\mu = -0''004474$	$\pi = 148\ \ 28\ \ 32.5$
$\delta e = -0.00000145$	$\Omega = \ 80\ \ 48\ \ \ 5.6$
$\delta\pi = +5'\ 14''4$	$i = \ 10\ \ 37\ \ \ 6.2$
$\delta i = \quad -3''9$	$e = 0.07844855$
$\delta\Omega = -2'\ 39''4$	$\mu = 770''718276$
	$\log a = 0.4420738$

The residuals left by this solution in heliocentric longitude and geocentric latitude, as they result from the equations of condition, are (Obs. — Calc.),

	Hel. Long.	Geoc. Lat.		Hel. Long.	Geoc. Lat.
1802	+10''0	+ 3''7	1866	+41''4	+13''2
1807	+13.9	+13.0	1873	— 1.8	+ 9.5
1830	—18.1	+ 8.3	1883	— 6.3	+ 4.7
1857	—40.2	— 8.6	1885	—22.0	— 0.1
1863	—15.0	+ 7.3	1890	+36.0	—19.7

They are larger than one would be inclined to attribute to the neglect of the square of the disturbing force. However, the derived mean elements, on account of the long period of time embraced, will have some value.

MEMOIR No. 62.

On the Values of the Eccentricities and Longitudes of the Perihelia of Jupiter and Saturn for Distant Epochs.

(Astronomical Journal, Vol. XVII, pp. 81-87, 1897.)

The values of these elements derived from the formulas given by LeVerrier and Stockwell, in their general treatment of the secular variations of the elements of the large planets of the solar system, are considerably in error at remote epochs on account of the neglect by these investigators of the terms arising from the squares and products of disturbing forces. Thus, Stockwell puts the length of the period of the principal inequalities of the eccentricities and perihelia of Jupiter and Saturn at 69140 years; but 55000 years is a much nearer approximation to it.

In ascertaining the modifications the perturbative function undergoes, by reason of terms of two dimensions with respect to disturbing forces, we are obliged to have recourse to the labors of LeVerrier, as no one else has derived them in the form which is necessary. I have given the results in this *Journal*, No. 204. From the values of the coefficients of R, the perturbative function, I deduce the following expression:

$$\begin{aligned} R = {} & 0''.002906504e^2 + 0''.002631048e'^2 + 0''.00859488e^4 + 0''.0926819e^2e'^2 \\ & + 0''.0591391e'^4 + 0''.28648e^6 + 7''.2004e^4e'^2 + 19''.8285e^2e'^4 + 5''.9589e'^6 \\ & - [0''.003565305 + 0''.0561361e^2 + 0''.145101e'^2 + 2''.87238e^4 + 23''.9129e^2e'^2 \\ & + 21''.7899e'^4]\, ee' \cos(\omega' - \omega) + [0''.0436099 + 4''.77063e^2 \\ & + 13''.1094e'^2]\, e^2e'^2 \cos 2(\omega' - \omega) - 2''.61437\, e^3e'^3 \cos 3(\omega' - \omega). \end{aligned}$$

The coefficients of this expression are generally the mean of the values deduced from LeVerrier's four equations; but, for the coefficient of $ee' \cos(\omega' - \omega)$, I have preferred the value which results from the equations for Saturn, as the motion of the eccentricities at the epoch 1850, thus obtained, more nearly agrees with that of my New Theory of Jupiter and Saturn; and, as the motion of the perihelion of Saturn at the same epoch, so far as it results from the inter-action of Jupiter and Saturn, from the same theory is $19''.95426$, to bring about an agreement, I have supposed that LeVerrier's coefficient of the term $+18''.12312\,e'$ in the expression for $\frac{e'}{\cos\psi'}\frac{d\omega'}{dt}$ ought to be increased by $0''.3$, and thus be read $+18''.42312$.

Desiring to pursue a similar course to that in a previous paper (*Annals of Mathematics*, Vol. V, p. 177), and get a form of the perturbative function in a measure independent of the values assumed for the masses of the planets and of the linear unit, I multiply the coefficients of the preceding expression by a factor whose logarithm is 1.9212280, and put $e^2 = 4\theta^2 - 4\theta^4$, $e'^2 = 4\theta'^2 - 4\theta'^4$, and express the development in powers of $\cos(\omega' - \omega)$ instead of in cosines of multiples of $\omega' - \omega$. Thus we have

$$\begin{aligned}\frac{a'}{mm'}\Omega = {} & 0.9697480\,\theta^2 + 0.8778427\,\theta'^2 + 10.5009\,\theta^4 + 65.4910\,\theta^2\theta'^2 + 78.0629\,\theta'^4 \\ & + 1506.4\,\theta^6 + 12905.5\,\theta^4\theta'^2 + 35803.5\,\theta^2\theta'^4 + 31652.9\,\theta'^6 \\ & - [1.1895555 + 58.9054\,\theta^2 + 193.0457\,\theta'^2 + 15244.4\,\theta^4 + 85660.1\,\theta^2\theta'^2 \\ & + 116032.8\,\theta'^4]\,\theta\theta' \cos(\omega' - \omega) + [116.4027 + 50818.3\,\theta^2 \\ & + 139849.0\,\theta'^2]\,\theta^2\theta'^2 \cos^2(\omega' - \omega) - 55825.8\,\theta^3\theta'^3 \cos^3(\omega' - \omega).\end{aligned}$$

If the coefficients of this expression are compared with those of the similar expression in the memoir quoted (p. 195) it will be seen how much the convergence of the series has been diminished by the inclusion of terms of two dimensions with respect to disturbing forces.

Denoting $m\sqrt{\mu a}$, $m'\sqrt{\mu' a'}$ severally by $\frac{1}{\lambda^2}$, $\frac{1}{\lambda'^2}$, we have the integral

$$\left(\frac{\theta}{\lambda}\right)^2 + \left(\frac{\theta'}{\lambda'}\right)^2 = K,$$

K being an arbitrary constant whose value must be ascertained by substituting in the equation for θ and θ' the values they have at a determinate epoch, as 1850. The values we attribute to the masses of Jupiter and Saturn are severally $\frac{1}{1047.355}$ and $\frac{1}{3501.6}$, the first being Bessel's value augmented by a 2000th part. We take a' as the linear unit, and thence log $a = 9.7367410$. Thus log $\lambda = 1.5757581$, log $\lambda' = 1.7721022$. We assume the following values of the eccentricities and longitudes of the perihelia for 1850:

$$e = 0.04825336, \qquad e' = 0.05605744,$$
$$\omega = 11^\circ\ 54'\ 26''.77, \qquad \omega' = 90^\circ\ 6'\ 38''.45.$$

But these values of course include the effect of the presence of Uranus, Neptune, etc., and, desiring to work as though Jupiter and Saturn were alone, we subtract from the values of $e\cos\omega$, $e\sin\omega$, etc., the terms due to the presence of the other planets. The values of the former, at the epoch 1850, are

$$e\cos\omega = +0.04721505, \qquad e\sin\omega = +0.00995615,$$
$$e'\cos\omega' = -0.00010829, \qquad e'\sin\omega' = +0.05605730,$$

and the terms of these due to the other planets, derived from Stockwell's formulas, are

$$-0.00057480, +0.00194960, -0.00053070, +0.00180250.$$

By subtracting these we obtain

$$e \cos \omega = +0.04778985, \qquad e \sin \omega = +0.00800655,$$
$$e' \cos \omega' = +0.00042241, \qquad e' \sin \omega' = +0.05425484,$$

and thus result

$$e = 0.04845509, \qquad e' = 0.05425649,$$
$$\omega = 9^\circ\ 30'\ 38''.93, \qquad \omega' = 89^\circ\ 33'\ 14''.12.$$

The substitution of these values of e and e' in the integral equation gives $\log K = 93.7956760$. Then we can replace the variables θ and θ' by the expressions

$$\theta = [8.4735961] \sin \tfrac{1}{2}\nu \quad , \quad \theta' = [8.6699402] \cos \tfrac{1}{2}\nu.$$

The potential function Ω becomes then divisible by K, and we will put $\Omega = KR$, where this R must not be confounded with the R we used at the outset. Then, writing x for $\cos \nu$, we get

$$\begin{aligned} R = {} & 0.0006895184 + 0.0003724115x + 0.0000727015x^2 + 0.0000109446x^3 \\ & - [0.0004889780 + 0.0001386728x + 0.0000304563x^2] \sin \nu \cos \gamma \\ & + [0.0000616764 + 0.0000275660x] \sin^2\nu \cos^2\gamma \\ & - 0.0000082095 \sin^3\nu \cos^3\gamma , \end{aligned}$$

where, for brevity, γ is put for $\omega' - \omega$. The partial derivative of Ω with respect to K is needed in the discussion, and we have

$$\begin{aligned} \frac{\partial (KR)}{\partial K} = {} & 0.0008002223 + 0.0005743155x + 0.0001902984x^2 + 0.0000328338x^3 \\ & - [0.0006723170 + 0.0003598214x + 0.0000913689x^2] \sin \nu \cos\gamma \\ & + [0.0001604290 + 0.0000826980x] \sin^2\nu \cos^2\gamma \\ & - 0.0000246285 \sin^3\nu \cos^3\gamma \end{aligned}$$

As the terms of these two equations, written above, are only the beginnings of infinite series we will add to them some additional terms obtained by induction, as thereby the results are probably rendered more exact. By substituting in the first the values of $x = \cos \nu$ and $\sin \nu \cos \gamma$ which hold at 1850, the value of the constant C, in the integral equation $R = C$, is ascertained. At that time we have $\nu = 109^\circ\ 3'\ 31''.62$, $\gamma = 80^\circ\ 2'\ 35''.19$. For convenience' sake we divide the equation $R = C$ by -0.000488978, the coefficient of $\sin \nu \cos \gamma$ in it. Thus, this coefficient

is unity in the equation which follows. For brevity we write y for $\sin \nu \cos \gamma$. Then

$$
\begin{array}{rllll}
0.3800515 = & -[9.8817338]x & +[0.0000000]y & -[9.1008297]y^2 & +[8.225028]y^3 \\
& -[9.1722541]x^2 & +[9.4527020]xy & -[8.7510845]xy^2 & +[7.825028]xy^3 \\
& -[8.3508344]x^3 & +[8.7934640]x^2y & -[8.2893993]x^2y^2 & +[7.22503\ \ ]x^2y^3 \\
& -[7.41747\ \ ]x^4 & +[8.0222860]x^3y & -[7.71577\ \ ]x^3y^2 & +[6.4250\ \ ]x^3y^3 \\
& -[6.3722\ \ ]x^5 & +[7.13917\ \ ]x^4y & -[7.03021\ \ ]x^4y^2 & +[5.4250\ \ ]x^4y^3 \\
& -[5.2149\ \ ]x^6 & +[6.1441\ \ ]x^5y & -[6.2327\ \ ]x^5y^2 & +[4.22\ \ ]x^5y^3 \\
& -[3.95\ \ ]x^7 & +[5.0371\ \ ]x^6y & -[5.3233\ \ ]x^6y^2 & \\
& & +[3.82\ \ ]x^7y & -[4.3019\ \ ]x^7y^2 & \\
& & & -[3.17\ \ ]x^8y^2 &
\end{array}
$$

$$
\begin{array}{lllll}
-[7.28163]y^4 & +[6.2828]y^5 & -[5.2210]y^6 & +[4.0961]y^7 & -[2.91]y^8 \\
-[6.6816\ \ ]xy^4 & +[5.4828]xy^5 & -[4.2210]xy^6 & +[3.10\ \ ]xy^7 & \\
-[5.8816\ \ ]x^2y^4 & +[4.4828]x^2y^5 & -[3.02\ \ ]x^2y^6 & & \\
-[4.8816\ \ ]x^3y^4 & +[3.28\ \ ]x^3y^5 & & & \\
-[3.68\ \ ]x^4y^4 & & & &
\end{array}
$$

The foregoing may be regarded as the equation in rectangular coordinates of a plane curve. It is obvious that we need consider only the portion between the limits $\pm$ 1 of the abscissa. We therefore compute for every 0.1 of x between these limits the corresponding value of y, and get the following table, where, in order to exhibit the quality of the function, the differences are also noted:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
−1.0	−0.3171623					
		+707853				
−0.9	0.2463770		+ 2330			
		710183		− 227		
−0.8	0.1753587		2103		+ 22	
		712286		205		+ 6
−0.7	0.1041301		1898		28	
		714184		177		12
−0.6	−0.0327117		1721		40	
		715905		137		8
−0.5	+0.0388788		1584		48	
		717489		89		10
−0.4	0.1106277		1495		58	
		718984		− 31		13
−0.3	0.1825261		1464		71	
		720448		+ 40		10
−0.2	0.2545709		1504		81	
		721952		121		20
−0.1	0.3267661		1625		101	
		723577		222		21
0.0	0.3991238		1847		122	
		725424		344		23
0.1	0.4716662		2191		145	
		727615		489		22
0.2	0.5444277		2680		167	
		730295		656		37
0.3	0.6174572		3336		204	
		733631		860		42
0.4	0.6908203		4196		246	
		737827		1106		50
0.5	0.7646030		5302		296	
		743129		1402		80
0.6	0.8389159		6704		376	
		749833		1778		70
0.7	0.9138992		8482		446	
		758315		2224		+57
0.8	0.9897307		10706		+503	
		769021		+2727		
0.9	1.0666328		+13433			
		+782454				
1.0	+1.1448782					

The value of y expanded in powers of x, which might be derived from the foregoing table, forms a series too slowly convergent to be used, but the table will serve instead. The variables x and y do not move through the whole range of the table, but only between the two limits which satisfy the equation $x^2 + y^2 = 1$. By trial from the data of the table we find for the first limit

$$x = \cos \nu = -0.9578284, \quad y = \sin \nu \cos \gamma = -0.2873410, \quad \gamma = 180°;$$

and, for the second limit

$$x = \cos \nu = +0.5735289, \quad y = \sin \nu \cos \gamma = +0.8191854, \quad \gamma = 0°.$$

The angle ν then moves between the extreme values $\nu = 55° \ 0' \ 11''.97$ and $\nu = 163° \ 18' \ 4''.20$. These values correspond to the maximum and minimum values of the eccentricities, and we have

When	$\gamma = 0°$	$e = 0.02747989$	$e' = 0.08289313.$
"	$\gamma = 180°$	$e = 0.05885862$	$e' = 0.01358174.$

But these are the values which result when the inter-action of Jupiter and Saturn is alone considered. To obtain the actual maximum and minimum, the sum of the coefficients of the terms arising from the presence of the other planets must be applied. According to Stockwell's investigation these are:

For *Jupiter* 0.0020673 ; *Saturn* 0.0019165.

Thence we have the maximum and minimum values as follows:

	Maximum Eccentricity	Minimum Eccentricity
Jupiter	0.06092592	0.02541259
Saturn	0.08480963	0.01166524

By putting

$$1 - x^2 - y^2 = \sin^2\nu \sin^2\gamma = (0.5735289 - x)(x + 0.9578284)\,Q$$

Q is a factor whose variation is small relatively to its magnitude.

From the table giving the values of y corresponding to the set values of x may be obtained the values of Q corresponding to the same argument. Calling h the factor -0.000488978, by which we have divided the equation $R = C$, we get the values of $\frac{1}{h}\frac{\partial R}{\partial y}$ corresponding to the same argument, by differentiating the preceding equation connecting x and y, with respect to y, and substituting in the result the value of y from the preceding table. In this way are derived the following values of log Q and log $\frac{1}{h}\frac{\partial R}{\partial y}$:

x	$\log Q$	$\log\left(\frac{1}{h}\frac{\partial R}{\partial y}\right)$
-1.0	0.1806684	9.9174264
-0.8	0.1815019	9.9249009
-0.6	0.1822846	9.9327715
-0.4	0.1830558	9.9409367
-0.2	0.1838695	9.9492582
0.0	0.1847981	9.9575473
0.2	0.1859370	9.9655508
0.4	0.1874115	9.9729259
0.6	0.1893944	9.9792066

As the variable ν does not move through the whole circumference, we substitute for it an auxiliary variable ψ possessing this property, such that

$$x = \cos\nu = -0.1921497 - 0.7656786\cos\psi.$$

Then for the elaboration of the equation

$$\frac{dt}{d\psi} = -\frac{\frac{dy}{dC}}{\sqrt{Q}}$$

we have the following table of values, given for intervals of 0.1 in x, in order to facilitate the following change to the variable ψ:

x	$-\frac{\frac{dy}{dC}}{\sqrt{Q}}$	Δ	Δ^2	Δ^3	Δ^4
-1.0	2008.862				
		-17.937			
-0.9	1990.925		-298		
		18.235		$+47$	
-0.8	1972.690		251		$+3$
		18.486		50	
-0.7	1954.204		201		9
		18.687		59	
-0.6	1935.517		142		4
		18.829		63	
-0.5	1916.688		79		8
		18.908		71	
-0.4	1897.780		-8		10
		18.916		81	
-0.3	1878.864		$+73$		3
		18.843		84	
-0.2	1860.021		157		17
		18.686		101	
-0.1	1841.335		258		7
		18.428		108	
0.0	1822.907		366		18
		18.062		126	
0.1	1804.845		492		17
		17.570		143	
0.2	1787.275		635		17
		16.935		160	
0.3	1770.340		795		24
		16.140		184	
0.4	1754.200		979		$+30$
		15.161		$+214$	
0.5	1739.039		$+1.193$		
		-13.968			
0.6	1725.071				

By interpolating in the preceding table we obtain the values of the function corresponding to equidistant values of the angle ψ, as follows:

ψ	x	$-\dfrac{\frac{dy}{d\theta}}{\sqrt{Q}}$
0°	-0.9578284	2001.337
15	0.9317385	1996.652
30	0.8552468	1982.799
45	0.7335662	1960.435
60	0.5749890	1930.819
75	0.3903219	1895.948
90	-0.1921497	1858.547
105	$+0.0060225$	1821.808
120	0.1906896	1788.887
135	0.3492668	1762.278
150	0.4709474	1743.331
165	0.5474391	1732.249
180	$+0.5735289$	1728.638

By applying mechanical quadratures to the values in the last column we obtain for the periodic series representing the latter function,

$$\frac{dt}{d\psi} = 1861.5617 + 138.234 \cos\psi + 3.217 \cos 2\psi - 1.861 \cos 3\psi + 0.205 \cos 4\psi - 0.024 \cos 5\psi + 0.003 \cos 6\psi.$$

Whence, by integration,

$$t + c = 1861.5617\psi + 138.234 \sin\psi + 1.608 \sin 2\psi - 0.620 \sin 3\psi + 0.051 \sin 4\psi - 0.005 \sin 5\psi.$$

Calling to mind that we have put $a' = 1$, it is obvious that the mean motion of Saturn is denoted by $\sqrt{\frac{3502.6}{3501.6}}$. But, when the Julian year is adopted as the unit of time, the mean motion of this planet is 43996″.08. The mean motion of the argument ψ in a year is then

$$\frac{43996''.08}{1861.5617}\sqrt{\frac{3501.6}{3502.6}} = 23''.63059;$$

and the period in which the principal inequalities of the eccentricities go through their round of values is 54844.16 years. Putting M for $23''.63059\,(t + c)$, we have

$$M = \psi + 15316''.6 \sin\psi + 178''.2 \sin 2\psi - 68''.7 \sin 3\psi + 5''.7 \sin 4\psi - 0''.5 \sin 5\psi + 0''.1 \sin 6\psi.$$

In order to have the proper value of M corresponding to a given date, from the value of ν previously given for 1850.0, for the same epoch, we

derive $\psi = 280° \ 6' \ 31''.13$; whence $M = 275° \ 53' \ 15''.3$. Thus the formula for M is

$$M = 275° \, 53' 15''.3 + 23''.63059 \, (t - 1850).$$

It remains to devise some method of obtaining separately the quantities ω and ω'. For this purpose we employ the equation

$$\frac{d\,(\omega + \omega' - Ct)}{d\psi} = \frac{K\dfrac{dy}{dK} - x\dfrac{dy}{dx} - \dfrac{x^2 y}{1 - x^2}}{\sqrt{Q}}.$$

Prolonging the series previously given for $\dfrac{d\,(KR)}{dK}$ by induction, we have

$$\begin{aligned}
\frac{d\,(KR)}{dK} = \; & 0.0008002223 - [6.8275741]y + [6.2052829]y^2 - [5.3914379]y^3 \\
& + [6.7591505]x - [6.5560870]xy + [5.9174950]xy^2 - [5.0904079]xy^3 \\
& + [6.2794352]x^2 - [5.9607984]x^2y + [5.37190\ \ ]x^2y^2 - [4.52938\ \ ]x^2y^3 \\
& + [5.5163212]x^3 - [5.18949\ \ ]x^3y + [4.56630\ \ ]x^3y^2 - [3.70835\ \ ]x^3y^3 \\
& + [4.49829\ \ ]x^4 - [4.1581\ \ ]x^4y + [3.5007\ \ ]x^4y^2 - [2.6273\ \ ]x^4y^3 \\
& + [3.2683\ \ ]x^5 - [2.88\ \ ]x^5y + [2.1851\ \ ]x^5y^2 - [1.2963\ \ ]x^5y^3 \\
& + [4.39477]y^4 - [3.1733]y^5 + [1.8417]y^6 - [0.2102]y^7 \\
& + [4.04477]xy^4 - [2.7933]xy^5 + [1.4417]xy^6 \\
& + [3.43477]x^2y^4 - [2.1533]x^2y^5 + [0.7817]x^2y^6 \\
& + [2.5748\ \ ]x^3y^4 - [1.2533]x^3y^5 \\
& + [1.4748\ \ ]x^4y^4
\end{aligned}$$

By attributing to x values evenly spaced from $x = -1.0$ to $x = 0.6$ and to y its corresponding values we get the following table of values for this function:

x	$1000\,\dfrac{d\,(KR)}{dK}$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
−1.0	+0.5203093					
		+32024				
−0.9	0.5235117		+ 499			
		32523		+139		
−0.8	0.5267640		638		− 14	
		33161		125		+ 2
−0.7	0.5300801		763		12	
		33924		113		− 3
−0.6	0.5334725		876		15	
		34800		98		2
−0.5	0.5369525		974		17	
		35774		81		4
−0.4	0.5405299		1055		21	
		36829		60		13
−0.3	0.5442128		1115		34	
		37944		+ 26		10
−0.2	0.5480072		1141		44	
		39085		− 18		13
−0.1	0.5519157		1123		57	
		40208		75		16
0.0	0.5559365		1048		73	
		41256		148		19
0.1	0.5600621		900		92	
		42156		240		25
0.2	0.5642777		660		117	
		42816		357		29
0.3	0.5685593		+ 303		146	
		43119		503		−28
0.4	0.5728712		− 200		−174	
		42919		−677		
0.5	0.5771631		− 877			
		+42042				
0.6	+0.5813673					

We can write the differential equations determining ω and ω' thus:

$$\frac{d\omega}{d\psi} = -\tfrac{1}{2}\frac{d(KR)}{dK}\frac{\frac{dy}{dC}}{\sqrt{Q}} - \tfrac{1}{2}\frac{(1+x)\frac{dy}{dx} + \frac{xy}{1-x}}{\sqrt{Q}}$$

$$\frac{d\omega'}{d\psi} = -\tfrac{1}{2}\frac{d(KR)}{dK}\frac{\frac{dy}{dC}}{\sqrt{Q}} + \tfrac{1}{2}\frac{(1-x)\frac{dy}{dx} + \frac{xy}{1+x}}{\sqrt{Q}}$$

or

$$\frac{d\omega}{d\psi} = -\frac{[9.6598021]}{1-x} + F$$

$$\frac{d\omega'}{d\psi} = \frac{[9.1099174]}{1+x} + F'$$

where the values of F and F' corresponding to special values of x are contained in the following table:

x	F	Δ	Δ^2	F'	Δ	Δ^2
−1.0	+0.6866546			+0.6801311		
		−12070			− 1122	
−0.9	0.6854176		−1818	0.6800189		−3268
		13888			4390	
−0.8	0.6840588		1379	0.6795799		2769
		15267			7159	
−0.7	0.6825321		964	0.6788640		2234
		16231			9393	
−0.6	0.6809090		588	0.6779247		1674
		16819			11067	
−0.5	0.6792271		270	0.6768180		1101
		17089			12168	
−0.4	0.6775182		− 13	0.6756012		− 537
		17102			12705	
−0.3	0.6758080		+ 159	0.6743307		0
		16943			12705	
−0.2	0.6741137		216	0.6730602		+ 506
		16727			12199	
−0.1	0.6724410		+ 145	0.6718403		966
		16582			11233	
0.0	0.6707828		− 128	0.6707170		1373
		16710			9860	
0.1	0.6691118		628	0.6697310		1689
		17338			8171	
0.2	0.6673780		1397	0.6689139		1876
		18735			6295	
0.3	0.6655045		2505	0.6682844		1891
		21240			4404	
0.4	0.6633805		4042	0.6678440		1713
		25282			2691	
0.5	0.6608523		−6058	0.6675749		+1290
		−31340			− 1401	
0.6	+0.6577183			+0.6674348		

It will be perceived that $\frac{d\omega}{d\psi}$ and $\frac{d\omega'}{d\psi}$ become infinite when in the first $x = 1$ and in the second $x = -1$. This is explained by the circumstance that, in this case, the one and the other of the eccentricities become zero, and, consequently, the positions of the perihelia indeterminate. Although,

in the present application, x does not attain either of these values, yet it is necessary to separate, in the formulas, the portions which become infinite, viz.: the terms $\frac{f}{1-x}$ and $\frac{f'}{1+x}$, f and f' being constants. It thus becomes necessary to discover the values of f and f'. This we do by assimilating $\frac{1}{2}\frac{xy}{\sqrt{Q}}$, which is to be divided by $1-x$ or $1+x$, to a polynomial in x. The theory of algebraical equations shows that the remainders, left after the several divisions, are what the function divided becomes, when, in the first case, we make $x=1$, and, in the second, $x=-1$. But we have

$$\frac{1}{2}\frac{xy}{\sqrt{Q}} = \frac{1}{2}xy\sqrt{\frac{(\mathrm{a}-x)(x-\mathrm{b})}{1-x^2-y^2}},$$

where a and b are respectively the superior and inferior limiting values of x. Thus the remainders are severally $\frac{1}{2}\sqrt{(1-\mathrm{a})(1-\mathrm{b})}$ and $\frac{1}{2}\sqrt{(1+\mathrm{a})(1+\mathrm{b})}$; and we have

$$f = -\tfrac{1}{2}\sqrt{(1-\mathrm{a})(1-\mathrm{b})}, \quad f' = +\tfrac{1}{2}\sqrt{(1+\mathrm{a})(1+\mathrm{b})}.$$

Interpolating the values of F and F' to correspond to equal intervals of the auxiliary angle ψ, precisely as was done for the function $\frac{dt}{d\psi}$, we get

ψ	F	F'
0°	0.6861706	0.6801257
15	0.6858527	0.6800921
30	0.6848457	0.6798599
45	0.6830572	0.6791319
60	0.6804925	0.6776614
75	0.6773524	0.6754797
90	0.6739817	0.6729620
105	0.6706829	0.6706534
120	0.6675439	0.6689822
135	0.6644982	0.6680445
150	0.6616393	0.6676374
165	0.6594553	0.6674958
180	0.6586206	0.6674627

By the well known process we obtain for the periodic series representing F and F':

$$\begin{aligned} F &= 0.6734830 + 0.0134490\cos\psi - 0.0007873\cos 2\psi + 0.0002755\cos 3\psi \\ &\quad - 0.0002945\cos 4\psi + 0.0000505\cos 5\psi - 0.0000058\cos 6\psi, \\ F' &= 0.6734830 + 0.0070854\cos\psi + 0.0004198\cos 2\psi - 0.0007822\cos 3\psi \\ &\quad - 0.0001050\cos 4\psi + 0.0000282\cos 5\psi - 0.0000036\cos 6\psi. \end{aligned}$$

The constant terms of the two functions have come out equal, as they should, and thus we have a check on the accuracy of the computation.

Integrating the expressions for $\frac{d\omega}{d\psi}$ and $\frac{d\omega'}{d\psi}$, we get

$$\omega = c - \text{arc tan}\,[[9.6690575]\tan\tfrac{1}{2}\psi] + 0.6734830\,\psi + 2774''.1\sin\psi - 81''.2\sin 2\psi$$
$$+ 18''.9\sin 3\psi - 15''.2\sin 4\psi + 2''.1\sin 5\psi - 0''.2\sin 6\psi,$$
$$\omega' = c' + \text{arc tan}\,[[0.7859273]\tan\tfrac{1}{2}\psi] + 0.6734830\,\psi + 1461''.5\sin\psi + 43''.3\sin 2\psi$$
$$- 53''.8\sin 3\psi - 5''.4\sin 4\psi + 1''.2\sin 5\psi - 0''.1\sin 6\psi.$$

It will be perceived from these formulas that, in the period of 54844.16 years, in which the eccentricities of the planets go through their round of values, the perihelion of Saturn advances through an arc 422° 27′ 13″.9, and that of Jupiter through an arc a circumference less, or 62° 27′ 13″.9.

Although we now have all the formulas necessary for ascertaining at any time the considered elements of the planets, in making further investigations on the perturbations of Jupiter and Saturn it is very convenient to have the four functions $e\cos\omega$, $e\sin\omega$, $e'\cos\omega'$, and $e'\sin\omega'$ expressed explicitly in terms of the time. For this purpose we derive the two expressions

$$\sqrt{1-x}\,{}^{\cos}_{\sin}\,\omega = +\,0.3730883\,{}^{\cos}_{\sin}\,[1.1734830\psi + c + 2774''.1\sin\psi - 81''.2\sin 2\psi + \ldots]$$
$$+\,1.0261360\,{}^{\cos}_{\sin}\,[0.1734830\psi + c + 2774''.1\sin\psi - 81''.2\sin 2\psi + \ldots]$$
$$\sqrt{1+x}\,{}^{\cos}_{\sin}\,\omega' = -\,0.7298805\,{}^{\cos}_{\sin}\,[1.1734830\psi + c + 1461''.5\sin\psi + 43''.3\sin 2\psi - \ldots]$$
$$+\,0.5245233\,{}^{\cos}_{\sin}\,[0.1734830\psi + c + 1461''.5\sin\psi + 43''.3\sin 2\psi - \ldots]$$

We transform these expressions so that they stand thus:

$$e\cos\omega = L\cos\chi - N\sin\chi + P\cos\chi' - S\sin\chi'$$
$$e\sin\omega = L\sin\chi + N\cos\chi + P\sin\chi' + S\cos\chi'$$
$$e'\cos\omega' = L'\cos\chi - N'\sin\chi + P'\cos\chi' - S'\sin\chi'$$
$$e'\sin\omega' = L'\sin\chi + N'\cos\chi + P'\sin\chi' + S'\cos\chi'$$

where χ denotes $1.1734830\,M + c$, and χ' $0.1734830\,M + c$, and compute the special values of L, N, etc., corresponding to equidistant values of M. The values of ψ, correspondent to the values of M, are

M	ψ	M	ψ
0°	0° 0′ 0″.0	105°	100° 49′ 19″.9
15	13 57 42.7	120	116 13 1.0
30	27 58 49.6	135	131 53 34.0
45	42 6 45.5	150	147 47 51.2
60	56 24 52.4	165	163 51 35.4
75	70 56 21.4	180	180 0 0.0
90	85 43 56.6		

The resulting values of L, N, etc., follow; they have been computed for every 15° of M, but it is thought sufficient here to give them for every 30°.

M	L	N	P	S
0	+0.01569403	0.00000000	+0.04316460	0.00000000
30	0.01568453	—0.00055468	0.04316546	—0.00000442
60	0.01566453	0.00098081	0.04316786	+0.00000199
90	0.01565339	0.00116239	0.04317138	0.00001605
120	0.01566451	0.00102276	0.04317516	0.00003573
150	0.01568794	—0.00058755	0.04317810	+0.00004339
180	+0.01569935	0.00000000	+0.04317924	0.00000000

M	L'	N'	P'	S'
0	—0.04827219	0.00000000	+0.03469046	0.00000000
30	0.04823472	+0.00184077	0.03468862	—0.00010047
60	0.04815368	0.00324916	0.03468358	0.00016630
90	0.04809893	0.00385871	0.03467646	0.00019278
120	0.04812057	0.00343532	0.03466902	0.00018175
150	0.04819279	+0.00201363	0.03466343	—0.00011503
180	—0.04823174	0.00000000	+0.03466139	0.00000000

The periodic developments of these quantities result as follow:

$$L = +0.01567527 - 186 \cos M + 2167 \cos 2M - 90 \cos 3M - 23 \cos 4M + 10 \cos 5M - 2 \cos 6M$$

$$N = -0.00115622 \sin M + 2159 \sin 2M + 672 \sin 3M - 262 \sin 4M + 55 \sin 5M - 9 \sin 6M$$

$$P = +0.04317165 - 730 \cos M + 27 \cos 2M - 1 \cos 3M$$

$$S = +0.00002272 \sin M - 2351 \sin 2M + 765 \sin 3M - 412 \sin 4M + 97 \sin 5M - 11 \sin 6M$$

$$L' = -0.04817544 - 2436 \cos M - 7656 \cos 2M + 430 \cos 3M - 1 \cos 4M - 15 \cos 5M + 4 \cos 6M$$

$$N' = +0.00385831 \sin M - 10364 \sin 2M - 145 \sin 3M + 385 \sin 4M - 103 \sin 5M + 16 \sin 6M$$

$$P' = +0.03467617 + 1454 \cos M - 27 \cos 2M - 1 \cos 3M + 2 \cos 4M$$

$$S' = -0.00020065 \sin M + 866 \sin 2M - 759 \sin 3M - 25 \sin 4M + 29 \sin 5M - 7 \sin 6M$$

The coefficients of these expressions are given uniformly to 8 decimals. From these 8 equations we derive

$$\begin{aligned}
e \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \omega = &+ 0.01568298 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \chi &&+ 0.04374883 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \chi' \\
&- 0.00059066 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (2\chi - \chi') &&- 0.00001497 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (2\chi' - \chi) \\
&+ 0.00002545 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (3\chi - 2\chi') &&+ 0.00000808 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (3\chi' - 2\chi) \\
&+ 0.00000085 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (4\chi - 3\chi') &&- 0.00000263 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (4\chi' - 3\chi) \\
&- 0.00000094 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (5\chi - 4\chi') &&+ 0.00000183 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (5\chi' - 4\chi) \\
&+ 0.00000027 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (6\chi - 5\chi') &&- 0.00000045 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (6\chi' - 5\chi) \\
&- 0.00000004 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (7\chi - 6\chi') &&+ 0.00000005 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (7\chi' - 6\chi)
\end{aligned}$$

$$\begin{aligned}
e' \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \omega' = &- 0.04826849 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \chi &&+ 0.03273483 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \chi' \\
&+ 0.00192117 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (2\chi - \chi') &&+ 0.00012113 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (2\chi' - \chi) \\
&- 0.00009390 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (3\chi - 2\chi') &&- 0.00000159 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (3\chi' - 2\chi) \\
&+ 0.00000131 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (4\chi - 3\chi') &&+ 0.00000186 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (4\chi' - 3\chi) \\
&+ 0.00000206 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (5\chi - 4\chi') &&+ 0.00000057 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (5\chi' - 4\chi) \\
&- 0.00000062 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (6\chi - 5\chi') &&- 0.00000020 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (6\chi' - 5\chi) \\
&+ 0.00000010 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (7\chi - 6\chi') &&+ 0.00000003 \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (7\chi' - 6\chi)
\end{aligned}$$

By the substitution of the value of ψ which belongs to 1850 in the formulas for ω and ω', we ascertain that the arbitrary constants c and c' have the values

$$c = 340^\circ\ 15'\ 45''.2 \quad , \quad c' = 160^\circ\ 15'\ 46''.2.$$

These values should differ by exactly 180°; the small deviation is attributable to accumulated errors of calculation; we adopt $c = 340^\circ\ 15'\ 45''.7$. The expressions for the two arguments χ and χ' are then

$$\chi = 304^\circ\ 0'\ 43''.5 + 27''.73010\ (t - 1850),$$
$$\chi' = 28^\circ\ 7'\ 28''.2 + 4''.09951\ (t - 1850).$$

We see that, in spite of the extreme slowness of convergence exhibited by the coefficients of the original form of R, the final expressions we have arrived at for determining the elements appear tolerably convergent.

In his recent prize memoir (*Die Säcularen Veränderungen der Bahnen der Grossen Planeten*, p. 256), Mr. Paul Harzer has, instead of our annual motions of the arguments χ and χ', the values $31''.456547$ and $3''.918590$. The somewhat large difference between the values of the motion of χ is due, I think, to a cause which Mr. Harzer has noted in the *Nachtrag* to his memoir, as well as to his neglect of second order terms which arise from the argument twice the movement of Saturn minus that of Jupiter.*

The expressions we have obtained need to be completed by the addition of the terms arising from the presence of the six planets not considered in this investigation. Adapted from the formulas of Stockwell, they are

$$\begin{aligned}
e \tfrac{\sin}{\cos}\, \omega = & -0.0000061 \tfrac{\sin}{\cos}(87^\circ\ 18' + 5''.50543t) + 0.0000104 \tfrac{\sin}{\cos}(15^\circ\ 38' + 7''.32997t) \\
& -0.0000012 \tfrac{\sin}{\cos}(331^\circ\ 16' + 17''.37128t) + 0.0000006 \tfrac{\sin}{\cos}(136^\circ\ 5' + 18''.03675t) \\
& +0.0000645 \tfrac{\sin}{\cos}(71^\circ\ 51' + 0''.63867t) + 0.0019845 \tfrac{\sin}{\cos}(107^\circ\ 41'\ 40'' + 2''.70598t)\ , \\
e' \tfrac{\sin}{\cos}\, \omega' = & -0.0000053 \tfrac{\sin}{\cos}(87^\circ\ 18' + 5''.50543t) + 0.0000107 \tfrac{\sin}{\cos}(15^\circ\ 38' + 7''.32997t) \\
& -0.0000077 \tfrac{\sin}{\cos}(331^\circ\ 16' + 17''.37128t) + 0.0000076 \tfrac{\sin}{\cos}(136^\circ\ 5' + 18''.03675t) \\
& +0.0000725 \tfrac{\sin}{\cos}(71^\circ\ 51' + 0''.63867t) + 0.0018127 \tfrac{\sin}{\cos}(107^\circ\ 41'\ 40'' + 2''.70598t).
\end{aligned}$$

It should be mentioned that, in the preceding investigation, the effect of the terms of two dimensions as to planetary masses in the inter-action of Jupiter and Saturn on the expressions just written has been neglected. Also the influence of the terms of the third order with respect to eccentricities and inclinations has been neglected, except when they involved solely the eccentricities of Jupiter and Saturn. These effects are, however, probably small in comparison with those which are considered.

---

* On the latter point consult *Astron. Papers of the American Ephemeris*, Vol. IV, p. 251.

MEMOIR No. 63.

On Intermediary Orbits in the Lunar Theory.

(Astronomical Journal, Vol. XVIII, pp. 81–87, 1897.)

The difficulties investigators have met in the endeavor to push the approximation to the coordinates of a planet beyond the simple Keplerian theory, by reason that the time appeared as a multiplier outside of the trigonometrical functions, are at once removed if we consent to divide the potential function otherwise than is commonly done. This modification has given rise to what are known as intermediary orbits. But the way Gyldén and others have introduced them seems unnecessarily obscure, as they are often discussed without the slightest reference to the law of gravitation.

In the lunar theory much will be gained if we can make the perigee and node have a uniform motion already in the first approximation. Now the disturbing function of the solar action may be considered as involving a term which is rigorously proportional to the square of the radius and another proportional to the square of the moon's perpendicular distance from the plane of the ecliptic. The first will give rise to a movement of the perigee, while the second will impart a movement to the node.

We may then write the potential function belonging to our problem

$$\Omega = \frac{\mu}{r} + \tfrac{1}{2}\kappa\,(x^2 + y^2) + \tfrac{1}{2}\kappa'\,z^2$$

where μ denotes the sum of the masses of the earth and moon and κ, κ' are two constants to be so taken that the perigee and node obtain their actual motions. Then, in considering the further effects of solar perturbation, we subtract from the ordinarily given disturbing function the terms

$$\tfrac{1}{2}\kappa\,(x^2 + y^2) + \tfrac{1}{2}\kappa'\,z^2$$

The differential equations of our problem in terms of rectangular coordinates are

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} = \kappa x$$

$$\frac{d^2y}{dt^2} + \frac{\mu y}{r^3} = \kappa y$$

$$\frac{d^2z}{dt^2} + \frac{\mu z}{r^3} = \kappa' z$$

They admit the two integrals of conservation of areas in the plane xy and of the conservation of living forces; or, as they may be written,

$$\frac{xdy - ydx}{dt} = h$$

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = 2\,(\Omega + C)$$

Here the time is the independent variable, but the equations to be treated are, to a considerable degree, simplified if we adopt for this the true longitude of the moon. Let, as usual, u denote the reciprocal of the moon's curtate radius, s the tangent of the latitude and v the true longitude. Then, bearing in mind that here $\frac{\partial \Omega}{\partial v} = 0$, the differential equations, usually given for these variables and the independent variable*, take the form

$$\frac{d^2u}{dv^2} + u - \frac{1}{h^2}\frac{\partial \Omega}{\partial u} - \frac{s^2}{h^2 u}\frac{\partial \Omega}{\partial s} = 0$$

$$\frac{d^2s}{dv^2} + s - \frac{s}{h^2 u}\frac{\partial \Omega}{\partial u} - \frac{1+s^2}{h^2u^2}\frac{\partial \Omega}{\partial s} = 0$$

$$\frac{dt}{dv} = \frac{1}{hu^2}$$

But $$\Omega = \mu u\,(1 + s^2)^{-\frac{1}{2}} + \tfrac{1}{2}\,\varkappa u^{-2} + \tfrac{1}{2}\,\varkappa' u^{-2}\,s^2;$$

if we substitute this value in the first and second equations of the preceding group, and, for simplicity, put $\mu = kh^2$, $\varkappa = \alpha h^2$, $\varkappa - \varkappa' = \beta h^2$, where k, α, β are constants, we get

$$\frac{d^2u}{dv^2} + u + \alpha u^{-3} - k\,(1 + s^2)^{-\frac{3}{2}} = 0$$

$$\frac{d^2s}{dv^2} + s + \beta u^{-4}\,s = 0$$

We may get rid of the constant k by substituting ku for u and then writing α for αk^{-4} and β for βk^{-4}. Thus we have

$$\left.\begin{aligned} \frac{d^2u}{dv^2} + [1 + \alpha u^{-4}]\,u - (1 + s^2)^{-\frac{3}{2}} &= 0 \\ \frac{d^2s}{dv^2} + [1 + \beta u^{-4}]\,s &= 0 \end{aligned}\right\} \quad (1)$$

These equations admit the integral

$$\frac{du^2 + (uds - sdu)^2}{dv^2} = 2u\,(1 + s^2)^{-\frac{1}{2}} - u^2 + [\alpha + (\alpha - \beta)\,s^2]\,u^{-2} + C$$

Since the left member of this cannot be negative, if we construct the algebraic surface whose equation is

$$2u\,(1 + s^2)^{-\frac{1}{2}} - u^2 + [\alpha + (\alpha - \beta)\,s^2]\,u^{-2} + C = 0$$

*TISSERAND, *Mécanique Céleste*, Tom. I, p. 90.

the moon must move on that side of this surface for which the left member of this equation remains positive. But, for the value of the arbitrary constant C which obtains in the lunar theory, this surface will have a fold which, enveloping the origin, is closed. Moreover, for any given time, the moon will be found to be moving within this fold. This affords us an inferior limit to the value of u.

The latter of equations (1), which more especially determines s, is satisfied by the value $s = 0$. While s does not vanish in the actual lunar theory it is nevertheless quite small, and we can see that its expression has a constant factor, which if small, will render s small. Thus, in a first approximation, we can assume $s = 0$. The introduction of this value into the first of equations (1), or into the integral equation, shows that, in this approximation, v is determined as a function of u by the equation

$$\frac{dv}{du} = \frac{u}{\sqrt{a + Cu^2 + 2u^3 - u^4}} = \frac{u}{\sqrt{\Delta}} \tag{2}$$

involving elliptic integrals. This equation has been treated by Legendre* and Peirce†.

After the integration of this equation the resulting value of u can be substituted in the second equation of (1), and we shall have a linear differential equation of the second order with variable coefficients for determining s. This being integrated, we can substitute the resulting value of s in the first of (1), and thus a more approximate equation will be had for determining u. This alternate use of each of the equations (1) will, it is evident, give rise to an elaboration of the values of u and s of the following form: let $\tan i$ denote the small constant which factors s, i being in the neighborhood of what is known as the inclination of the lunar orbit, then

$$u = U_0 + U_1 \tan^2 i + U_2 \tan^4 i + \ldots$$
$$s = S_0 \tan i + S_1 \tan^3 i + S_2 \tan^5 i + \ldots$$

where the U and S will be independent of the arbitrary constant $\tan i$, save that, involving two arguments, ξ, which may be called the moon's anomaly, and η, which may be called the argument of latitude, these arguments are of the form $\xi = cv + \text{const.}$, $\eta = gv + \text{const.}$, c and g are constants which are developed in the series

$$c = c_0 + c_1 \tan^2 i + c_2 \tan^4 i + \ldots$$
$$g = g_0 + g_1 \tan^2 i + g_2 \tan^4 i + \ldots$$

where $c_0, c_1, \ldots\; g_0, g_1 \ldots$ are independent of $\tan i$.

---

* *Traité des Fonctions Elliptiques*, Tom. I, p. 561.

† *Analytic Mechanics*, pp. 394, 396.

As to the constants α and C which appear in Eq. (2), it is plain we are not obliged to retain them, but may substitute two others for them. Let us adopt a and e, such that

$$\alpha = (1 - e^2)^2 (1 - a) a^3, \quad C = -(3 + e^2) a + 2(1 + e^2) a^2,$$

where a is a little less than unity, and e is small. Then the factors of Δ are thus shown:

$$\Delta = [a^2 e^2 - (u - a)^2][u^2 - 2(1 - a) u - a(1 - a)(1 - e^2)]$$

Eq. (2) can be somewhat simplified by adopting a new variable w such that $u = a\mathrm{w}$, and putting $\gamma = \dfrac{1 - a}{a}$ which gives

$$\frac{dv}{d\mathrm{w}} = \frac{\mathrm{w}}{\sqrt{e^2 - (\mathrm{w} - 1)^2}\sqrt{\mathrm{w}^2 - 2\gamma \mathrm{w} - \gamma(1 - e^2)}}$$

The adoption of a new variable E, such that

$$\mathrm{w} = (1 + e)\cos^2 \tfrac{1}{2}E + (1 - e)\sin^2 \tfrac{1}{2}E = 1 + e\cos E,$$

produces the equation

$$\frac{dv}{dE} = \frac{\mathrm{w}}{\sqrt{\mathrm{w}^2 - 2\gamma \mathrm{w} - \gamma(1 - e^2)}}$$

where, as v and E will be supposed to increase together, the radical should always receive the positive sign. In order to avoid the appearance of radicals in the expressions which follow, we adopt a new constant to replace γ such that

$$\gamma = \frac{4\delta^2}{(1 - \delta^2)^2}(1 - e^2)$$

Here e and δ are small positive quantities which may be regarded as of the first order. Then the last equation can be written

$$\frac{dv}{dE} = \frac{\mathrm{w}}{\sqrt{\mathrm{w} + \dfrac{2\delta}{(1 + \delta)^2}(1 - e^2)}\sqrt{\mathrm{w} - \dfrac{2\delta}{(1 - \delta)^2}(1 - e^2)}}$$

If we adopt three new constants, g, h, H, such that they are determined by the equations

$$\frac{2e}{1 - g} = 1 + e + \frac{2\delta}{(1 + \delta)^2}(1 - e^2)$$

$$\frac{2e}{1 - h} = 1 + e - \frac{2\delta}{(1 - \delta)^2}(1 - e^2)$$

$$H = \frac{1 + e}{2e}\sqrt{(1 - g)(1 - h)}$$

the last equation can be given the form

$$\frac{dv}{dE} = H\,\frac{1 + \dfrac{1 - e}{1 + e}\tan^2 \tfrac{1}{2}E}{\sqrt{1 + g\tan^2 \tfrac{1}{2}E}\sqrt{1 + h\tan^2 \tfrac{1}{2}E}}$$

This equation can be further transformed by adopting a new variable ψ to supersede E, such that

$$\tan \tfrac{1}{2}E = \frac{1}{\sqrt{g}} \tan \psi$$

And, employing the four constants k, n, f, K determined by the equations

$$k^2 = 1 - \frac{h}{g} = \frac{8\delta(1+\delta^2)e}{(1+2e\delta+\delta^2)^2 - 16\delta^2}, \qquad n = \frac{1}{g} - 1$$

$$\frac{f}{1-f} = \frac{2\delta}{(1+\delta)^2}(1-e), \qquad K = \frac{1}{\sqrt{g}}\sqrt{\frac{1-h}{1-g}}$$

the differential equation takes the form

$$dv = 2K\left[\frac{1}{1+n\sin^2\psi} - f\right]\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

It will be perceived that the constants k, n, f, are positive quantities of the first order of smallness, while K differs from unity by a quantity of the same order.

The form to which the equation is now reduced is that of Legendre, and it shows that the integral involves the first and third species of elliptic integrals. But we may here introduce the Θ function of Jacobi. If we put

$$\psi = am\,(x)$$

the differential equation takes the form

$$dv = 2K\left[\frac{1}{1+n\,sn^2x} - f\right]dx$$

Deriving the constant argument a from the equation, $(i^2 = -1)$,

$$sn^2(ia) = -\frac{n}{k^2} = \frac{g-1}{g-h}$$

we have, in terms of the Θ functions,*

$$\int_0^x \frac{dx}{1-k^2\,sn^2(ia)\,sn^2x} = \frac{1}{\dfrac{d.\log sn\,(ia)}{d\,(ia)}}\left[\frac{\theta'(ia)}{\theta(ia)}x + \tfrac{1}{2}\log\frac{\theta(x-ia)}{\theta(x+ia)}\right] + x$$

But
$$\frac{d.\log sn\,(ia)}{d\,(ia)} = \frac{cn\,(ia)}{sn\,(ia)}dn\,(ia) = \frac{1}{\sqrt{g}}\sqrt{\frac{1-h}{g-1}} = \frac{K}{i}$$

Thus
$$\int_0^x \frac{2K dx}{1+n\,sn^2x} = 2i\left[\frac{\theta'(ia)}{\theta(ia)}x + \tfrac{1}{2}\log\frac{\theta(x-ia)}{\theta(x+ia)}\right] + 2Kx$$

and, putting ν for $2K(1-f)$,

$$v + \text{const.} = \left[2\frac{d\log\theta(ia)}{da} + \nu\right]x + \frac{1}{i}\log\frac{\theta(x+ia)}{\theta(x-ia)}$$

---

* BERTRAND, *Calcul Intégral*, p. 653.

The well-known periodic development of $\Theta(x)$ is

$$\theta(x) = 1 - 2q\cos\frac{\pi x}{K} + 2q^4\cos\frac{2\pi x}{K} - 2q^9\cos\frac{3\pi x}{K} + \dots$$

For most of the applications q is so small that not more than two or three terms of this series need be considered. The constants involved have the significations

$$K = \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - k^2\sin^2\beta}}, \quad K' = \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - k'^2\sin^2\beta}}, \quad q = \epsilon^{-\pi\frac{K'}{K}}$$

As the Θ functions in our problem have complex arguments, it is necessary to separate the real from the imaginary portions. This has been done by Jacobi*, and putting $b = \frac{a}{K'}$, we have

$$\frac{d\,.\log\theta(ia)}{da} = -\frac{\pi}{K}\left\{\frac{q^{1-b}}{1-q^{1-b}} - \frac{q^{1+b}}{1-q^{1+b}} + \frac{q^{3-b}}{1-q^{3-b}} - \frac{q^{3+b}}{1-q^{3+b}} + \dots\right\}$$

And if we make

$$m = 2\frac{d\log\theta(ia)}{da} + \nu$$

we have

$$v + \text{const.} = mx + 2\arctan\frac{q^{1-b}\sin\frac{\pi x}{K}}{1 - q^{1-b}\cos\frac{\pi x}{K}} + 2\arctan\frac{q^{3-b}\sin\frac{\pi x}{K}}{1 - q^{3-b}\cos\frac{\pi x}{K}} + \dots$$

$$- 2\arctan\frac{q^{1+b}\sin\frac{\pi x}{K}}{1 - q^{1+b}\cos\frac{\pi x}{K}} - 2\arctan\frac{q^{3+b}\sin\frac{\pi x}{K}}{1 - q^{3+b}\cos\frac{\pi x}{K}} - \dots$$

This equation gives the value of v in terms of x; by the inversion of the series x can be obtained in terms of v; from which can be found $cn\,x$, and thence the value of u from the equation

$$u = a\text{w} = a\frac{1 - e + [g - 1 + (g+1)e]\,cn^2x}{1 + (g-1)\,cn^2x}$$

The employment of elliptic functions in the treatment of the problem does not appear to be an improvement, since, depending on two arguments, they can not be tabulated to any great extent, and thus interpolation between the given values is necessarily difficult. In addition to this these functions are not so easily multiplied together as circular functions. Under these circumstances it seems the employment of periodic series throughout will have the greater advantage.

---

* *Journal für die Mathematik*, Vol. XXXIX, p. 345.

I will now show by a numerical example how much easier is the treatment by computing special values of the functions involved and thence deducing their periodic developments. It is true that this procedure demands that we know at the outset the values of the two independent constants involved. But this limitation need not greatly trouble us.

Numerical Illustration.

Let us assume the values $e = 0.055$ and $\delta = 0.1$. This value of e is about that which obtains in the actual lunar theory. The value of δ considerably exceeds that which would give the observed motion of the lunar perigee; but, by its employment, we shall better exemplify the advantages of the numerical method.

We compute to the argument E for every 30° in the semi-circumference the values of w and $\frac{dv}{dE}$ from the formulas

$$\text{w} = 1 + e\cos E, \quad \frac{dv}{dE} = \frac{\text{w}}{\sqrt{\text{w}^2 - 2\gamma \text{w} - \gamma(1-e^2)}}$$

The computation of the latter equation will be facilitated by using the quantities θ and x derived from

$$\sin^2\theta = \frac{\gamma}{\text{w}}\left(2 + \frac{1-e^2}{\text{w}}\right), \quad x = \frac{\sin^2\theta}{2\cos^2\frac{1}{2}\theta\cos\theta}$$

when $\frac{dv}{dE} = 1 + x$. The following are the values arrived at:

E	w	$\frac{dv}{dE}$
0°	1.05500 00000	1.06213 69983
30	1.04763 13972	1.06277 08524
60	1.02750 00000	1.06456 66064
90	1.00000 00000	1.06718 23976
120	0.97250 00000	1.07000 58547
150	0.95236 86028	1.07221 86201
180	0.94500 00000	1.07306 18242

From the values of $\frac{dv}{dE}$ we obtain the periodic development

$$\frac{dv}{dE} = \left\{\begin{array}{l} 1.06739\,06237 \\ -0.00545\,46813\cos E \\ +0.00020\,85064\cos 2E \\ -0.00000\,77215\cos 3E \\ +0.00000\,02807\cos 4E \\ -0.00000\,00101\cos 5E \\ +0.00000\,00004\cos 6E \end{array}\right\}$$

By integration of this and putting

$$\xi = 0.93686\,41412\,v + \text{const.},$$

we get

$$\xi = E + \left\{\begin{array}{rr} - & 1054''07407 \sin E \\ + & 20.14611 \sin 2E \\ - & 0.49737 \sin 3E \\ + & 0.01356 \sin 4E \\ - & 0.00039 \sin 5E \\ + & 0.00001 \sin 6E \end{array}\right\}$$

By a very easy tentative process we ascertain the values of the excess of the angle E over ξ to the argument ξ for every 30° of the semi-circumference. These, with the correspondent values of w follow:

ξ	$E - \xi$	w
0°	0''00000	1.05500 00000
30	+512.29165	1.04756 29497
60	897.78659	1.02729 24201
90	1053.76895	0.99971 90163
120	928.00622	0.97228 59807
150	+542.53732	0.95229 64346
180	0.00000	0.94500 00000

From these special values we derive the periodic series

$$E = \xi + \left\{\begin{array}{rr} & 1054''12213 \sin \xi \\ - & 17.45483 \sin 2\xi \\ + & 0.35334 \sin 3\xi \\ - & 0.00704 \sin 4\xi \\ + & 0.00016 \sin 5\xi \end{array}\right\}, \quad \mathrm{w} = \left\{\begin{array}{l} \,0.99985\,94669 \\ + 0.05500\,21473 \cos \xi \\ + 0.00014\,04918 \cos 2\xi \\ - 0.00000\,21465 \cos 3\xi \\ + 0.00000\,00413 \cos 4\xi \\ - 0.00000\,00008 \cos 5\xi \end{array}\right\}$$

The latter expression conjoined with the equation giving ξ in terms of v may be regarded as the integral of the differential equation between w and v. It will be perceived how rapidly convergent the series is. There is here certainly no need for the introduction of elliptic functions.

Turning now our attention to the treatment of the differential equation for s, let us assume that, in the equation $\frac{d^2s}{dv^2} + \left[1 + \frac{\beta}{\mathrm{w}^4}\right] s = 0$, β has the value 0.01, nearly that which prevails in the actual lunar theory. We compute then the following special values of $\frac{\beta}{\mathrm{w}^4}$:

ξ	$\frac{\beta}{\mathrm{w}^4}$	ξ	$\frac{\beta}{\mathrm{w}^4}$
0°	0.00807 21674	120°	0.01118 98407
30	0.00830 38497	150	0.01215 93782
60	0.00897 89110	180	0.01253 92848
90	0.01001 12472		

From these special values we deduce

$$1+\frac{\beta}{\mathrm{w}^4}=\left\{\begin{array}{l} \quad 1.01015\,81588 \\ -0.00222\,60030 \cos \xi \\ +0.00014\,72390 \cos 2\xi \\ -0.00000\,92096 \cos 3\xi \\ +0.00000\,03278 \cos 4\xi \\ -0.00000\,00126 \cos 5\xi \\ +0.00000\,00004 \cos 6\xi \end{array}\right\}$$

Changing the independent variable from v to ξ we have the equation:

$$\frac{d^2 s}{d\xi^2}+\left\{\begin{array}{l} \quad 1.15089\,61649 \\ -0.00253\,61359 \cos \xi \\ +0.00016\,77528 \cos 2\xi \\ -0.00001\,04927 \cos 3\xi \\ +0.00000\,03735 \cos 4\xi \\ -0.00000\,00144 \cos 5\xi \\ +0.00000\,00005 \cos 6\xi \end{array}\right\} s=0$$

To integrate this we put $\zeta = \varepsilon^{\xi\sqrt{-1}}$ and assume that $s = \Sigma_i b_i \zeta^{g+i}$, i taking all integral values positive and negative. If $D = \frac{d}{d\xi}\sqrt{-1}$ the differential equation may be written $D^2 s = \Sigma_i \Theta_i \zeta^i . s$, where $\Theta_{-i} = \Theta_i$. Θ_0 is equal then to the first coefficient in the last equation written at length, but the following Θ's are equivalent to the halves of the succeeding coefficients taken in order. If we use the symbol $[i]$ to denote $(g+i)^2 - \Theta_0$, the constant coefficients b_j satisfy the equation

$$[g+j]^2 b_j - \Sigma_i \theta_{j-i} b_i = 0$$

which holds for all integral values of j positive and negative. An abbreviated specimen of the equations the b_j satisfy is shown as follows:

$$\begin{array}{l} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ldots + [-1]\, b_{-1} - \theta_1 b_0 - \theta_2 b_1 - \ldots = 0 \\ \ldots - \theta_1 b_{-1} + [0]\, b_0 - \theta_1 b_1 - \ldots = 0 \\ \ldots - \theta_2 b_{-1} - \theta_1 b_0 + [1]\, b_1 - \ldots = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

All the b's but one must be eliminated from this group of equations and the result is an equation which determines g. I have given this equation previously* to a degree of approximation which more than suffices for our present purpose. Employing it, we find that $g = 1.07279\,78607$; that is, calling the principal argument of the latitude η, we have

$$\eta = g\xi + \text{const.} = 1.00506\,58465\, v + \text{const.}$$

* *Annals of Mathematics*, Vol. IX, p. 36.

We can now solve the group of equations determining the b's in terms of one of them, preferably b_0. The solution gives us

$$s = b_0 \Sigma_i \frac{b_i}{b_0} \epsilon^{(g+i)\xi\sqrt{-1}} + b_0' \Sigma_i \frac{b_i}{b_0} \epsilon^{-(g+i)\xi\sqrt{-1}}$$

where b_0 and b_0' are the arbitrary constants introduced by the integration. But, in place of b_0 and b_0', we substitute the arbitrary constants i and λ, such that

$$b_0 = \tfrac{1}{2} \tan i \,.\, \epsilon^{\lambda\sqrt{-1}}, \quad b_0' = -\tfrac{1}{2} \tan i \,.\, \epsilon^{-\lambda\sqrt{-1}}$$

Then, η being equivalent to $g\xi + \lambda$,

$$s = \tan i \; \Sigma_i \frac{b_i}{b_0} \sin (\eta + i\xi)$$

We solve the group of equations giving the values of the $\frac{b_i}{b_0}$ by using the method of gradual approximations through recursion. Only one repetition of this is necessary to get the values of the $\frac{b_i}{b_0}$ correct to the tenth decimal. The value of s arrived at is:

$$s = \tan i \left\{ \begin{array}{ll} -0.00000\,00004 \sin (\eta - 5\xi) & -0.00040\,30991 \sin (\eta + \xi) \\ +0.00000\,00215 \sin (\eta - 4\xi) & +0.00001\,01773 \sin (\eta + 2\xi) \\ -0.00000\,18705 \sin (\eta - 3\xi) & -0.00000\,03429 \sin (\eta + 3\xi) \\ -0.00028\,32410 \sin (\eta - 2\xi) & +0.00000\,00077 \sin (\eta + 4\xi) \\ +0.00110\,66224 \sin (\eta - \xi) & -0.00000\,00002 \sin (\eta + 5\xi) \\ +1.00000\,00000 \sin \eta & \end{array} \right\}$$

That this series, infinite in two directions, is convergent scarcely needs demonstration.

The next step in the treatment of the problem is the determination of the part of w factored by $\tan^2 i$. The rigorous equation for w is

$$\frac{d^2 \text{w}}{dv^2} + \left[1 + \gamma \frac{(1-e^2)^2}{\text{w}^4}\right] \text{w} - (1+\gamma)(1+s^2)^{-\frac{3}{2}} = 0$$

or, neglecting the fourth and higher powers of s,

$$\frac{d^2 \text{w}}{dv^2} + \left[1 + \gamma \frac{(1-e^2)^2}{\text{w}^4}\right] \text{w} - (1+\gamma) + \tfrac{3}{2}(1+\gamma) s^2 = 0$$

Making ξ the independent variable and putting

$$W = \frac{[e^2 - (\text{w}-1)^2][\text{w}^2 - 2\gamma \text{w} - \gamma(1-e^2)]}{2c^2 \text{w}^2}, \quad X = -\tfrac{3}{2} \frac{1+\gamma}{c^2} \int s^2 \frac{d\text{w}}{d\xi} d\xi$$

the integral of this equation takes the form

$$\tfrac{1}{2} \frac{d\text{w}^2}{d\xi^2} = W + X$$

If w must receive the increment δw on account of X hitherto neglected, we must have

$$\frac{d\mathrm{w}}{d\xi}\frac{d\,.\,\delta\mathrm{w}}{d\xi}=\frac{dW}{d\mathrm{w}}\delta\mathrm{w}+X$$

But, by the theory of the variation of arbitrary constants, δw may be supposed to result from the application of a correction to the arbitrary constant which completes the value of the argument ξ. This correction can as well be supposed to be applied to ξ. Calling it $\delta\xi$, we have $\delta\mathrm{w}=\frac{d\mathrm{w}}{d\xi}\,\delta\xi$, and, making this substitution in the last equation, we have

$$\frac{d\mathrm{w}^2}{d\xi^2}\frac{d\,.\,\delta\xi}{d\xi}+\frac{d\mathrm{w}}{d\xi}\frac{d^2\mathrm{w}}{d\xi^2}\delta\xi=\frac{dW}{d\xi}\delta\xi+X$$

Cancelling the terms in both members of this which are equivalent, we have

$$\frac{d\mathrm{w}^2}{d\xi^2}\frac{d\,.\,\delta\xi}{d\xi}=X,\quad \delta\xi=\int X\left(\frac{d\mathrm{w}}{d\xi}\right)^{-2}d\xi$$

The last gives

$$\delta\mathrm{w}=\frac{d\mathrm{w}}{d\xi}\int X\left(\frac{d\mathrm{w}}{d\xi}\right)^{-2}d\xi=-\tfrac{3}{2}\,\frac{1+\gamma}{c^2}\frac{d\mathrm{w}}{d\xi}\int\left(\frac{d\mathrm{w}}{d\xi}\right)^{-2}\int s^2\,\frac{d\mathrm{w}}{d\xi}\,d\xi^2$$

As the factor $\frac{d\mathrm{w}}{d\xi}$ enters into this expression an equivalent number of times when we consider both sides of the integral signs, we can reject from it a factor which makes it small. However, this factor must be restored when we derive from the expression for $\delta\xi$ the part of it proportional to ξ, that is, the part of the motion of the perigee proportional to $\tan^2 i$.

From the preceding value of s we derive

$$-\tfrac{3}{2}\,\frac{1+\gamma}{c^2}\,s^2=\tan^2 i\left\{\begin{array}{ll}
-\,0.88926\,1535 & +\,0.00000\,0002\cos(2\eta-5\xi)\\
-\,0.00125\,0668\cos\xi & -\,0.00000\,0546\cos(2\eta-4\xi)\\
+\,0.00048\,6447\cos 2\xi & -\,0.00000\,3882\cos(2\eta-3\xi)\\
+\,0.00000\,3714\cos 3\xi & -\,0.00050\,2660\cos(2\eta-2\xi)\\
-\,0.00000\,0048\cos 4\xi & +\,0.00196\,8352\cos(2\eta-\xi)\\
 & +\,0.88925\,9439\cos 2\eta\\
 & -\,0.00071\,6900\cos(2\eta+\xi)\\
 & +\,0.00017\,8311\cos(2\eta+2\xi)\\
 & -\,0.00000\,0617\cos(2\eta+3\xi)\\
 & +\,0.00000\,0014\cos(2\eta+4\xi)
\end{array}\right\}$$

After rejecting from $\frac{d\mathrm{w}}{d\xi}$ the constant factor $-0.05500\,21473$ it becomes:

$$k\frac{d\mathrm{w}}{d\xi}=\left\{\begin{array}{r}
\sin\xi\\
+\,0.00510\,8593\sin 2\xi\\
-\,0.00011\,7076\sin 3\xi\\
+\,0.00000\,3000\sin 4\xi\\
-\,0.00000\,0077\sin 5\xi\\
+\,0.00000\,0002\sin 6\xi
\end{array}\right\}=\sin\xi\left\{\begin{array}{l}
0.99988\,2847\\
+\,0.01022\,3190\cos\xi\\
-\,0.00023\,4306\cos 2\xi\\
+\,0.00000\,6004\cos 3\xi\\
-\,0.00000\,0154\cos 4\xi\\
+\,0.00000\,0004\cos 5\xi
\end{array}\right\}$$

If, in the periodic development of s^2, we limit ourselves to the portion independent of the argument η, we obtain, after multiplication and integration, the expression

$$-\tfrac{3}{2}\frac{k}{c^2}(1+\gamma)\int s^2\frac{dw}{d\xi}\,d\xi = \tan^2 i\left\{\begin{array}{l} \ 0.88950\ 7990 \cos \xi \\ +\ 0.00258\ 4996 \cos 2\xi \\ -\ 0.00011\ 4720 \cos 3\xi \\ -\ 0.00000\ 0126 \cos 4\xi \\ -\ 0.00000\ 0005 \cos 5\xi \end{array}\right\}$$

This being multiplied by the factor $k^{-2}\left(\dfrac{dw}{d\xi}\right)^{-2}$ there results

$$-\tfrac{3}{2}\frac{1+\gamma}{kc^2}\left(\frac{dw}{d\xi}\right)^{-2}\int s^2\frac{dw}{d\xi}\,d\xi = \frac{\tan^2 i}{\sin^2 \xi}\left\{\begin{array}{l} -\ 0.00910\ 0625 \\ +\ 0.89010\ 8010 \cos \xi \\ -\ 0.00652\ 3274 \cos 2\xi \\ +\ 0.00013\ 7485 \cos 3\xi \\ -\ 0.00000\ 7173 \cos 4\xi \\ +\ 0.00000\ 0227 \cos 5\xi \\ -\ 0.00000\ 0007 \cos 6\xi \end{array}\right\}$$

For the integration which comes next in order let us put

$$\int[\gamma_0+\gamma_1\cos\xi+\gamma_2\cos 2\xi+\ldots]\frac{d\xi}{\sin^2\xi}=[\varepsilon_0+\varepsilon_1\cos\xi+\varepsilon_2\cos 2\xi+\ldots]\frac{1}{\sin\xi}+h\xi$$

By differentiating this equation and comparing the coefficients it will be found that

$$\begin{aligned} h &= -2\gamma_2 - 4\gamma_4 - 6\gamma_6 - \ldots \\ \varepsilon_0 &= -\gamma_1 - 3\gamma_3 - 5\gamma_5 - \ldots \\ \varepsilon_1 &= -\gamma_0 - \gamma_2 - 2\gamma_4 - 3\gamma_6 - \ldots \\ \varepsilon_2 &= \tfrac{6}{3}\gamma_3 + \tfrac{10}{3}\gamma_5 + \ldots \\ \varepsilon_3 &= \gamma_4 + \tfrac{3}{2}\gamma_6 + \ldots \\ \varepsilon_4 &= \tfrac{2}{3}\gamma_5 + \tfrac{14}{15}\gamma_7 + \ldots \\ \varepsilon_5 &= \tfrac{1}{2}\gamma_6 + \ldots \end{aligned}$$

Applying these formulas it is found in the first place that

$$h = +\ 0.01307\ 5285 \tan^2 i.$$

If this is multiplied by k, we shall have the portion of the motion of the perigee factored by $\tan^2 i$, which is

$$\delta c = -\ 0.23772317 \tan^2 i$$

The residual portion of the integral is

$$\frac{\tan^2 i}{\sin \xi}\left\{\begin{array}{l} -\ 0.89052\ 1597 \\ +\ 0.01563\ 8267 \cos \xi \\ +\ 0.00027\ 5726 \cos 2\xi \\ -\ 0.00000\ 7184 \cos 3\xi \\ +\ 0.00000\ 0151 \cos 4\xi \\ -\ 0.00000\ 0004 \cos 5\xi \end{array}\right\}$$

If this is multiplied by $k\dfrac{d\text{w}}{d\xi}$ we obtain

$$\delta\text{w} = \tan^2 i \left\{ \begin{array}{l} -0.89033\ 736 \\ +0.00653\ 204 \cos \xi \\ +0.00056\ 429 \cos 2\xi \\ -0.00001\ 295 \cos 3\xi \\ +0.00000\ 027 \cos 4\xi \end{array} \right\}$$

For the portion of δw which involves the argument η it is necessary to employ the linear differential equation of the second order

$$\frac{d^2\delta\text{w}}{d\xi^2} + \frac{1}{c^2}\left[1 - \frac{3\gamma(1-c^2)^2}{\text{w}^4}\right]\delta\text{w} + \tfrac{3}{2}\frac{1+\gamma}{c^2}s^2 = 0$$

In this we have

$$\frac{1}{c^2}\left[1 - \frac{3\gamma(1-e^2)^2}{\text{w}^4}\right] = \left\{ \begin{array}{l} \ 0.99890\ 38887 \\ +0.03077\ 06154 \cos \xi \\ -0.00203\ 53235 \cos 2\xi \\ +0.00012\ 73071 \cos 3\xi \\ -0.00000\ 45320 \cos 4\xi \\ +0.00000\ 01744 \cos 5\xi \\ -0.00000\ 00061 \cos 6\xi \end{array} \right\}$$

If we put

$$\delta\text{w} = \Sigma k_i \cos(2\eta + i\xi), \quad \frac{1}{c^2}\left[1 - \frac{3\gamma(1-e^2)^2}{\text{w}^4}\right] = \theta_0 + 2\theta_1 \cos\xi + 2\theta_2 \cos 2\xi + \ldots$$

$$\tfrac{3}{2}\frac{1+\gamma}{c^2}s^2 = \Sigma H_i \cos(2\eta + i\xi), \quad [i] = (2g+i)^2 - \theta_0$$

we have for finding the coefficients k_i the following group of linear equations:

$$\begin{array}{l} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots + [-1]\,k_{-1} - \theta_1 k_0 - \theta_2 k_1 - \ldots = H_{-1} \\ \ldots - \theta_1 k_{-1} + [0]\,k_0 - \theta_1 k_1 - \ldots = H_0 \\ \ldots - \theta_2 k_{-1} - \theta_1 k_0 + [1]\,k_1 - \ldots = H_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

These equations being solved we obtain

$$\delta\text{w} = \tan^2 i \left\{ \begin{array}{l} -0.00000\ 007 \cos(2\eta - 6\xi) \\ +0.00000\ 227 \cos(2\eta - 5\xi) \\ -0.00010\ 285 \cos(2\eta - 4\xi) \\ -0.01836\ 848 \cos(2\eta - 3\xi) \\ +0.00407\ 911 \cos(2\eta - 2\xi) \\ -0.29855\ 849 \cos(2\eta - \xi) \\ -0.29386\ 799 \cos 2\eta \\ -0.00043\ 539 \cos(2\eta + \xi) \\ +0.00000\ 630 \cos(2\eta + 2\xi) \\ -0.00000\ 070 \cos(2\eta + 3\xi) \\ +0.00000\ 002 \cos(2\eta + 4\xi) \end{array} \right\}$$

It will be perceived that this infinite series is amply convergent for practical purposes. The presence of the small divisors $[-3] = -0.019\ldots$ and $[-1] = +0.021\ldots$ has the effect of making the coefficients of the arguments $2\eta - 3\xi$ and $2\eta - \xi$ abnormally large in the series of coefficients, but the ratio of decrement in the coefficients is scarcely influenced by this circumstance.

The next step in the elaboration of the subject would be the determination of the change δs in s which is caused by taking into account the portion of w factored by $\tan^2 i$. Here we should have to deal with the linear differential equation

$$\frac{d^2\,\delta s}{dv^2} + \left[1 + \frac{\beta}{\mathrm{w}^4}\right]\delta s = \frac{4\beta}{\mathrm{w}^5}\, s\delta \mathrm{w}$$

But enough has been done to show the practical advantage of retaining periodic series involving circular functions.

MEMOIR No. 64.

Note on the Mass of Mercury.

(Astronomical Journal, Vol. XIX, pp. 157–158, 167, 1898.)

Desiring to make some investigations on the secular perturbations of the solar system, it was necessary to choose values for the masses of the planets. Mercury gives the most difficulty in this respect, the values attributed to its mass being so discrepant. They have been derived from its action on Venus or on Encke's comet. To weight them and so take a mean is a method of treatment which does not recommend itself. Under these circumstances it has seemed to me that a value of the mass deduced solely from analogical considerations might be acceptable.

If there is any truth in the nebular hypothesis, the materials forming the masses of the four planets Mercury, Venus, the Earth, Mars and the Earth's moon ought to have approximately the same chemical constitution. Now the matter composing the outside layer of the Earth has a density about 2.55 (that of water being unity) while the mean density is 5.67. We hence infer that the density varies in the interior through the effect of pressure, and the phenomena connected with the rotation of the Earth are very well satisfied by what is known as Legendre's law of density $\rho = k\frac{\sin mr}{r}$; where ρ denotes the density and r the distance of the matter considered from the center of the Earth, while k and m are constants, the latter showing the degree of compressibility. Without making a violent hypothesis this law may be extended from the Earth to the other four bodies under consideration, m being supposed the same for all, while k may be different for each. Knowing the masses of all the bodies but Mercury, we may derive the value of k for each of four different cases. Then we may attribute to Mercury a superficial density equal in succession to that of each of the four remaining bodies, and thus arrive at four different values for the mass of the former; and their agreement or disagreement will afford a criterion for the measure of success of the treatment.

I have compiled the following table of necessary data:

	Semi-diameter at distance 1	Reciprocal of mass	Log a	Log $\frac{R'}{R}$
Mercury	3″.34	......	9.58026	0.03020
Venus	8.546	408000	9.98827	0.31283
Earth	8.78	333470	0.00000	0.34632
Moon	2.393	27178000	9.43551	0.01636
Mars	5.051	3093500	9.76013	0.07321

The values of the semi-diameters are those which are employed in the *American Ephemeris.* By consulting Houzeau's *Vade-mecum de l'Astronome* it will be seen that astronomers are not well agreed upon these constants even in quite recent times. The value of the equatorial horizontal solar parallax has been diminished a little to make it correspond to the Earth's radius in latitude 35°. Taking this radius as the unit, the column headed "Log a" gives the logarithm of the radius a of each planet. Let R denote the density at the surface and R' the mean density. Then Legendre's law of density gives the equation

$$\frac{R'}{R} = \frac{3}{ma}\left[\frac{1}{ma} - \cot ma\right]$$

which, in the case of the Earth, becomes

$$\frac{1}{m}\left[\frac{1}{m} - \cot m\right] = \frac{5.67}{3 \times 2.55}$$

and from which $m = 2.5518 = 146° \, 12' \, 20''$. Having now the value of m as well as those of the several a, we compute the values of $\frac{R'}{R}$ for each planet, and their logarithms are inscribed in the last column of the table.

The details of the computation of the mass of Mercury severally from the four other bodies is thus shown:

	Venus	Earth	Moon	Mars
Log reciprocal of mass	5.61066	5.52306	7.43422	6.49045
Log $\left(\frac{a'}{a}\right)^3$	1.22403	1.25922	9.56575	0.53961
Log $\frac{R'}{R}$ of planet — Log $\frac{R'}{R}$ of Mercury	0.28263	0.31612	9.98616	0.04301
Log reciprocal of mass of Mercury	7.11732	7.09840	6.98613	7.07307
Mass of Mercury	$\frac{1}{13\,101\,500}$	$\frac{1}{12\,543\,000}$	$\frac{1}{9\,685\,750}$	$\frac{1}{11\,832\,400}$

The three planets give results quite accordant, but the one from the Moon is somewhat larger. Taking the mean of the four values we have for the mass of Mercury $\frac{1}{11\,834\,200}$.

Additional Note on the Mass of Mercury.

Since the appearance of the Note on the Mass of Mercury (*A.J.* No. 452) it has occurred to me that the cause of the Moon giving a larger mass for Mercury than the three other bodies treated is the neglect of the augmentation of density through the loss of interior heat. The smaller the dimensions of the body considered, the greater will be this augmentation, provided other conditions remain the same. Now the Moon is the smallest of the four bodies treated in the previous note. In the lack of any information as to the original temperature and the length of time the cooling has been going on, it seems the best we can do is to suppose that the effect on the general density of the body is proportional to some power of the radius. This power can be determined from the data given in the previous note, by adopting the principle that that value is to be used which will bring the results from the four bodies into closest agreement. Thus, employing the method of least squares, we find that the exponent 3 of the ratio $\frac{a'}{a}$ used in the former note, ought to be reduced to 2.92849. This is not an excessive correction to make for the effect of cooling. Substituting the latter value of the exponent for the former, the values of the mass of Mercury, severally from the four bodies treated, are

Venus	Earth	Moon	Mars
$\frac{1}{10\,710\,000}$	$\frac{1}{10\,194\,200}$	$\frac{1}{10\,403\,600}$	$\frac{1}{10\,826\,200}$

These values are in much better agreement than the former, and the mean gives for the mass of Mercury $\frac{1}{10\,530\,500}$.

MEMOIR No. 65.

On the Inequalities in the Lunar Theory Strictly Proportional to the Solar Eccentricity.

(Astronomical Journal, Vol. XX, pp. 115–124, 1899.)

This article is in continuation of that in *A. J.* No. 353, to which the reader is referred for the explanation of many of the symbols here used. I have there elaborated numerically the special case of the theory in which we have $e = 0$, $\gamma = 0$, $e' = 0$, the symbols having the signification Delaunay attributes to them.

In the further elaboration of the problem the order of the introduction of complexity into it is not indifferent. For it is plain that if, as the next step, we treat the case in which e' is to have a determinate value, but where still $e = 0$, $\gamma = 0$, we shall not be troubled with considering the motions of the perigee and node. Also the subject will be more easily handled if we do not attempt to take account at once of terms of all dimensions with respect to e', but in succession of those of one, two, three, etc., dimensions. Probably it will not be necessary to go beyond the fourth dimension.

After this is done it will be possible to consider a lunar theory in which e has a determinate value, but in which still $\gamma = 0$. The motion of the perigee must now be regarded, but that of the node need not yet be attended to. In like manner, as before, we divide the investigation so that terms severally proportional to e, e^2, e^3, etc., may be separately considered.

Finally, the complete form of the lunar theory is elaborated, in which γ has a determinate value and the latitude of the moon is considered. Here it is necessary to notice the motions of both perigee and node. For a like reason as before we divide the work so that terms proportional to γ, γ^2, γ^3, etc., may be separately obtained.

I.

In the former article it was supposed that the motion of the sun about the center of gravity of the earth and moon was circular, and we put

$$r' = a'(1 + K), \quad \lambda' = \epsilon' + n't$$

Here we propose to take into account the eccentricity e' of this orbit, neglecting, however, all powers of this quantity above the first. The differential equations for the moon's coordinates are the same as before, but we now assume that

$$r' = a'(1 + K) - a'e' \cos l', \quad \lambda' = \varepsilon' + n't + 2e' \sin l',$$

l' being the mean anomaly of the sun. Then the function R of the preceding article ought to be augmented by a quantity Z which is given by the equation

$$Z = -e' \cos l' \,.\, r' \frac{\partial R}{\partial r'} - 2e' \sin l' \,.\, \frac{\partial R}{\partial \phi}$$

Employing here the rectangular coordinates x, y, let us suppose that R and Z are expressed in terms of them. In forming the equations to *variation* we employ the notation

$$\frac{\partial^2 R}{\partial x^2} = H, \quad \frac{\partial^2 R}{\partial x \partial y} = J, \quad \frac{\partial^2 R}{\partial y^2} = K*$$

and call the augmentations of the coordinates which are strictly proportional to e' by the designations δx, δy. These equations are then

$$\frac{d^2 \delta x}{d\tau^2} = H\delta x + J\delta y + \frac{\partial Z}{\partial x}, \quad \frac{d^2 \delta y}{d\tau^2} = K\delta y + J\delta x + \frac{\partial Z}{\partial y}$$

In this particular case we no longer have the Jacobian integral in finite terms, but we may have it expressed by an infinite periodic series. Let us put

$$\frac{dM}{d\tau} = -e' \cos l' \,.\, \frac{d}{d\tau}\left[r' \frac{\partial R}{\partial r'}\right] - 2e' \sin l' \,.\, \frac{d}{d\tau}\left[\frac{\partial R}{\partial \phi}\right]$$

Although M cannot be had in finite terms, nevertheless, as the right member is a known function of τ, this quantity is expressible by an infinite periodic series. Then the equation, which here takes the place of the Jacobian integral is

$$\frac{dx^2 + dy^2}{2d\tau^2} - \mathrm{m} \frac{xdy - ydx}{d\tau} = R + M + C$$

In taking the variation of this equation we note that when the arbitrary constant C is developed in powers of e' only even powers present themselves; consequently $\delta C = 0$. By putting

$$F = \frac{dx}{d\tau} + \mathrm{m}y, \quad G = \frac{dy}{d\tau} - \mathrm{m}x$$

it is plain the variation of the last equation may be given the form

$$F\frac{d\delta x}{d\tau} + G\frac{d\delta y}{d\tau} - \frac{dF}{d\tau}\delta x - \frac{dG}{d\tau}\delta y = M$$

---

* The reader is asked to discriminate the two uses of K.

It is also evident that the quantities $F, G, H, J, K,$ satisfy the relations

$$\frac{d^2F}{d\tau^2} = HF + JG, \quad \frac{d^2G}{d\tau^2} = KG + JF$$

In place of the unknown δx, δy we propose to adopt ρ, σ such that

$$\delta x = F\rho, \quad \delta y = G\sigma$$

The three equations to variation then become

$$\frac{d}{d\tau}\left[F^2\frac{d\rho}{d\tau}\right] + JFG(\rho - \sigma) = F\frac{\partial Z}{\partial x}, \quad \frac{d}{d\tau}\left[G^2\frac{d\sigma}{d\tau}\right] + JFG(\sigma - \rho) = G\frac{\partial Z}{\partial y},$$

$$F^2\frac{d\rho}{d\tau} + G^2\frac{d\sigma}{d\tau} = M$$

of which the last is plainly a consequence of the first and second. For the sake of brevity employing the additional notation

$$L^2 = JFG, \quad N = \tfrac{1}{2}\left(F\frac{\partial Z}{\partial x} - G\frac{\partial Z}{\partial y}\right)$$

and introducing a single new variable w to take the place of ρ and σ, the third equation to *variation* is satisfied by making

$$F^2\frac{d\rho}{d\tau} = \tfrac{1}{2}M + L\mathrm{w}, \quad G^2\frac{d\sigma}{d\tau} = \tfrac{1}{2}M - L\mathrm{w}$$

From the first and second of the equations to *variation* we can obtain

$$\frac{d(L\mathrm{w})}{d\tau} + L^2(\rho - \sigma) = N$$

Dividing by L^2 and differentiating we get

$$\frac{d}{d\tau}\left[\frac{1}{L^2}\frac{d(L\mathrm{w})}{d\tau}\right] + \frac{d\rho}{d\tau} - \frac{d\sigma}{d\tau} = \frac{d}{d\tau}\left(\frac{N}{L^2}\right)$$

But

$$\frac{d\rho}{d\tau} - \frac{d\sigma}{d\tau} = \tfrac{1}{2}M\left(\frac{1}{F^2} - \frac{1}{G^2}\right) + L\left(\frac{1}{F^2} + \frac{1}{G^2}\right)\mathrm{w}$$

Substituting this in the preceding equation and putting

$$\theta = J\left[\frac{F}{G} + \frac{G}{F}\right] - L\frac{d^2L^{-1}}{d\tau^2}, \quad W = L\left[\frac{d}{d\tau}\left(\frac{N}{L^2}\right) - \tfrac{1}{2}M\left(\frac{1}{F^2} - \frac{1}{G^2}\right)\right]$$

the linear differential equation of the second order

$$\frac{d^2\mathrm{w}}{d\tau^2} + \theta\mathrm{w} = W$$

is obtained for the determination of w.

The value of w resulting from the integration of this equation would have the general form

$$\mathrm{w} = AS + BT + V,$$

A and B being the arbitrary constants. As S and T are periodic functions

involving the mean anomaly of the moon, and as the solution we need ought not to have terms of this sort, it is necessary here to assume that $A = 0$, $B = 0$. Thus $\mathrm{w} = V$, where V contains only terms of the same periods as those occurring in W.

After w has thus been found we have

$$\delta x = F \int \frac{\frac{1}{2} M + L\mathrm{w}}{F^2} d\tau, \quad \delta y = G \int \frac{\frac{1}{2} M - L\mathrm{w}}{G^2} d\tau$$

Here the two arbitrary constants must be so taken that δx and δy may have no terms independent of l', the mean anomaly of the sun. The variations of the coordinates usually adopted are then

$$\delta \left(\frac{a}{r}\right) = -\frac{a}{r^3}(x\delta x + y\delta y), \quad \delta\lambda = \frac{1}{r^2}(x\delta y - y\delta x)$$

II.

Although the preceding treatment constitutes a solution of the problem in an analytical sense, and possesses much elegance on account of the simplicity of the square of the velocity as expressed in terms of the differentials of the rectangular coordinates, it presents difficulties in practice when the quantities involved must be developed in infinite periodic series. The quantities F and G periodically vanish, which prevents the development of their reciprocals in infinite series. These difficulties can be overcome by putting the factor which makes the quantity vanish in evidence and executing the integration in accordance; the infinities then disappear. But the easier treatment is to have recourse to polar coordinates.

We adopt as variables for expressing the moon's position $\psi = \log r$, and ϕ. In order to avoid the writing it we suppose $a = 1$, and adopt the V of the preceding article. Then, supposing V and Z in terms of ψ and ϕ, the differential equations of motion are

$$\epsilon^{2\psi}\left[\frac{d^2\psi}{d\tau^2} + \frac{d\psi^2}{d\tau^2} - \frac{d\phi^2}{d\tau^2} - 2\mathrm{m}\frac{d\phi}{d\tau}\right] = \frac{1}{2}\frac{\partial V}{\partial\psi} + \frac{\partial Z}{\partial\psi}$$

$$\epsilon^{2\psi}\left[\frac{d^2\phi}{d\tau^2} + 2\frac{d\psi}{d\tau}\frac{d\phi}{d\tau} + 2\mathrm{m}\frac{d\psi}{d\tau}\right] = \frac{1}{2}\frac{\partial V}{\partial\phi} + \frac{\partial Z}{\partial\phi}$$

and the Jacobian integral

$$\epsilon^{2\psi}\frac{d\psi^2 + d\phi^2}{d\tau^2} = V + 2M$$

Let the increments of ψ and ϕ having e' as a factor be denoted by $\delta\psi$ and $\delta\phi$. To facilitate the investigation we can suppose the existence of two variables ρ and σ such that

$$\delta\psi = \frac{d\psi}{d\tau}\rho + \frac{d\phi}{d\tau}\sigma, \quad \delta\phi = \frac{d\phi}{d\tau}\rho - \frac{d\psi}{d\tau}\sigma$$

It is very plain that the values, $\rho =$ a constant, $\sigma = 0$, must satisfy the equations to *variation* which result from those of motion and their integral. Thus, when we obtain these equations expressed in terms of ρ and σ, in every case the factor multiplying ρ must vanish; we need not therefore go through the formality of deriving it. Hence, in making the transformation from $\delta\psi$, $\delta\phi$ to ρ, σ, we can limit ourselves to the expressions

$$\delta\psi = \frac{d\phi}{d\tau}\sigma, \quad \delta\phi = -\frac{d\psi}{d\tau}\sigma,$$

$$\frac{d\delta\psi}{d\tau} = \frac{d\psi}{d\tau}\frac{d\rho}{d\tau} + \frac{d\phi}{d\tau}\frac{d\sigma}{d\tau} + \frac{d^2\phi}{d\tau^2}\sigma, \quad \frac{d\delta\phi}{d\tau} = \frac{d\phi}{d\tau}\frac{d\rho}{d\tau} - \frac{d\psi}{d\tau}\frac{d\sigma}{d\tau} - \frac{d^2\psi}{d\tau^2}\sigma,$$

$$\frac{d^2\delta\psi}{d\tau^2} = \frac{d\psi}{d\tau}\frac{d^2\rho}{d\tau^2} + \frac{d\phi}{d\tau}\frac{d^2\sigma}{d\tau^2} + 2\frac{d^2\psi}{d\tau^2}\frac{d\rho}{d\tau} + 2\frac{d^2\phi}{d\tau^2}\frac{d\sigma}{d\tau} + \frac{d^3\phi}{d\tau^3}\sigma,$$

$$\frac{d^2\delta\phi}{d\tau^2} = \frac{d\phi}{d\tau}\frac{d^2\rho}{d\tau^2} - \frac{d\psi}{d\tau}\frac{d^2\sigma}{d\tau^2} + 2\frac{d^2\phi}{d\tau^2}\frac{d\rho}{d\tau} - 2\frac{d^2\psi}{d\tau^2}\frac{d\sigma}{d\tau} - \frac{d^3\psi}{d\tau^3}\sigma$$

In these expressions we can eliminate the second and third differentials of ψ and ϕ by means of the original equations of motion.

Subjecting the Jacobian integral to *variation* we find

$$\varepsilon^{2\psi}\left[\frac{d\psi}{d\tau}\frac{d\delta\psi}{d\tau} + \frac{d\phi}{d\tau}\frac{d\delta\phi}{d\tau}\right] = V\frac{d\rho}{d\tau} + \varepsilon^{2\psi}\left[\frac{d\psi}{d\tau}\frac{d^2\phi}{d\tau^2} - \frac{d\phi}{d\tau}\frac{d^2\psi}{d\tau^2}\right]\sigma$$

$$V\delta\psi = V\frac{d\phi}{d\tau}\sigma, \quad -\tfrac{1}{2}\,\delta V = \tfrac{1}{2}\left[\frac{\partial V}{\partial\phi}\frac{d\psi}{d\tau} - \frac{\partial V}{\partial\psi}\frac{d\phi}{d\tau}\right]\sigma$$

The sum of these three equations gives the *variation* of the Jacobian integral.

But we have

$$\varepsilon^{2\psi}\left[\frac{d\psi}{d\tau}\frac{d^2\phi}{d\tau^2} - \frac{d\phi}{d\tau}\frac{d^2\psi}{d\tau^2}\right] = \tfrac{1}{2}\left[\frac{\partial V}{\partial\phi}\frac{d\psi}{d\tau} - \frac{\partial V}{\partial\psi}\frac{d\phi}{d\tau}\right] - V\left(\frac{d\phi}{d\tau} + 2\,\mathrm{m}\right)$$

Consequently the equation to *variation* derived from the Jacobian integral is

$$V\frac{d\rho}{d\tau} + \left[\frac{\partial V}{\partial\phi}\frac{d\psi}{d\tau} - \frac{\partial V}{\partial\psi}\frac{d\phi}{d\tau} - 2\mathrm{m}V\right]\sigma = M$$

or, if we put

$$\varDelta_1 = \frac{1}{V}\left[\frac{\partial V}{\partial\psi}\frac{d\phi}{d\tau} - \frac{\partial V}{\partial\phi}\frac{d\psi}{d\tau}\right] + 2\mathrm{m}$$

it will take the form

$$\frac{d\rho}{d\tau} = \varDelta_1\sigma + \frac{M}{V}$$

It is necessary to derive still another equation to *variation* from the original equations of motion. Then, if we take the *variations* of both and multiply the first by $\frac{d\phi}{d\tau}$ and the second by $-\frac{d\psi}{d\tau}$, it is plain from the equiv-

alents we have given for $\frac{d^2\delta\psi}{d\tau^2}$ and $\frac{d^2\delta\phi}{d\tau^2}$, that not only does the term in ρ vanish, but also the term in $\frac{d^2\rho}{d\tau^2}$. Consequently the resulting equation will be of the form

$$A\frac{d^2\sigma}{d\tau^2}+B\frac{d\rho}{d\tau}+C\frac{d\sigma}{d\tau}+D\sigma=N$$

We can eliminate $\frac{d\rho}{d\tau}$ from this by means of the last equation, and thus shall have a linear equation of the second order for the determination of σ.

Proceeding to the derivation of this equation to *variation*, we divide the somewhat complex mass of terms into four parts.

1. Make the second derivatives in the equations alone to vary and we get the terms

$$\text{(I)}\qquad V\frac{d^2\sigma}{d\tau^2}+V\left[\Delta_1+2\left(\frac{d\phi}{d\tau}+\text{m}\right)\right]\frac{d\rho}{d\tau}+\left[\frac{dV}{d\tau}-2V\frac{d\psi}{d\tau}\right]\frac{d\sigma}{d\tau}+\varepsilon^{2\psi}\left[\frac{d\psi}{d\tau}\frac{d^3\psi}{d\tau^3}+\frac{d\phi}{d\tau}\frac{d^3\phi}{d\tau^3}\right]\sigma$$

But by differentiating the equations of motion it results that

$$\varepsilon^{2\psi}\left[\frac{d\psi}{d\tau}\frac{d^3\psi}{d\tau^3}+\frac{d\phi}{d\tau}\frac{d^3\phi}{d\tau^3}\right]=-2V\frac{d^2\psi}{d\tau^2}-\text{m}V\left[\Delta_1+2\left(\frac{d\phi}{d\tau}+\text{m}\right)\right]$$
$$-\frac{dV}{d\tau}\frac{d\psi}{d\tau}+\tfrac{1}{2}\left[\frac{\partial^2V}{\partial\psi^2}\frac{d\psi^2}{d\tau^2}+2\frac{\partial^2V}{\partial\psi\partial\phi}\frac{d\psi d\phi}{d\tau^2}+\frac{\partial^2V}{\partial\phi^2}\frac{d\phi^2}{d\tau^2}\right]$$

2. By making only the first derivatives in the equations to vary we get the terms

$$\text{(II)}\qquad -2V\left(\frac{d\phi}{d\tau}+\text{m}\right)\frac{d\rho}{d\tau}+2V\frac{d\psi}{d\tau}\frac{d\sigma}{d\tau}+V\left[\Delta_1+2\left(\frac{d\phi}{d\tau}+\text{m}\right)+2\frac{d^2\psi}{d\tau^2}\right]\sigma$$

3. By making only the common factor $\varepsilon^{2\psi}$ of the two equations to vary we get the terms

$$\text{(III)}\qquad V[\Delta_1-2\text{m}]\frac{d\phi}{d\tau}\sigma$$

4. In fine, by making only the second members of the equations to vary we get the terms

$$\text{(IV)}\qquad -\tfrac{1}{2}\left[\frac{\partial^2V}{\partial\psi^2}\frac{d\phi^2}{d\tau^2}-2\frac{\partial^2V}{\partial\psi\partial\phi}\frac{d\psi d\phi}{d\tau^2}+\frac{\partial^2V}{\partial\phi^2}\frac{d\psi^2}{d\tau^2}\right]\sigma$$

By adding these four divisions of terms we arrive at the equation

$$\frac{d}{d\tau}\left[V\frac{d\sigma}{d\tau}\right]+V\Delta_1\frac{d\rho}{d\tau}+\left[\tfrac{1}{2}\left(\frac{\partial^2V}{\partial\psi^2}-\frac{\partial^2V}{\partial\phi^2}-2\frac{\partial V}{\partial\psi}\right)\frac{d\psi^2-d\phi^2}{d\tau^2}+2\left(\frac{\partial^2V}{\partial\psi\partial\phi}-\frac{\partial V}{\partial\phi}\right)\frac{d\psi d\phi}{d\tau^2}\right]\sigma$$
$$=\frac{\partial Z}{\partial\psi}\frac{d\phi}{d\tau}-\frac{\partial Z}{\partial\phi}\frac{d\psi}{d\tau}=N$$

No symmetry is apparent in the coefficient of σ as written in this equation, but it may be given the following form:

$$-\left[\frac{\partial V}{\partial\psi}\frac{d\psi}{d\tau}+\frac{\partial V}{\partial\phi}\frac{d\phi}{d\tau}\right]\frac{d\psi}{d\tau}+\tfrac{1}{2}\left[\frac{\partial^2 V}{\partial\psi^2}\frac{d\psi^2}{d\tau^2}+2\frac{\partial^2 V}{\partial\psi\partial\phi}\frac{d\psi}{d\tau}\frac{d\phi}{d\tau}+\frac{\partial^2 V}{\partial\phi^2}\frac{d\phi^2}{d\tau^2}\right]$$
$$+\left[\frac{\partial V}{\partial\psi}\frac{d\phi}{d\tau}-\frac{\partial V}{\partial\phi}\frac{d\psi}{d\tau}\right]\frac{d\phi}{d\tau}-\tfrac{1}{2}\left[\frac{\partial^2 V}{\partial\psi^2}\frac{d\phi^2}{d\tau^2}-2\frac{\partial^2 V}{\partial\psi\partial\phi}\frac{d\psi}{d\tau}\frac{d\phi}{d\tau}+\frac{\partial^2 V}{\partial\phi^2}\frac{d\psi^2}{d\tau^2}\right]$$

Eliminating $\frac{d\rho}{d\tau}$ from the equation by means of the value given by the equation to *variation* derived from the Jacobian integral we have

$$\frac{d}{d\tau}\left[V\frac{d\sigma}{d\tau}\right]+\left[\tfrac{1}{2}\left(\frac{\partial^2 V}{\partial\psi^2}-\frac{\partial^2 V}{\partial\phi^2}-2\frac{\partial V}{\partial\psi}\right)\frac{d\psi^2-d\phi^2}{d\tau^2}+2\left(\frac{\partial^2 V}{\partial\psi\partial\phi}-\frac{\partial V}{\partial\phi}\right)\frac{d\psi d\phi}{d\tau^2}+V\Delta_1^2\right]\sigma=N-\Delta_1 M$$

In order to reduce this equation to the simplest possible form we introduce a new variable w such that

$$\sigma=V^{-\frac{1}{2}}\,\mathrm{w}$$

If we put Q for the coefficient of σ in the equation before $\frac{d\rho}{d\tau}$ was eliminated we have the equation

$$\frac{d^2\mathrm{w}}{d\tau^2}+\frac{1}{V}\left[Q+V\Delta_1^2+\tfrac{1}{4}\frac{1}{V}\frac{dV^2}{d\tau^2}-\tfrac{1}{2}\frac{d^2V}{d\tau^2}\right]\mathrm{w}=\frac{N-\Delta_1 M}{\sqrt{V}}$$

But

$$\tfrac{1}{2}\frac{d^2V}{d\tau^2}=\tfrac{1}{2}\left[\frac{\partial^2V}{\partial\psi^2}\frac{d\psi^2}{d\tau^2}+2\frac{\partial^2V}{\partial\psi\partial\phi}\frac{d\psi d\phi}{d\tau^2}+\frac{\partial^2V}{\partial\phi^2}\frac{d\phi^2}{d\tau^2}\right]+\tfrac{1}{2}\left[\frac{\partial V}{\partial\psi}\frac{d^2\psi}{d\tau^2}+\frac{\partial V}{\partial\phi}\frac{d^2\phi}{d\tau^2}\right]$$
$$=\tfrac{1}{2}\left[\frac{\partial^2V}{\partial\psi^2}\frac{d\psi^2}{d\tau^2}+2\frac{\partial^2V}{\partial\psi\partial\phi}\frac{d\psi d\phi}{d\tau^2}+\frac{\partial^2V}{\partial\phi^2}\frac{d\phi^2}{d\tau^2}\right]-\tfrac{1}{2}\left[\frac{\partial V}{\partial\psi}\frac{d\psi}{d\tau}+\frac{\partial V}{\partial\phi}\frac{d\phi}{d\tau}\right]\frac{d\psi}{d\tau}$$
$$+\tfrac{1}{2}\left[\frac{\partial V}{\partial\psi}\frac{d\phi}{d\tau}-\frac{\partial V}{\partial\phi}\frac{d\phi}{d\tau}\right]\left(\frac{d\phi}{d\tau}+\mathrm{m}\right)+\tfrac{1}{4}\,\varepsilon^{-2\psi}\left[\frac{\partial V^2}{\partial\psi^2}+\frac{\partial V^2}{\partial\phi^2}\right]$$

$$\tfrac{1}{4}V\Delta_1^2=\frac{1}{4V}\left[\frac{\partial V^2}{\partial\psi^2}\frac{d\phi^2}{d\tau^2}-2\frac{\partial V}{\partial\psi}\frac{\partial V}{\partial\phi}\frac{d\psi d\phi}{d\tau^2}+\frac{\partial V^2}{\partial\phi^2}\frac{d\psi^2}{d\tau^2}\right]+\mathrm{m}\left[\frac{\partial V}{\partial\psi}\frac{d\phi}{d\tau}-\frac{\partial V}{\partial\phi}\frac{d\psi}{d\tau}\right]+\mathrm{m}^2V$$

$$\tfrac{1}{4}\frac{1}{V}\frac{dV^2}{d\tau^2}=\frac{1}{4V}\left[\frac{\partial V^2}{\partial\psi^2}\frac{d\psi^2}{d\tau^2}+2\frac{\partial V}{\partial\psi}\frac{\partial V}{\partial\phi}\frac{d\psi d\phi}{d\tau^2}+\frac{\partial V^2}{\partial\phi^2}\frac{d\phi^2}{d\tau^2}\right]$$

By the assistance of these expressions we obtain the equation

$$Q+\tfrac{1}{4}V\Delta_1^2+\tfrac{1}{4}\frac{1}{V}\frac{dV^2}{d\tau^2}-\tfrac{1}{2}\frac{d^2V}{d\tau^2}=-\tfrac{1}{2}\left[\frac{\partial V}{\partial\psi}\frac{d\psi}{d\tau}+\frac{\partial V}{\partial\phi}\frac{d\phi}{d\tau}\right]\frac{d\psi}{d\tau}+\tfrac{1}{2}\left[\frac{\partial V}{\partial\psi}\frac{d\phi}{d\tau}-\frac{\partial V}{\partial\phi}\frac{d\psi}{d\tau}\right]\frac{d\phi}{d\tau}$$
$$-\tfrac{1}{2}\left[\frac{\partial^2V}{\partial\psi^2}\frac{d\phi^2}{d\tau^2}-2\frac{\partial^2V}{\partial\psi d\phi}\frac{d\psi d\phi}{d\tau^2}+\frac{\partial^2V}{\partial\phi^2}\frac{d\psi^2}{d\tau^2}\right]+\mathrm{m}^2V$$

Consequently, if we put

$$\Delta_2=-\frac{1}{2V}\left[\left(\frac{\partial^2V}{\partial\psi^2}-\frac{\partial V}{\partial\psi}\right)\frac{d\phi^2}{d\tau^2}-2\left(\frac{\partial^2V}{\partial\psi\partial\phi}-\frac{\partial V}{\partial\phi}\right)\frac{d\psi d\phi}{d\tau^2}+\left(\frac{\partial^2V}{\partial\phi^2}+\frac{\partial V}{\partial\psi}\right)\frac{d\psi^2}{d\tau^2}\right]+\mathrm{m}^2$$

$$\theta=\tfrac{3}{4}\Delta_1^2+\Delta_2$$

$$W=\frac{N-\Delta_1 M}{\sqrt{V}}$$

the equation determining w is

$$\frac{d^2\mathrm{w}}{d\tau^2} + \theta\mathrm{w} = W$$

Attending to the integration of this we put

$$\mathrm{w} = \Sigma \mathrm{b}_i \cos (l' + i\tau), \quad W = \Sigma W_i \cos (l' + i\tau), \quad \theta = \Sigma \theta_i \cos i\tau$$

These summations are extended from $i = -\infty$ to $i = +\infty$, and, in the last equation we have $\Theta_{-i} = \Theta_i$. Thus, for the determination of the unknown coefficients b_i, we have the group of linear equations represented generally by

$$-(j + \mathrm{m})^2 \mathrm{b}_j + \Sigma . \theta_i \mathrm{b}_{j-i} = W_j$$

For the degree of approximation we wish to attain it is necessary to go from b_{-10} to b_{10}; thus we have 21 equations with 21 unknowns. These we solve by steps of approximation. First we limit ourselves to the 5 unknowns from b_{-2} to b_2 and the 5 equations which specially determine them; afterwards, with these values, it is easy to obtain values of the remaining unknowns to the corresponding degree of approximation. With these values the final terms of the 5 equations may be corrected and the operation of solution repeated. About three repetitions of this procedure suffice to obtain the desired degree of precision.

III.

In reducing the preceding results to numbers we employ the values of the constants given in the former article. We there gave the values of $\frac{a}{r}$ and ϕ for every 15° in the semi-circumference of the argument τ. The expression for $\frac{1}{2}V$ being

$$\begin{aligned}\tfrac{1}{2}V = C + (1 + \mathrm{m})^2\frac{a}{r} + \tfrac{1}{2}\mathrm{m}^2\frac{r^2}{a^2} &+ \alpha_1\frac{r^2}{a^2}[\tfrac{3}{4}\cos 2\phi + \tfrac{1}{4}] \\ &+ \alpha_2\frac{r^3}{a^3}[\tfrac{5}{8}\cos 3\phi + \tfrac{3}{8}\cos\phi] \\ &+ \alpha_3\frac{r^4}{a^4}[\tfrac{35}{64}\cos 4\phi + \tfrac{5}{16}\cos 2\phi + \tfrac{9}{64}] \\ &+ \alpha_4\frac{r^5}{a^5}[\tfrac{63}{128}\cos 5\phi + \tfrac{35}{128}\cos 3\phi + \tfrac{15}{64}\cos\phi]\end{aligned}$$

for facilitating the computation of special values of this function and its

several partial derivatives with respect to r, r', and ϕ, we give the following tables of the values of its several terms:

τ	$(1+\mathrm{m})^2\left(\frac{a}{r}-1\right)$	$\frac{1}{2}\mathrm{m}^2\frac{r^2}{a^2}$	$\frac{3}{4}\alpha_1\frac{r^2}{a^2}\cos 2\phi$	$\frac{1}{4}\alpha_1\frac{r^2}{a^2}$	$\frac{5}{8}\alpha_2\frac{r^3}{a^3}\cos 3\phi$
°	11	11	12	12	12
0	+917 68107 2	+321 75273 4	+482 62760 36	+160 87586 76	+1 00211 19
15	803 54383 9	322 37745 1	416 34669 54	161 18822 54	+ 69995 23
30	494 00132 3	324 08088 3	+235 80106 01	162 03993 89	— 2606 91
45	+ 76 72033 2	326 39865 9	— 9 57873 67	163 19882 30	74493 68
60	—333 46944 1	328 70136 5	253 56129 83	164 35017 28	1 03443 01
75	627 27574 9	330 36573 4	431 35825 30	165 18235 45	72727 17
90	728 23916 4	330 94059 6	496 40898 06	165 46978 45	— 192 20
105	609 52328 6	330 26481 1	431 80183 08	165 13189 30	+ 72428 55
120	—299 36347 7	328 50897 5	254 30444 60	164 25397 80	1 03343 56
135	+124 57742 0	326 13157 8	— 10 40897 12	163 06528 30	74582 34
150	552 13811 3	323 75992 5	+235 05589 87	161 87946 01	+ 2785 70
165	867 98717 5	322 02450 5	415 73832 07	161 01175 27	— 69813 25
180	+984 24369 5	+321 38924 9	+482 08237 79	+160 69412 59	—1 00041 43

τ	$\frac{3}{8}\alpha_2\frac{r^3}{a^3}\cos\phi$	$\frac{35}{64}\alpha_3\frac{r^4}{a^4}\cos 4\phi$	$\frac{5}{16}\alpha_3\frac{r^4}{a^4}\cos 2\phi$	$\frac{9}{64}\alpha_3\frac{r^4}{a^4}$	$\frac{63}{128}\alpha_4\frac{r^5}{a^5}\cos 5\phi$	$\frac{35}{128}\alpha_4\frac{r^5}{a^5}\cos 3\phi$	$\frac{15}{64}\alpha_4\frac{r^5}{a^5}\cos\phi$
°	12	12	12	12	12	12	12
0	+60126 71	+221 23	+126 42	+56 89	+50	+28	+24
15	58168 61	+107 18	109 27	57 11	+12	+20	23
30	52374 80	—118 82	+ 62 21	57 71	—45	— 1	21
45	43013 14	227 49	— 2 55	58 54	—35	—21	17
60	30596 28	—108 76	67 85	59 37	+28	—29	12
75	15920 21	+120 21	116 01	59 97	+51	—21	+ 7
90	+ 38 44	234 04	133 74	60 18	0	0	0
105	—15841 42	+121 08	116 09	59 94	—51	+21	— 6
120	30512 44	—107 77	68 01	59 30	—29	+29	12
135	42922 81	227 08	— 2 76	58 45	+35	+21	17
150	52278 46	—119 04	+ 61 95	57 60	+45	+ 1	21
165	58068 18	+106 71	108 99	56 98	—12	—19	23
180	—60024 86	+220 73	+126 13	+56 76	—50	—28	—24

τ	$-\frac{3}{2}\alpha_1\frac{r^2}{a^2}\sin 2\phi$	$-\frac{15}{8}\alpha_2\frac{r^3}{a^3}\sin 3\phi$	$-\frac{3}{8}\alpha_2\frac{r^3}{a^3}\sin\phi$	$-\frac{35}{16}$ ..	$-\frac{5}{8}$ ..	$-\frac{315}{128}$ ..	$-\frac{105}{128}$ ..	$-\frac{15}{64}$ ..
°	11	11	11	11	11	11	11	11
15	—491 89521 3	—2 16365 5	—15897 5	—778 0	—129 1	—25	—6	—1
30	850 20076 8	3 03801 8	30840 7	—761 6	224 3	—12	9	1
45	979 00551 6	—2 10733 7	43863 1	+ 35 6	260 1	+19	—6	2
60	845 70794 4	+ 7709 2	54022 3	814 7	226 3	+22	0	2
75	487 83993 1	2 24124 0	60497 4	799 4	131 2	— 8	+6	2
90	— 1 21698 6	3 13601 7	62720 4	+ 2 3	— 3	27	9	3
105	+485 65631 7	2 24793 4	60488 5	—796 7	+130 6	— 8	+6	2
120	844 14115 4	+ 8684 2	54007 1	815 5	225 7	+22	0	2
135	978 17019 3	—2 09900 7	43846 0	— 38 7	260 3	+19	—6	2
150	849 92555 2	3 03336 0	30826 4	+759 0	224 0	—12	8	1
165	+491 87278 5	—2 16205 3	—15889 5	+776 9	+128 9	—24	—6	—1

τ	$\log \frac{d\psi}{d\tau}$	$\log \frac{d\phi}{d\tau}$
0°	. . . . .	0.00859 82784
15	7.8590 4872	0.00740 23642
30	8.0940 2940	0.00414 82420
45	8.1511 7491	9.99973 58223
60	8.0818 5997	9.99536 74425
75	7.8314 6683	9.99221 44191
90	6.4712 7940n	9.99111 30885
105	7.8663 1621n	9.99236 04082
120	8.0993 3173n	9.99564 41626
135	8.1629 8202n	0.00011 68426
150	8.1032 1855n	0.00460 21576
165	7.8669 8852n	0.00789 81733
180	. . . . .	0.00910 75964

Where it is necessary the order of the last decimal is expressed by the small figures at the top of the column. In order to have the special values of the various derivatives of $\frac{1}{2} V$, correspondent to the indicated values of τ, it is necessary only to multiply the quantities given in the preceding tables by certain positive or negative integers, in all cases less than 25 (their values are perceived at a glance) and sum the products. In this way have been formed the following values of Δ_1, Δ_2 and Θ:

τ	Δ_1	Δ_2	Θ
0°	—2.14482 69404 8	—2.40513 46257 4	1.04507 73277 1
15	2.14686 78054 4	2.39669 23142 2	1.06008 87163 1
30	2.15260 71888 3	2.37396 30611 0	1.10132 52209 1
45	2.16083 67957 6	2.34370 65632 7	1.15820 51801 6
60	2.16952 31500 4	2.31438 96004 2	1.21573 34234 9
75	2.17619 04936 5	2.29363 64155 1	1.25821 73830 0
90	2.17875 27123 2	2.28650 95831 7	1.27371 29529 9
105	2.17640 68220 4	2.29472 39660 8	1.25783 60251 8
120	2.17000 52862 7	2.31645 86141 2	1.21523 35926 8
135	2.16163 95914 4	2.34656 95375 1	1.15794 47549 4
150	2.15373 42653 2	2.37739 03147 5	1.10153 81494 8
165	2.14823 94105 1	2.40044 88286 3	1.06075 05950 0
180	—2.14628 96921 7	—2.40899 82779 1	1.04592 13041 6

From the special values of Θ is derived the following periodic series representing it, which may be compared with that given in the memoir "On the Motion of the Lunar Perigee."* The differences are due to the inclusion here of terms dependent on $\frac{a}{a'}$ and μ:

$$\theta = \left\{ \begin{array}{lrl} & 1.15884\,04425 & \\ - & 11\,88248 \cos & \tau \\ - & 11408\,84846 \cos & 2\tau \\ - & 30\,72745 \cos & 3\tau \\ + & 76\,55834 \cos & 4\tau \\ + & 42061 \cos & 5\tau \\ - & 1\,83319 \cos & 6\tau \\ - & 966 \cos & 7\tau \\ + & 1085 \cos & 8\tau \\ + & 15 \cos & 9\tau \\ - & 20 \cos & 10\tau \end{array} \right\}$$

Attending next to the determination of M we obtain

$$\frac{dM}{d\tau} = - \left\{ \begin{array}{lrl} & 0.00003\,87849 \sin & \tau \\ + & 2922\,21996 \sin & 2\tau \\ + & 10\,80772 \sin & 3\tau \\ + & 17\,81067 \sin & 4\tau \\ + & 8725 \sin & 5\tau \\ + & 12406 \sin & 6\tau \\ + & 70 \sin & 7\tau \\ + & 91 \sin & 8\tau \\ + & 1 \sin & 9\tau \\ + & 1 \sin & 10\tau \end{array} \right\} e' \cos l' + \left\{ \begin{array}{lrl} & 0.00002\,89353 \cos & \tau \\ + & 3914\,43255 \cos & 2\tau \\ + & 16\,51032 \cos & 3\tau \\ + & 23\,79574 \cos & 4\tau \\ + & 13207 \cos & 5\tau \\ + & 16565 \cos & 6\tau \\ + & 106 \cos & 7\tau \\ + & 122 \cos & 8\tau \\ + & 1 \cos & 9\tau \\ + & 1 \cos & 10\tau \end{array} \right\} e' \sin l'$$

It will be noticed that M does not contain any term having the argument l'. After integration,

$$M = \left\{ \begin{array}{lrl} & 0.00003\,22820 \sin & \tau \\ + & 2019\,58114 \sin & 2\tau \\ + & 5\,60460 \sin & 3\tau \\ + & 6\,04140 \sin & 4\tau \\ + & 2670 \sin & 5\tau \\ + & 2789 \sin & 6\tau \\ + & 15 \sin & 7\tau \\ + & 15 \sin & 8\tau \end{array} \right\} e' \sin l' + \left\{ \begin{array}{lrl} & 0.00004\,13948 \cos & \tau \\ + & 1542\,75047 \cos & 2\tau \\ + & 3\,75361 \cos & 3\tau \\ + & 4\,57478 \cos & 4\tau \\ + & 1788 \cos & 5\tau \\ + & 2105 \cos & 6\tau \\ + & 10 \cos & 7\tau \\ + & 11 \cos & 8\tau \end{array} \right\} e' \cos l'$$

*_Acta Mathematica_, Vol. VIII, p. 34.

From this we deduce the following

SPECIAL VALUES OF M

τ		
0°	$+0.01555\,25750\, e' \cos l'$	$0\, e' \sin l'$
15	$1345\,00568\, e' \cos l'$	$+0.01019\,87512\, e' \sin l'$
30	$+\ 772\,63606\, e' \cos l'$	$1761\,47242\, e' \sin l'$
45	$-\ 4\,31439\, e' \cos l'$	$2025\,77999\, e' \sin l'$
60	$775\,31651\, e' \cos l'$	$1746\,54941\, e' \sin l'$
75	$1335\,33942\, e' \cos l'$	$+\ 1003\,74843\, e' \sin l'$
90	$1538\,19663\, e' \cos l'$	$-\ 2\,34984\, e' \sin l'$
105	$1332\,20811\, e' \cos l'$	$1005\,42422\, e' \sin l'$
120	$771\,96674\, e' \cos l'$	$1741\,00398\, e' \sin l'$
135	$-\ 4\,83494\, e' \cos l'$	$2013\,32650\, e' \sin l'$
150	$+\ 765\,49740\, e' \cos l'$	$1747\,00847\, e' \sin l'$
165	$1331\,69118\, e' \cos l'$	$-\ 1010\,22609\, e' \sin l'$
180	$+\ 1539\,43534\, e' \cos l'$	$0\, e' \sin l'$

In the next place are obtained the following

SPECIAL VALUES OF W

τ		
0°	$+0.07206\,87725\,1\, e' \cos l'$	$0\, e' \sin l'$
15	$6373\,32334\,1\, e' \cos l'$	$+0.04138\,52211\,3\, e' \sin l'$
30	$4088\,83403\,1\, e' \cos l'$	$7183\,16977\,7\, e' \sin l'$
45	$+\ 950\,87713\,2\, e' \cos l'$	$8317\,24605\,4\, e' \sin l'$
60	$-\ 2206\,67047\,7\, e' \cos l'$	$7219\,14643\,7\, e' \sin l'$
75	$4527\,76228\,5\, e' \cos l'$	$+\ 4166\,67510\,0\, e' \sin l'$
90	$5372\,55207\,6\, e' \cos l'$	$-\ 16\,50666\,6\, e' \sin l'$
105	$4509\,37054\,1\, e' \cos l'$	$4183\,13967\,5\, e' \sin l'$
120	$-\ 2185\,72203\,8\, e' \cos l'$	$7196\,39977\,7\, e' \sin l'$
135	$+\ 952\,71800\,7\, e' \cos l'$	$8257\,34027\,0\, e' \sin l'$
150	$4057\,57123\,0\, e' \cos l'$	$7111\,96683\,0\, e' \sin l'$
165	$6311\,42691\,8\, e' \cos l'$	$-\ 4090\,85272\,9\, e' \sin l'$
180	$+\ 7132\,62653\,6\, e' \cos l'$	$0\, e' \sin l'$

From these special values we derive the following development of W in periodic series:

$$W = \left\{ \begin{array}{llll} +0.00925\,20209\,4\, e' \cos l' & & & \\ + & 15\,41991\,7\, e' \cos (l' - \tau) & +0.00002\,48859\,0\, e' \cos (l' + \tau) & \\ + & 7279\,31393\,7\, e' \cos (l' - 2\tau) & - & 1008\,74636\,6\, e' \cos (l' + 2\tau) \\ + & 24\,29723\,0\, e' \cos (l' - 3\tau) & - & 4\,93926\,3\, e' \cos (l' + 3\tau) \\ - & 30\,68123\,3\, e' \cos (l' - 4\tau) & + & 4\,08240\,3\, e' \cos (l' + 4\tau) \\ - & 17058\,8\, e' \cos (l' - 5\tau) & + & 2614\,9\, e' \cos (l' + 5\tau) \\ + & 67579\,0\, e' \cos (l' - 6\tau) & - & 9142\,5\, e' \cos (l' + 6\tau) \\ + & 404\,1\, e' \cos (l' - 7\tau) & - & 68\,2\, e' \cos (l' + 7\tau) \\ - & 386\,6\, e' \cos (l' - 8\tau) & + & 51\,0\, e' \cos (l' + 8\tau) \\ - & 3\,9\, e' \cos (l' - 9\tau) & + & 6\, e' \cos (l' + 9\tau) \\ + & 6\,2\, e' \cos (l' - 10\tau) & - & 1\,3\, e' \cos (l' + 10\tau) \end{array} \right\}$$

By the integration of the differential equation which determines w we have the following expression:

$$
\mathrm{w} = \left\{ \begin{array}{rrrr}
+ & 0{\cdot}00674\;50610\;2\, e' \cos l' & & \\
+ & 2\;39399\;9\, e' \cos (l' - \tau) & - & 0.00252\;38205\;5\, e' \cos (l' + \tau) \\
- & 2899\;18882\;6\, e' \cos (l' - 2\tau) & + & 305\;65254\;1\, e' \cos (l' + 2\tau) \\
- & 3\;32273\;3\, e' \cos (l' - 3\tau) & + & 2\;30508\;2\, e' \cos (l' + 3\tau) \\
+ & 13\;82463\;3\, e' \cos (l' - 4\tau) & - & 1\;36791\;6\, e' \cos (l' + 4\tau) \\
+ & 3513\;1\, e' \cos (l' - 5\tau) & - & 1215\;0\, e' \cos (l' + 5\tau) \\
- & 7615\;3\, e' \cos (l' - 6\tau) & + & 779\;6\, e' \cos (l' + 6\tau) \\
- & 33\;4\, e' \cos (l' - 7\tau) & + & 11\;0\, e' \cos (l' + 7\tau) \\
+ & 65\;2\, e' \cos (l' - 8\tau) & - & 6\;6\, e' \cos (l' + 8\tau) \\
+ & 3\, e' \cos (l' - 9\tau) & - & 1\, e' \cos (l' + 9\tau) \\
- & 4\, e' \cos (l' - 10\tau) & + & 1\, e' \cos (l' + 10\tau)
\end{array} \right\}
$$

From this are derived the following

Special Values of w

τ				
0°	—	0.02157 58418 7 $e' \cos l'$		0 $e' \sin l'$
15		1807 47900 1 $e' \cos l'$	—	0.01527 35055 8 $e' \sin l'$
30	—	844 90341 0 $e' \cos l'$		2640 55345 2 $e' \sin l'$
45	+	486 01318 6 $e' \cos l'$		3028 64441 3 $e' \sin l'$
60		1841 03223 7 $e' \cos l'$		2568 06398 4 $e' \sin l'$
75		2862 85315 8 $e' \cos l'$	—	1365 61467 1 $e' \sin l'$
90		3280 56804 9 $e' \cos l'$	+	260 41159 9 $e' \sin l'$
105		2990 75178 6 $e' \cos l'$		1865 71010 9 $e' \sin l'$
120		2088 92223 8 $e' \cos l'$		3009 19711 1 $e' \sin l'$
135	+	838 08675 4 $e' \cos l'$		3380 87043 2 $e' \sin l'$
150	—	411 94126 3 $e' \cos l'$		2884 08159 3 $e' \sin l'$
165		1323 18927 0 $e' \cos l'$	—	1651 34320 0 $e' \sin l'$
180	—	1655 69829 2 $e' \cos l'$		0 $e' \sin l'$

From the data previously obtained we get the following

Special Values of $\frac{d\rho}{d\tau}$.

τ				
0°	+	0.06091 01912 9 $e' \cos l'$		0 $e' \sin l'$
15		5158 78573 3 $e' \cos l'$	+	0.04244 94386 6 $e' \sin l'$
30	+	2573 26074 0 $e' \cos l'$		7395 94234 9 $e' \sin l'$
45	—	1055 74291 8 $e' \cos l'$		8582 53152 6 $e' \sin l'$
60		4812 57284 5 $e' \cos l'$		7388 50262 6 $e' \sin l'$
75		7677 79144 4 $e' \cos l'$	+	4038 47342 0 $e' \sin l'$
90		8832 37137 3 $e' \cos l'$	—	577 90963 9 $e' \sin l'$
105		7955 32897 2 $e' \cos l'$		5141 45591 3 $e' \sin l'$
120		5350 01353 2 $e' \cos l'$		8345 26567 4 $e' \sin l'$
135	—	1817 73486 6 $e' \cos l'$		9329 38253 0 $e' \sin l'$
150	+	1638 36514 7 $e' \cos l'$		7900 93575 2 $e' \sin l'$
165		4115 17345 9 $e' \cos l'$	—	4497 95751 8 $e' \sin l'$
180	+	5010 36052 9 $e' \cos l'$		0 $e' \sin l'$

Applying to these values the operation of mechanical quadratures we get

$$\frac{d\rho}{d\tau} = \left\{\begin{array}{lr} -0.01538\ 77367\ 0 & \\ + & 539\ 56432\ 0 \cos \tau \\ + & 7190\ 05560\ 1 \cos 2\tau \\ + & 96295\ 1 \cos 3\tau \\ - & 102\ 05093\ 8 \cos 4\tau \\ - & 20314\ 4 \cos 5\tau \\ + & 1\ 47482\ 3 \cos 6\tau \\ + & 524\ 7 \cos 7\tau \\ - & 1616\ 2 \cos 8\tau \\ - & 8\ 0 \cos 9\tau \\ + & 17\ 6 \cos 10\tau \end{array}\right\} e' \cos l' + \left\{\begin{array}{lr} -0.00552\ 99581\ 4 \sin \tau & \\ + & 8957\ 77624\ 9 \sin 2\tau \\ + & 24\ 30526\ 0 \sin 3\tau \\ - & 126\ 13921\ 9 \sin 4\tau \\ - & 59785\ 4 \sin 5\tau \\ + & 1\ 81944\ 3 \sin 6\tau \\ + & 1055\ 8 \sin 7\tau \\ - & 1988\ 2 \sin 8\tau \\ - & 15\ 2 \sin 9\tau \\ + & 22\ 2 \sin 10\tau \end{array}\right\} e' \sin l' .$$

By integration there results

$$\rho = \left\{\begin{array}{lr} -0.19032\ 70207\ 1 & \\ + & 512\ 72406\ 0 \cos \tau \\ - & 4631\ 78417\ 9 \cos 2\tau \\ - & 8\ 11629\ 9 \cos 3\tau \\ + & 32\ 06357\ 3 \cos 4\tau \\ + & 12025\ 9 \cos 5\tau \\ - & 30660\ 8 \cos 6\tau \\ - & 151\ 7 \cos 7\tau \\ + & 250\ 6 \cos 8\tau \\ + & 1\ 7 \cos 9\tau \\ - & 2\ 2 \cos 10\tau \end{array}\right\} e' \sin l' + \left\{\begin{array}{lr} +0.00498\ 11112\ 6 \sin \tau & \\ + & 3782\ 26520\ 7 \sin 2\tau \\ + & 53971\ 5 \sin 3\tau \\ - & 26\ 16081\ 1 \sin 4\tau \\ - & 4257\ 3 \sin 5\tau \\ + & 24993\ 5 \sin 6\tau \\ + & 76\ 7 \sin 7\tau \\ - & 204\ 6 \sin 8\tau \\ - & 9 \sin 9\tau \\ + & 1\ 8 \sin 10\tau \end{array}\right\} e' \cos l'$$

Hence are derived the following

SPECIAL VALUES OF ρ.

τ		
0°	—0.23128 00028 1 $e' \sin l'$	0 $e' \cos l'$
15	22538 36845 4 $e' \sin l'$	+0.01997 98674 4 $e' \cos l'$
30	20920 39137 4 $e' \sin l'$	3502 45719 9 $e' \cos l'$
45	18696 55948 6 $e' \sin l'$	4134 64423 7 $e' \cos l'$
60	16468 61193 5 $e' \sin l'$	3729 60631 5 $e' \cos l'$
75	14866 86931 2 $e' \sin l'$	2394 78619 4 $e' \cos l'$
90	14368 54518 3 $e' \sin l'$	+ 497 52806 2 $e' \cos l'$
105	15143 98827 2 $e' \sin l'$	— 1433 29429 4 $e' \cos l'$
120	16997 68730 1 $e' \sin l'$	2866 77746 9 $e' \cos l'$
135	19432 96679 0 $e' \sin l'$	3429 38634 3 $e' \cos l'$
150	21808 24983 0 $e' \sin l'$	3003 30996 5 $e' \cos l'$
165	23517 46011 4 $e' \sin l'$	— 1739 46295 6 $e' \cos l'$
180	—0.24137 45332 1 $e' \sin l'$	0 $e' \cos l'$

Having now the special values of ρ and w we get the special values of $\delta\lambda$ and $\delta\left(\frac{a}{r}\right)$ by the formulas

$$\delta\lambda = \frac{d\phi}{d\tau}\rho - \frac{d\psi}{d\tau}\frac{\mathrm{w}}{\sqrt{V}}$$

$$\delta\left(\frac{a}{r}\right) = -\frac{a}{r}\left[\frac{d\psi}{d\tau}\rho + \frac{d\phi}{d\tau}\frac{\mathrm{w}}{\sqrt{V}}\right]$$

Thus far we have kept e' indeterminate, but now it is judged advisable to attribute to it the value 0.01677106, which is the same as that used by Delaunay. Moreover, the coefficients of $\delta\lambda$ are transformed into arc.

SPECIAL VALUES.

τ	$\delta\lambda$		$\delta\frac{a}{r}$	
	″	″	″	
0°	—816.05984 4 sin l'	0 cos l'	+36 75568 5 cos l'	0 sin l'
15	792.69006 7 sin l'	+ 70.75146 7 cos l'	30 48711 6 cos l'	+28 71933 1 sin l'
30	729.51228 3 sin l'	122.68357 8 cos l'	+13 55645 2 cos l'	49 03195 6 sin l'
45	644.88729 3 sin l'	142.70351 7 cos l'	— 9 14362 9 cos l'	55 29927 7 sin l'
60	562.56978 3 sin l'	126.87367 1 cos l'	31 45083 3 cos l'	46 14576 9 sin l'
75	504.82460 4 sin l'	80.69018 6 cos l'	47 76870 9 cos l'	+24 33929 0 sin l'
90	486.97775 1 sin l'	+ 16.89635 2 cos l'	54 33233 6 cos l'	— 4 38397 9 sin l'
105	514.25781 6 sin l'	— 47.94719 4 cos l'	49 81041 3 cos l'	32 82054 8 sin l'
120	580.81136 9 sin l'	97.26512 1 cos l'	35 45410 3 cos l'	53 77915 8 sin l'
135	670.71818 9 sin l'	118.24174 1 cos l'	—14 92205 9 cos l'	61 56411 8 sin l'
150	761.18733 6 sin l'	105.17933 6 cos l'	+ 6 33176 0 cos l'	53 48420 2 sin l'
165	828.04911 1 sin l'	— 61.61056 4 cos l'	22 30531 6 cos l'	—31 03229 5 sin l'
180	—852.67727 6 sin l'	0 cos l'	+28 23767 1 cos l'	0 sin l'

By the application of mechanical quadratures we get the following series. Delaunay's values of the coefficients are added for the sake of comparison in the case of the longitude; in the case of the radius, Delaunay having stopped with terms of the fifth order, it seems hardly worth while to make comparison. They have been obtained by making in his expressions $e = 0$, $\gamma = 0$, and by substituting for his value of $\frac{a}{a'}$ the value which corresponds to the constant 8″.8 of solar parallax; and the terms which involve the simple power of this factor have been multiplied by $1 - 2\mu$ in order to take into account the moon's mass; but no inductive terms have been added.

$$\delta\lambda = \left\{\begin{array}{lr} & \text{Del. Coeff.} \\ -0.00009'' \sin(l' - 8\tau) & \\ -0.00002 \sin(l' - 7\tau) & \\ -0.01054 \sin(l' - 6\tau) & -0.0038'' \\ -0.00144 \sin(l' - 5\tau) & -0.0014 \\ -1.25517 \sin(l' - 4\tau) & -1.1916 \\ -0.09012 \sin(l' - 3\tau) & -0.1160 \\ -152.08250 \sin(l' - 2\tau) & -152.1127 \\ +0.59511 \sin(l' - \tau) & +0.5718 \\ -659.23785 \sin l' & -659.2305 \\ +17.69186 \sin(l' + \tau) & +17.5918 \\ -21.60085 \sin(l' + 2\tau) & -21.6338 \\ +0.11250 \sin(l' + 3\tau) & +0.1045 \\ -0.18003 \sin(l' + 4\tau) & -0.1809 \\ +0.00082 \sin(l' + 5\tau) & +0.0006 \\ -0.00152 \sin(l' + 6\tau) & +0.0004 \\ +0.00001 \sin(l' + 7\tau) & \\ -0.00001 \sin(l' + 8\tau) & \end{array}\right.$$

$$\delta\left(\frac{a}{r}\right) = \left\{\begin{array}{lr} + & {}^{10}7 \cos(l' - 8\tau) \\ + & 1 \cos(l' - 7\tau) \\ + & 699 \cos(l' - 6\tau) \\ + & 74 \cos(l' - 5\tau) \\ + & 65281 \cos(l' - 4\tau) \\ + & 3124 \cos(l' - 3\tau) \\ + & 50\,92412 \cos(l' - 2\tau) \\ - & 11723 \cos(l' - \tau) \\ - & 11\,47540 \cos l' \\ + & 4\,28970 \cos(l' + \tau) \\ - & 7\,51559 \cos(l' + 2\tau) \\ + & 5400 \cos(l' + 3\tau) \\ - & 9530 \cos(l' + 4\tau) \\ + & 53 \cos(l' + 5\tau) \\ - & 102 \cos(l' + 6\tau) \\ & 0 \cos(l' + 7\tau) \\ - & 1 \cos(l' + 8\tau) \end{array}\right.$$

If no errors have been committed in these computations, and pains have been taken to detect and eliminate them, the coefficients set down should correspond to the values of the elements employed with an uncertainty of not more than a unit in the last decimal.

MEMOIR No. 66.

On the Extension of Delaunay's Method in the Lunar Theory to the General Problem of Planetary Motion.

(Transactions of the American Mathematical Society, Vol. I, pp. 205–242, 1900.)

Part I.—Exposition of the Theory.

The method of integrating the differential equations of motion, adopted by Delaunay for the elaboration of his lunar theory, as it is explained by him, demands its division into several cases, and is established through very tedious transformations. These disadvantages disappear when the greatest generality is given to the procedure. Hence, an explanation of the method, as it would be applied to the motion of a planetary system like the solar, will, doubtless, be welcome to astronomers.

I.

Let T denote the living force of the system, Ω the potential function, and, with Poincaré, put

$$F = \Omega - T.$$

The k variables, necessary for completely defining the position of the system, may be denoted by $q_1, q_2, \ldots, q_k$. Use accents to denote complete differentiation with respect to the time of the latter, and we have

$$T = \text{function}\,(q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k).$$

The partial derivatives of this function with respect to the k variables q'_i are to be used as variables instead of the latter, and we put

$$p_i = \frac{\partial T}{\partial q'_i} \qquad (i = 1, 2, \ldots, k).$$

By means of these k equations the q'_i can be eliminated from T, and thus will result:

$$T = \text{function}\,(q_1, q_2, \ldots, q_k, p_1, p_2, \ldots, p_k).$$

Then the system of differential equations for determining the variables p_i and q_i is:

(1) $$\frac{dp_i}{dt} = \frac{\partial F}{\partial q_i}, \quad \frac{dq_i}{dt} = -\frac{\partial F}{\partial p_i}, \qquad (i = 1, 2, \ldots, k).$$

Let us suppose that Ω is separated into the two parts Ω_0 and Ω_1, and that, when we neglect Ω_1 in F, the equations (1) can be completely integrated, and their integrals expressed in terms of two sets of k quantities each, symbolized thus:

$$L_1,\ L_2,\ \ldots,\ L_k,$$
$$\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_k,$$

of which the first set are constants, and the second set linear functions of the time of the form $n_i t + c_i$, n_i being a function of the L_i, and c_i an arbitrary constant. Nothing forbids our taking the L_i such that they may be the elements severally conjugate to the λ_i.

Now, desiring to integrate the equations (1) when F has its complete value, we may adopt the L_i and the λ_i as the dependent variables to be employed. The differential equations of the problem are then:

$$(2) \qquad \frac{dL_i}{dt} = \frac{\partial F}{\partial \lambda_i}, \quad \frac{d\lambda_i}{dt} = -\frac{\partial F}{\partial L_i}, \qquad (i = 1, 2, \ldots, k).$$

Here the function F has been made to involve the L_i and λ_i by eliminating the old variables p_i and q_i from it by means of their values given by the integrals derived on the supposition that $\Omega = \Omega_0$. As

$$F = \text{a constant}$$

is an integral of the problem, and $\Omega_0 - T =$ a constant, when Ω_1 is neglected, it is quite evident that when we substitute in $\Omega_0 - T$ for p_i and q_i their values in terms of the L_i and λ_i, the λ_i completely disappear and $\Omega_0 - T$ becomes a function of the L_i only. Thus, in the second form for F, the variables λ_i enter into it solely through the portion Ω_1.

II.

In order to exemplify we will adduce the solar system composed of the Sun and the eight major planets. We will suppose that the masses of the Sun, Mercury, Venus, . . ., Neptune are denoted by m_0, m_1, m_2, . . ., m_8, and will put

$$\mu_i = m_0 + m_1 + m_2 + \ldots + m_i, \quad \varkappa_i = \frac{m_i}{\mu_i} \qquad (i = 0, 1, \ldots, 8).$$

Let the type of representation of the rectangular coordinates of the planets relative to the Sun be as follows:

Mercury	x_1,
Venus	$x_2 + \varkappa_1 x_1$,
Earth	$x_3 + \varkappa_2 x_2 + \varkappa_1 x_1$.
....	....
Neptune	$x_8 + \varkappa_7 x_7 + \ldots + \varkappa_1 x_1$.

The differential equations these variables satisfy are:

$$(3)\qquad \begin{cases} \mu_{i-1}\varkappa_i \dfrac{d^2 x_i}{dt^2} = \dfrac{\partial \Omega}{\partial x_i}, \\ \mu_{i-1}\varkappa_i \dfrac{d^2 y_i}{dt^2} = \dfrac{\partial \Omega}{\partial y_i}, \\ \mu_{i-1}\varkappa_i \dfrac{d^2 z_i}{dt^2} = \dfrac{\partial \Omega}{\partial z_i}, \end{cases} \qquad (i = 1, 2, \ldots, 8).$$

Here Ω denotes the sum of the products of every two masses of the system divided by their distance, a relation we will write thus:

$$(4)\qquad \Omega = m_0 \sum \frac{m_i}{\Delta_{0,i}} + \sum \frac{m_i m_j}{\Delta_{i,j}},$$

Suppose that the portion to be separated from Ω is

$$(5)\qquad \Omega_0 = m_0 \sum \frac{m_i}{r_i},$$

r_i standing for $\sqrt{x_i^2 + y_i^2 + z_i^2}$. Then, if Ω_0 is substituted for Ω in equations (3), and the members are divided by $\mu_{i-1}\varkappa_i$, we get

$$(6)\qquad \begin{cases} \dfrac{d^2 x_i}{dt^2} + m_0 \dfrac{\mu_i}{\mu_{i-1}} \dfrac{x_i}{r_i^3} = 0, \\ \dfrac{d^2 y_i}{dt^2} + m_0 \dfrac{\mu_i}{\mu_{i-1}} \dfrac{y_i}{r_i^3} = 0, \\ \dfrac{d^2 z_i}{dt^2} + m_0 \dfrac{\mu_i}{\mu_{i-1}} \dfrac{z_i}{r_i^3} = 0. \end{cases} \qquad (i = 1, 2, \ldots, 8).$$

It will be seen that each group of these equations, corresponding to the same value of i, is independent of all the rest, and that it differs from the group of equations of relative motion of two bodies only in that the constant $m_0\mu_i/\mu_{i-1}$ takes the place of $m_0 + m_i$.

Let a_i be the semi-axis major, e_i the eccentricity, ϕ_i the inclination, l_i the mean anomaly, g_i the angular distance of the perihelion from the node, and h_i the longitude of the node. Put

$$(7)\qquad \begin{cases} L_i = \sqrt{m_0 \dfrac{\mu_i}{\mu_{i-1}} a_i}, \\ G_i = L_i \sqrt{1 - e_i^2}, \\ H_i = G_i \cos \phi_i. \end{cases} \qquad (i = 1, 2, \ldots, 8).$$

Then, when the elements become variable by reason of the addition of Ω_1 to Ω_0, they will satisfy the differential equations:

$$(8)\qquad \begin{cases} \dfrac{dL_i}{dt} = \dfrac{\partial R_i}{\partial l_i}, & \dfrac{dl_i}{dt} = -\dfrac{\partial R_i}{\partial L_i}, \\ \dfrac{dG_i}{dt} = \dfrac{\partial R_i}{\partial g_i}, & \dfrac{dg_i}{dt} = -\dfrac{\partial R_i}{\partial G_i}, \\ \dfrac{dH_i}{dt} = \dfrac{\partial R_i}{\partial h_i}, & \dfrac{dh_i}{dt} = -\dfrac{\partial R_i}{\partial H_i}, \end{cases} \qquad (i = 1, 2, \ldots, 8).$$

where R_i will be, in terms of Ω_1, mentioned above,

$$R_i = \frac{m_0 \frac{\mu_i}{\mu_{i-1}}}{2a_i} + \frac{\Omega_1}{\mu_{i-1}\varkappa_i}, \qquad (i = 1, 2, \ldots, 8). \tag{9}$$

Desiring to have the same perturbative function, whatever may be the integer i, we multiply the values (7) of L_i, G_i, H_i, as also the value (9) of R_i by the constant $\mu_{i-1}\varkappa_i$, which does not alter the form of equations (8). We now have:

$$\left\{\begin{aligned} L_i &= m_i \sqrt{m_0 \frac{\mu_{i-1}}{\mu_i} a_i}, \\ G_i &= L_i \sqrt{1 - e_i^2}, \\ H_i &= G_i \cos \phi_i, \end{aligned}\right. \qquad (i = 1, 2, \ldots, 8). \tag{10}$$

as also:

$$F = m_0 \sum_{i=1}^{i=8} \frac{m_i}{2a_i} + m_0 \sum_{i=2}^{i=8} m_i \left[\frac{1}{\Delta_{0,i}} - \frac{1}{r_i}\right] + \sum \frac{m_i m_j}{\Delta_{i,j}}. \tag{11}$$

If the planetary coordinates in the last equation are replaced by the elements (10) and the l_i, g_i, h_i, the differential equations of the system are:

$$\left\{\begin{aligned} \frac{dL_i}{dt} &= \frac{\partial F}{\partial l_i}, & \frac{dl_i}{dt} &= -\frac{\partial F}{\partial L_i}, \\ \frac{dG_i}{dt} &= \frac{\partial F}{\partial g_i}, & \frac{dg_i}{dt} &= -\frac{\partial F}{\partial G_i}, \\ \frac{dH_i}{dt} &= \frac{\partial F}{\partial h_i}, & \frac{dh_i}{dt} &= -\frac{\partial F}{\partial H_i}. \end{aligned}\right. \qquad (i = 1, 2, \ldots, 8). \tag{12}$$

In the second term of the right member of (11) the quantities $1/\Delta_{0,2}$, $1/\Delta_{0,3}$, $\ldots$, $1/\Delta_{0,8}$, can be developed in infinite series, the first terms of which are $1/r_2$, $1/r_3$, $\ldots$, $1/r_8$, and thus are cancelled by the term $1/r_i$. Then the two latter terms of (11) are of the second order with reference to planetary masses.

III.

In order to make the application of Delaunay's method it appears necessary that F should be developed in a series, finite or infinite and periodic with respect to the variables l_i, g_i, h_i, which have been named the angular variables. In astronomical problems the series is generally infinite. For legitimate employment this series must remain convergent throughout the whole duration of motion, while t is passing from $-\infty$ to $+\infty$. It becomes then pertinent to ask what conditions must be fulfilled in order that this series may be convergent. It is well known that the reciprocal of the distance between two planets can be developed in a convergent infinite series, periodic with respect to the mean anomalies of the planets, provided that the orbits, as they stand in space, have no point in common, or when

the reciprocal of the distance never becomes infinite. The condition of convergence in the present case is precisely similar to this. Here, however, not only the mean anomalies l_i are left indeterminate in the series, but also the remaining angular variables g_i and h_i which define the position of the perihelia and nodes. Hence, in the present case, there must not only be no actual intersection of the orbits, but none when the perihelia and nodes are shifted in every possible way, the linear variables, or the mean distances, eccentricities and inclinations retaining their actual values. In the Delaunay development of the reciprocal of the distance between two planets, it is necessary and it suffices for convergence that the perihelion radius of one of the planets should always exceed the aphelion radius of the other.

We may consider this subject under a more general aspect. Let F have the periodic development

$$F = \Sigma A \cos [j_1\lambda_1 + j_2\lambda_2 + \ldots + j_k\lambda_k], \tag{13}$$

where $\lambda_1 \ldots \lambda_k$ are the angular variables, the j positive or negative integers, and A is a function of the linear variables L only. That this infinite series may be convergent, F must not only actually never become infinite, but never even potentially so. It is necessary here to explain what we mean by the qualifying epithet "potentially." If, while the linear variables L are supposed to maintain their actual values, and, consequently, the coefficients A their actual values, we allow to all the angular variables λ the complete swing of movement from 0 to 2π, F remains always finite, we say it never *potentially* becomes infinite. In order that F may not *actually* become infinite it is necessary and sufficient that the velocities of and the distances between the points of the system should remain finite. In order that F may not *potentially* become infinite, it is necessary and sufficient that the values of the linear variables $L_1, L_2, \ldots, L_k$ should remain within a certain domain. The definition of this domain is very complex after a Delaunay transformation has been operated, but is quite simple in terms of the original Keplerian linear variables $L_1, L_2, \ldots, L_k$.

We may illustrate this subject by bringing forward the case of the solar system as it has been described in § II. Employing the linear elements a_i and e_i of equation (10) or of equations (7), the inequalities which define this domain are:

$$\tag{14} \begin{cases} a_1(1-e_1) & > 0, \\ a_2(1-e_2) - (1-x_1)\,a_1(1+e_1) & > 0, \\ a_3(1-e_3) - (1-x_2)\,a_2(1+e_2) & > 0, \\ \ldots\ldots\ldots\ldots\ldots & \\ a_8(1-e_8) - (1-x_7)\,a_7(1+e_7) & > 0, \\ a_8(1+e_8) & < \infty, \end{cases}$$

or, if we prefer to use the elements L_i and G_i of equation (10), the inequalities are:

$$(15) \quad \left\{ \begin{array}{l} L_1 [L_1 - \sqrt{L_1^2 - G_1^2}] > 0, \\ \frac{\mu_2}{m_2^2} L_2 [L_2 - \sqrt{L_2^2 - G_2^2}] > \frac{\mu_1}{m_1^2} L_1 [L_1 + \sqrt{L_1^2 - G_1^2}], \\ \frac{\mu_3}{m_3^2} L_3 [L_3 - \sqrt{L_3^2 - G_3^2}] > \frac{\mu_2}{m_2^2} L_2 [L_2 + \sqrt{L_2^2 - G_2^2}], \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{\mu_8}{m_8^2} L_8 [L_8 - \sqrt{L_8^2 - G_8^2}] > \frac{\mu_7}{m_7^2} L_7 [L_7 + \sqrt{L_7^2 - G_7^2}], \\ L_8 + \sqrt{L_8^2 - G_8^2} > \infty. \end{array} \right.$$

It is remarkable that the inclinations of the orbits play no part in these inequalities.

IV.

As, in general, it is not necessary to distinguish between the three kinds l_i, g_i, h_i of angular variables, nor between the three kinds L_i, G_i, H_i, of linear variables, for simplicity of notation we shall suppose that the angular variables are denoted by $l_1, l_2, \ldots, l_k$, and their corresponding conjugate linear variables by $L_1, L_2, \ldots, L_k$.

Selecting a particular linear combination θ of the angular variables so that

$$\theta = j_1 l_1 + j_2 l_2 + \ldots + j_k l_k,$$

the j being positive or negative integers prime to each other, Delaunay's method, somewhat generalized, consists in making such a transformation of variables as would constitute a complete solution of the problem if F, in its periodic development, contained as arguments only integral multiples of θ. That is, in this special case, the new linear variables would turn out constants, and the new angular variables would be of the form $n(t+c)$, n and c being likewise constants. It is clear that when we make such a transformation in F, the terms in the former periodic development involving the cosines of the finite multiples of θ will disappear, but the absolute term will receive a modification. A little consideration will make it evident that the derivation of such a transformation is dependent on quadratures.

The discussion of this derivation is greatly facilitated by making a linear transformation of variables, in the cases of both angular and linear variables. In this it is evident that we can take θ as one of the angular variables; then let Θ be its conjugate linear variable. Thus, we may have the following as the variables involved in the problem:

Linear variables, $\Theta, \Lambda_1, \Lambda_2, \ldots, \Lambda_{k-1}$;
Angular variables, $\theta, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}$.

And the canonical system of differential equations will be:

$$
(16)\qquad \left\{\begin{array}{ll}
\dfrac{d\Theta}{dt} = \dfrac{\partial F}{\partial \theta}, & \dfrac{d\theta}{dt} = -\dfrac{\partial F}{\partial \Theta}, \\[2mm]
\dfrac{d\Lambda_1}{dt} = \dfrac{\partial F}{\partial \lambda_1}, & \dfrac{d\lambda_1}{dt} = -\dfrac{\partial F}{\partial \Lambda_1}, \\
\cdots & \cdots \\
\dfrac{d\Lambda_{k-1}}{dt} = \dfrac{\partial F}{\partial \lambda_{k-1}}, & \dfrac{d\lambda_{k-1}}{dt} = -\dfrac{\partial F}{\partial \Lambda_{k-1}}.
\end{array}\right.
$$

Let us now consider the mean value of the function F relatively to the angular variables $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$. Then, since F as a periodic function involves only cosines of arguments, if $[F]$ denote the mentioned mean value, we shall have

$$
(17)\qquad [F] = \left[\frac{1}{\pi}\int_0^{\pi}\right]^{k-1} F d\lambda_1 d\lambda_2 \ldots d\lambda_{k-1},
$$

where the first factor of the right member denotes an operation repeated $k-1$ times, once in reference to each of the variables $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$. As F remains finite whatever values $\theta, \lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ may assume, it follows that $[F]$ is finite whatever may be the value of θ. Thus $[F]$ is developable as a periodic function of θ involving only cosines; and we may write:

$$
(18)\qquad [F] = -B - A_1 \cos\theta - A_2 \cos 2\theta - A_3 \cos 3\theta - \ldots,
$$

where $B, A_1, A_2, A_3, \ldots$ are functions of the linear variables $\Theta, \Lambda_1, \Lambda_2, \ldots, \Lambda_{k-1}$.

Let us now suppose that, in equations (16), $[F]$ is substituted for F. They then become:

$$
(19)\qquad \left\{\begin{array}{ll}
\dfrac{d\Theta}{dt} = \dfrac{\partial [F]}{\partial \theta}, & \dfrac{d\theta}{dt} = -\dfrac{\partial [F]}{\partial \Theta}, \\[2mm]
\dfrac{d\Lambda_1}{dt} = 0, & \dfrac{d\lambda_1}{dt} = -\dfrac{\partial [F]}{\partial \Lambda_1}, \\
\cdots & \cdots \\
\dfrac{d\Lambda_{k-1}}{dt} = 0, & \dfrac{d\lambda_{k-1}}{dt} = -\dfrac{\partial [F]}{\partial \Lambda_{k-1}}.
\end{array}\right.
$$

$\Lambda_1, \Lambda_2, \ldots, \Lambda_{k-1}$ are therefore constants, and the two equations of the first line contain no other variables than Θ and θ, and thus form a distinct system by themselves and determine these two variables; after which, by substitution of values, the remaining differential equations for $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ determine these variables through quadratures.

V.

As $[F]$ involves only two variables Θ and θ, the two equations which begin (19) have the integral, C being an arbitrary constant,

$$[F] + C = 0. \tag{20}$$

This integral constitutes a relation between the two variables Θ and θ; and, if the latter are regarded as coordinates defining the position of a point in a plane, (20) is the equation of a plane curve. For this graphical exhibition of the connection between the two variables, we might adopt that in which they are the polar coordinates of a point, Θ being the radius and θ the angle. But, in some cases Θ may pass through zero. This difficulty may be obviated by adding to it a sufficiently large positive constant and thus it be rendered uniformly positive. This can be done provided it does not go to negative infinity. However, all circumstances considered, it will probably be a better course to adopt a representation in rectangular coordinates, θ being the abscissa and Θ the ordinate.

If we derive from (20) an expression for θ in terms of Θ and substitute it in the first equation of (19) and take the reciprocal of both members we shall have the time in terms of Θ by a quadrature, and, by the inversion of this, Θ as a function of t. On the other hand, if we derive from (20) an expression for Θ in terms of θ, and substitute it in the second equation of (19) we shall have the time in terms of θ by a quadrature, and, by the inversion of this, θ as a function of t.

We proceed to note some of the properties of the curve whose equation is (20). In the first place it must be stated that if the differential equations of (19), which determine the variables Θ and θ, compel the first of these to take on values rendering the right member of (18) a divergent series, we agree to set aside such cases as nugatory. Singularities of a certain kind are therefore excluded. The curve cannot have a *point d'arrêt*, for, at this point, we should have simultaneously $d\Theta/dt = 0$ and $d\theta/dt = 0$; and in consequence all succeeding derivatives of these variables would vanish. Thus, at this point Θ and θ would be invariable, which is impossible. It cannot have a multiple point, since, for given values of Θ and θ, there is but one value of each of the quantities $d\Theta/dt$ and $d\theta/dt$. If the curve pass through a point, it must proceed thence until it returns to that point or goes on to infinity. In the latter case, taking a polar representation for the moment, it may either have two infinite branches, or may make an infinite number of turns about the pole, or, in other words, be a spiral. But, since equation (20) involves only cosines of θ without sines of the same, the curve must

needs be symmetrically situated with respect to the axis from which θ is measured. Hence, the last supposition must be rejected; that is, it cannot be a spiral, nor can it have more than one distinct turn about the pole.

The curves graphically representing (20) may be divided into three classes. Here, for convenience, we adopt a rectangular representation. Let us suppose that an infinite number of values between the limits 0 and π are substituted for θ in (20); the result will be an infinite number of equations for determining the corresponding values of Θ. Let one of these be satisfied by a real value of Θ. Then it may happen that all the remaining equations are satisfied by real values of this variable continuous among themselves and with the value first mentioned. The variable θ can then move from $-\infty$ to $+\infty$ and there will always be a corresponding real value for Θ. The first equation of (19) shows that Θ will be at a maximum or minimum when $\theta = i\pi$, i being a positive or negative integer. As, in the equation

$$(21) \qquad \frac{d\theta}{dt} = A_1 \sin\theta + 2A_2 \sin 2\theta + 3A_3 \sin 3\theta + \ldots,$$

the quantity A_1 is, in general, larger than A_2, A_3, ..., it follows that Θ will have no other maximum or minimum values than those just mentioned. In addition, if a maximum value occurs for $\theta = 2i\pi$, then will a minimum value occur for $\theta = (2i+1)\pi$, and *vice versa*. If, in (20) we put, in succession $\theta = 0$, $\theta = \pi$, we shall have the two equations:

$$(22) \qquad \begin{cases} C - B = \quad A_1 + A_2 + A_3 + \ldots, \\ C - B = -A_1 + A_2 - A_3 + \ldots. \end{cases}$$

And if Θ be regarded as the unknown to be determined by them, it is plain that the maximum value of Θ will be a root of one of them and the minimum value a root of the other. Again Θ cannot be constant unless all the coefficients A vanish. It is quite evident that, in this case, the values of Θ and θ can be represented by the infinite periodic series:

$$(23) \qquad \begin{cases} \Theta = \quad \Theta_0 \quad + \Theta_1 \cos[\theta_0(t+c)] + \Theta_2 \cos 2[\theta_0(t+c)] + \ldots, \\ \theta = \theta_0(t+c) + \theta_1 \sin[\theta_0(t+c)] + \theta_2 \sin 2[\theta_0(t+c)] + \ldots. \end{cases}$$

These two equations are to be regarded as the integrals of the first and second differential equations of the group (19); c is one of the arbitrary constants introduced by the integration, the other may be supposed to be either the C of (20) or the Θ_0 of the first of (23). But while C and c are conjugate to each other, this is not necessarily the case with the elements Θ_0 and $\theta_0(t+c)$. The remaining coefficients of (23), Θ_1, Θ_2, ..., θ_0, θ_1, θ_2, ..., are functions of C or Θ_0. On account of the form of the curve which represents (20) in this case the latter may be called the sinusoid case.

We come now to consider the second case of the representation of (20) by a curve. Here, if we give to θ its range of values between 0 and π, we shall find that the equations determining the corresponding values of Θ have two real roots for an arc of values for θ which either begins at 0 or ends at π; and, in the first case the arc terminates, or, in the second case, begins, at the same intermediate point. At this point the two real roots become equal, and, for the remainder of the semi-circumference, they are imaginary. Consequently, at this point, θ attains either a maximum or minimum value. Because the equation contains only cosines of multiples of θ, in the one case, the right line $\theta = 0$, and, in the other, the right line $\theta = \pi$, divides the area embraced by the curve symmetrically. The maximum and minimum values of Θ are given by the roots of that one of the two equations of (22) which has two real roots. In this case, θ cannot be represented by series like the second of (23), but, in general, we may give the integrals of the problem the form:

$$(24)\qquad \begin{cases} \theta\cos\theta = P_0 + P_1\cos[\theta_0(t+c)] + P_2\cos 2[\theta_0(t+c)] + \ldots, \\ \theta\sin\theta = \qquad Q_1\sin[\theta_0(t+c)] + Q_2\sin 2[\theta_0(t+c)] + \ldots, \end{cases}$$

where θ_0, P_0, P_1, P_2, ..., Q_1, Q_2, ..., are constant coefficients and functions of the C of (20), while c, as before, is the other arbitrary constant. It will be perceived that, in the former case, the integral equations (23) can be given the form (24) if one chooses; and Delaunay has always adopted it where the eccentricity e would appear as a divisor in the first form. At the two points, at which θ has attained its maximum or minimum value, we have $d\theta/dt = 0$, or

$$(25)\qquad \frac{dB}{d\theta} + \frac{dA_1}{d\theta}\cos\theta + \frac{dA_2}{d\theta}\cos 2\theta + \ldots = 0.$$

When $dB/d\Theta$ and $dA_1/d\Theta$ are quantities of the same order of magnitude, the second case is likely to occur. As the curve, which here represents the connection of the variables Θ and θ, is a closed one, this case may be called the ovaloid case. This kind of motion in the variables is, however, generally termed a libration. Observation has not yet shown that it occurs in the system of the eight major planets of the solar system, although it is possible it may exist for very large values of the integers j_i. However, should this prove true, the influence of this circumstance on the motion of the system would be quite insignificant.

The third case in the graphical representation of (20) occurs when, in a certain range of values for θ, bisected by the value $\theta = 0$ or by the value $\theta = \pi$, we find a real value for Θ, but this value tends towards positive or negative infinity as the limits are approached. Here there is one maximum

and no minimum for Θ or one minimum and no maximum. As in the previous cases, these values occur when $\theta = 0$ or $\theta = \pi$. As long as the instantaneous orbits of the planets composing the system are elliptic in their nature this case cannot present itself. And Θ cannot go beyond a certain limit without some of the elements becoming imaginary. In order, therefore, to prevent the occurrence of functions of complex variables, a modified system has to be adopted. But an illustration of this case can very easily be constructed. In order to escape the difficulty of divergence when $|\Theta|$ exceeds a certain limit, let us suppose that $[F]$ is finite and does not run into an infinite series, and that all the quantities A_i beyond A_1 vanish. Then the equation (20), being solved with reference to $\cos\theta$, gives

$$\cos\theta = \frac{C - B}{A_1} = \frac{f(\theta)}{F(\theta)}.$$

Let Θ_0 be the value of Θ when $\theta = 0$; in order to have the present case we ought to have

$$\frac{f(\theta_0)}{F(\theta_0)} = 1, \quad \frac{f(\infty)}{F(\infty)} = a,$$

a being less than unity. We may suppose that Θ is involved linearly in B and A_1, so that a, b, c, d being constants,

$$f(\theta) = a + b\theta, \quad F(\theta) = c + d\theta.$$

Then

$$\theta_0 = \frac{a - c}{d - b}, \quad a = \frac{b}{d}.$$

All the conditions will be fulfilled if we put

$$f(\theta) = 3 + 4\theta, \quad F(\theta) = 2 + 5\theta;$$

whence $\Theta_0 = 1$ and $a = \frac{4}{5}$. Θ is thus continuous while θ is contained between the two values given by the equation $\cos\theta = \frac{4}{5}$. At the limits Θ becomes infinite. In a system of polar coordinates, if Θ is the radius and θ the angle, the equation of the curve graphically exhibiting the connection of the variables Θ and θ is:

$$\cos\theta = \frac{3 + 4\theta}{2 + 5\theta}, \text{ or } \theta = \frac{-3 + 2\cos\theta}{4 - 5\cos\theta}.$$

It is thus a quartic curve whose equation in rectangular coordinates is:

$$[2x - 4(x^2 + y^2)]^2 = (3 - 5x)^2(x^2 + y^2),$$

whose course resembles that of a hyperbola. The formula for the time is:

$$t + c = \int \frac{d\theta}{\sqrt{(2 + 5\theta)^2 - (3 + 4\theta)^2}} = \tfrac{1}{3}\int \frac{d\theta}{\sqrt{(\theta - \frac{2}{9})^2 - (\frac{7}{9})^2}}.$$

If this be integrated between the limits $\Theta = 1$ and $\Theta = \Theta$ it will give the time required to describe the curve from the point $\theta = 0$ to the point having the radius Θ.

VI.

Let us now suppose that by the integration of the system of differential equations (19), it is proposed to remove from F the periodic terms having the argument θ, that is, those contained in $[F]$. We confine ourselves to the first case as that will usually be the one which presents itself. The integrals of (19) will evidently have the form:

$$(26)\quad \begin{cases} \Theta = \Theta_0 + \Theta_1 \cos[\theta_0(t+c)] + \Theta_2 \cos 2[\theta_0(t+c)] + \ldots, \\ \Lambda_1 = \text{a constant}, \quad \Lambda_2 = \text{a constant}, \quad \ldots, \quad \Lambda_{k-1} = \text{a constant}, \\ \theta = \theta_0(t+c) + \theta_1 \sin[\theta_0(t+c)] + \theta_2 \sin 2[\theta_0(t+c)] + \ldots, \\ \lambda_1 = (\lambda_1) + \lambda_1^{(0)}(t+c) + \lambda_1^{(1)} \sin[\theta_0(t+c)] + \lambda_1^{(2)} \sin 2[\theta_0(t+c)] + \ldots, \\ \lambda_2 = (\lambda_2) + \lambda_2^{(0)}(t+c) + \lambda_2^{(1)} \sin[\theta_0(t+c)] + \lambda_2^{(2)} \sin 2[\theta_0(t+c)] + \ldots, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \lambda_{k-1} = (\lambda_{k-1}) + \lambda_{k-1}^{(0)}(t+c) + \lambda_{k-1}^{(1)} \sin[\theta_0(t+c)] + \lambda_{k-1}^{(2)} \sin 2[\theta_0(t+c)] + \ldots, \end{cases}$$

where c, (λ_1), (λ_2), $\ldots$, (λ_{k-1}), Λ_1, Λ_2, $\ldots$, Λ_{k-1}, and one other involved in Θ_0, Θ_1, $\ldots$, θ_0, $\lambda_1^{(0)}$, $\lambda_1^{(1)}$, $\ldots$, complete the number of $2k$ arbitrary constants. If then we take formulas of transformation as follows, replacing

$$(27)\quad \begin{cases} \Theta & \text{by } \Theta_0 + \Theta_1 \cos\theta + \Theta_2 \cos 2\theta + \ldots, \\ \Lambda_1 & \text{by } \Lambda_1, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \\ \Lambda_{k-1} & \text{by } \Lambda_{k-1}, \\ \theta & \text{by } \theta + \theta_1 \sin\theta + \theta_2 \sin 2\theta + \ldots, \\ \lambda_1 & \text{by } \lambda_1 + \lambda_1^{(1)} \sin\theta + \lambda_1^{(2)} \sin 2\theta + \ldots, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \\ \lambda_{k-1} & \text{by } \lambda_{k-1} + \lambda_{k-1}^{(1)} \sin\theta + \lambda_{k-1}^{(2)} \sin 2\theta + \ldots, \end{cases}$$

the terms of F involving θ as the argument will disappear from it, their mean values going over into the non-periodic term — B.

We shall now have a new set of angular variables θ, λ_1, λ_2, $\ldots$, λ_{k-1} and it will be pertinent to inquire what are the linear variables conjugate to them, so that we may still have differential equations of the canonical form. Taking the two equations of (27) which define Θ and θ in terms of the new variables, let us suppose that the latter are Θ_0 and θ' (an accent is applied to the latter for the sake of discrimination). Then we have

$$(28)\quad \begin{cases} \Theta = \Theta_0 + \Theta_1 \cos\theta' + \Theta_2 \cos 2\theta' + \ldots, \\ \theta = \theta' + \theta_1 \sin\theta' + \theta_2 \sin 2\theta' + \ldots, \end{cases}$$

Then, by the substitution of these values in the two differential equations of (16) which determine Θ and θ, we get

$$(29)\qquad \begin{cases} \dfrac{\partial\theta}{\partial\theta_0}\dfrac{d\theta_0}{dt}+\dfrac{\partial\theta}{\partial\theta'}\dfrac{d\theta'}{dt}= \dfrac{\partial F}{\partial\theta_0}\dfrac{\partial\theta_0}{\partial\theta}+\dfrac{\partial F}{\partial\theta'}\dfrac{\partial\theta'}{\partial\theta}, \\ \dfrac{\partial\theta}{\partial\theta_0}\dfrac{d\theta_0}{dt}+\dfrac{\partial\theta}{\partial\theta'}\dfrac{d\theta'}{dt}=-\dfrac{\partial F}{\partial\theta_0}\dfrac{\partial\theta_0}{\partial\theta}-\dfrac{\partial F}{\partial\theta'}\dfrac{\partial\theta'}{\partial\theta}. \end{cases}$$

By multiplying these equations by the proper factors, and putting Δ for the functional determinant or Jacobian:

$$(30)\qquad \Delta=\frac{\partial\theta}{\partial\theta_0}\frac{\partial\theta}{\partial\theta'}-\frac{\partial\theta}{\partial\theta'}\frac{\partial\theta}{\partial\theta_0},$$

we have

$$(31)\qquad \Delta\frac{d\theta_0}{dt}=\frac{\partial F}{\partial\theta'},\quad \Delta\frac{d\theta'}{dt}=-\frac{\partial F}{\partial\theta_0}.$$

But

$$(32)\qquad \begin{aligned}\Delta=&\left[1+\frac{\partial\theta_1}{\partial\theta_0}\cos\theta'+\frac{\partial\theta_2}{\partial\theta_0}\cos 2\theta'+\ldots\right]\left[1+\theta_1\cos\theta'+2\theta_2\cos 2\theta'+\ldots\right]\\ &+\left[\theta_1\sin\theta'+2\theta_2\sin 2\theta'+\ldots\right]\left[\frac{\partial\theta_1}{\partial\theta_0}\sin\theta'+\frac{\partial\theta_2}{\partial\theta_0}\sin 2\theta'+\ldots\right].\end{aligned}$$

According to the theorem of Poisson, Δ is independent of t, or what in this case amounts to the same thing, of θ'. Hence, in computing its value, we have regard only to the absolute term. Thus

$$(33)\qquad \begin{aligned}\Delta&=1+\tfrac{1}{2}\left[\theta_1\frac{\partial\theta_1}{\partial\theta_0}+2\theta_2\frac{\partial\theta_2}{\partial\theta_0}+\ldots+\theta_1\frac{\partial\theta_1}{\partial\theta_0}+2\theta_2\frac{\partial\theta_2}{\partial\theta_0}+\ldots\right]\\ &=1+\tfrac{1}{2}\frac{\partial}{\partial\theta_0}\left[\theta_1\theta_1+2\theta_2\theta_2+3\theta_3\theta_3+\ldots\right].\end{aligned}$$

Then, if we adopt a new variable Θ' in place of Θ_0 such that

$$(34)\qquad \theta'=\int\Delta\, d\theta_0=\theta_0+\tfrac{1}{2}\left[\theta_1\theta_1+2\theta_2\theta_2+3\theta_3\theta_3+\ldots\right],$$

equations (31) will be transformed into:

$$(35)\qquad \frac{d\theta'}{dt}=\frac{\partial F}{\partial\theta'},\quad \frac{d\theta'}{dt}=-\frac{\partial F}{\partial\theta'},$$

which have the canonical form. As to the remaining linear variables Λ_1, Λ_2, ..., Λ_{k-1}, which are identical with the former variables denoted by the same symbols, it is evident that they remain the conjugates of the new variables λ_1, λ_2, ..., λ_{k-1}.

VII.

As it is somewhat difficult to discover the linear transformation of variables required to pass from the set

$$\left.\begin{matrix}L_1, & L_2, & \ldots, & L_k\\ l_1, & l_2, & \ldots, & l_k\end{matrix}\right\}\text{ to the set }\left.\begin{matrix}\theta, & \Lambda_1, & \Lambda_2, & \ldots, & \Lambda_{k-1}\\ \theta, & \lambda_1, & \lambda_2, & \ldots, & \lambda_{k-1}\end{matrix}\right\},$$

suppose we adhere to the use of the first set. Then we have the differential equations

$$\frac{d\Theta}{dt} = \frac{\partial [F]}{\partial \theta}, \quad \frac{d\theta}{dt} = -\frac{\partial [F]}{\partial \Theta}, \tag{36}$$

$$\frac{dL_i}{dt} = j_i \frac{\partial [F]}{\partial \theta}, \quad \frac{dl_i}{dt} = -\frac{\partial [F]}{\partial L_i}, \qquad (i = 1, 2, \ldots, k). \tag{37}$$

Then

$$L_i = (L_i) + j_i \Theta, \qquad (i = 1, 2, \ldots, k). \tag{38}$$

where the (L_i) are arbitrary constants. By means of these equations the linear variables L_i may be eliminated from the right members of all the differential equations (36) and (37), and thus be replaced by the single variable Θ. Thus the mentioned right members become functions only of the two variables Θ and θ; and, after Θ and θ are found in terms of t from the integration of (36), the L_i will be found from (38), and the l_i from a quadrature operated on the second equations of (37).

Then we can suppose that the integrals of (36) and (37) are written thus:

$$\left\{\begin{aligned}
\Theta &= \Theta \cos [\theta_0(t + c)] + \Theta_2 \cos 2 [\theta_0(t + c)] + \ldots, \\
L_1 &= (L_1) + j_1 \Theta_1 \cos [\theta_0(t + c)] + j_1 \Theta_2 \cos 2 [\theta_0(t + c)] + \ldots, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
L_k &= (L_k) + j_k \Theta_1 \cos [\theta_0(t + c) + j_k \Theta_2 \cos 2 [\theta_0](t + c)] + \ldots, \\
\theta &= \theta_0(t + c) + \theta_1 \sin [\theta_0(t + c)] + \theta_2 \sin 2 [\theta_0(t + c) + \ldots, \\
l_1 &= (l_1) + l_1^{(0)}(t + c) + l_1^{(1)} \sin [\theta_0(t + c)] + l_1^{(2)} \sin 2 [\theta_0(t + c)] + \ldots, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
l_k &= (l_k) + l_k^{(0)}(t + c) + l_k^{(1)} \sin [\theta_0(t + c)] + l_k^{(2)} \sin 2 [\theta_0(t + c)] + \ldots,
\end{aligned}\right. \tag{39}$$

where the (L_1), (L_2), ..., (L_k), (l_1), (l_2), ..., (l_k) are arbitrary constants. The latter set are not independent as they must satisfy the relation:

$$j_1 (l_1) + j_2 (l_2) + \ldots + j_k (l_k) = 0.$$

As there is the additional arbitrary constant c, the number of independent constants is $2k$ as it should be. The coefficients θ_0, $l_i^{(0)}$, $l_i^{(1)}$, $l_i^{(2)}$, ..., which may be regarded as functions of the (L_i), satisfy the relations:

$$\begin{aligned}
j_1 l_1^{(0)} + j_2 l_2^{(0)} + \ldots + j_k l_k^{(0)} &= \theta_0, \\
j_1 l_1^{(1)} + j_2 l_2^{(1)} + \ldots + j_k l_k^{(1)} &= \theta_1, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot
\end{aligned}$$

The formulas of transformation are then:

$$\left\{\begin{array}{l}
\text{Replace } L_1 \text{ by } L_1 + j_1 \Theta_1 \cos \theta + j_1 \Theta_2 \cos 2\theta + \ldots, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\text{replace } L_k \text{ by } L_k + j_k \Theta_1 \cos \theta, + j_k \Theta_2 \cos 2\theta + \ldots, \\
\text{replace } l_1 \text{ by } l_1 + l_1^{(1)} \sin \theta + l_1^{(2)} \sin 2\theta + \ldots, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\text{replace } l_k \text{ by } l_k + l_k^{(1)} \sin \theta + l_k^{(2)} \sin 2\theta + \ldots.
\end{array}\right. \tag{40}$$

We have now to find what linear variables are conjugate to the new angular variables $l_1, l_2, \ldots, l_k$. They are discovered immediately from a comparison of (34) and (38). As, from (34), it appears that Θ_0 the arbitrary constant, which may be conceived as annexed to the series for Θ, is not the element conjugate to the angular element $\theta_0(t+c)$, but that the expression

$$\tfrac{1}{2}[\theta_1\theta_1 + 2\theta_2\theta_2 + 3\theta_3\theta_3 + \ldots]$$

must be added to it to produce the required conjugate, it is plain from (38) that, after the transformation, the new L_i is no longer the exact conjugate of l_i, but that we have for that element the value

$$L_i + \tfrac{1}{2}j_i[\theta_1\theta_1 + 2\theta_2\theta_2 + 3\theta_3\theta_3 + \ldots] \qquad (i = 1, 2, \ldots, k). \tag{41}$$

VIII.

In making one of Delaunay's transformations it is not absolutely necessary that we should employ the linear variables L_i, which are the conjugates of the angular variables l_i, in the development of the various series needed; we may use any others connected with the former by known relations. Then equation (41) will inform us, at any stage of the transformations, what function the conjugates of the angular variables are of the used linear variables. Thus, in making his developments in the lunar theory, Delaunay has not used the elements he calls L, G, H, but has substituted for them others which he names a, e, γ.

Let us suppose that the new set of linear elements we determine to use are denoted by the symbols $e_1, e_2, \ldots, e_k$. Then, in order to form the equivalents for the dl_i/dt, it will be necessary to know the values of the partial derivatives $\partial e_i/\partial L_j$ in terms of the e_i. The number of these derivatives is k^2, and we shall have as many equations for determining them. Having the L_i in terms of the e_i, the general form of these equations will be:

$$\frac{\partial L_i}{\partial e_1}\frac{\partial e_1}{\partial L_j} + \frac{\partial L_i}{\partial e_2}\frac{\partial e_2}{\partial L_j} + \ldots + \frac{\partial L_i}{\partial e_k}\frac{\partial e_k}{\partial L_j} = 0 \text{ or } 1, \tag{42}$$

according as i and j are different or are the same. These equations divide themselves into k groups each containing k equations; each group serving to determine the k partial derivatives of the e_i with reference to one of the L_i which we denote by L_j. The functional determinant of each group of equations is the same, being the Jacobian of the variables $L_1, L_2, \ldots, L_k$ with reference to the variables $e_1, e_2, \ldots, e_k$. Calling this determinant Δ, we shall have

$$\frac{\partial e_i}{\partial L_j} = \frac{1}{\Delta}\frac{\partial \Delta}{\partial\left(\dfrac{\partial L_j}{\partial e_i}\right)}. \tag{43}$$

We shall much abbreviate the calculation of these partial derivatives by agreeing to compute $\partial e_i/\partial L_j$ not directly but by a correction to be added to the value it had before the transformation was made. This correction will be denoted by $D(\partial e_i/dL_j)$. Now the transformation being that made to remove from F all the periodic terms having the argument

$$j_1 l_1 + j_2 l_2 + \ldots + j_k l_k,$$

suppose that, in this transformation, we put

$$K = \tfrac{1}{2}[\theta_1\Theta_1 + 2\theta_2\Theta_2 + 3\theta_3\Theta_3 + \ldots], \tag{44}$$

and also

$$-K_m = \frac{\partial K}{\partial e_1}\left(\frac{\partial e_1}{\partial L_m}\right) + \frac{\partial K}{\partial e_2}\left(\frac{\partial e_2}{\partial L_m}\right) + \ldots + \frac{\partial K}{\partial e_k}\left(\frac{\partial e_k}{\partial L_m}\right), \tag{45}$$

where $(\partial e_i/\partial L_m)$ denotes the value of the partial derivative before the transformation; then we shall have the group of linear equations:

$$\left\{\begin{array}{l}
\dfrac{\partial L_1}{\partial e_1} D\left(\dfrac{\partial e_1}{\partial L_m}\right) + \dfrac{\partial L_1}{\partial e_2} D\left(\dfrac{\partial e_2}{\partial L_m}\right) + \ldots + \dfrac{\partial L_1}{\partial e_k} D\left(\dfrac{\partial e_k}{\partial L_m}\right) = j_1 K_m, \\
\dfrac{\partial L_2}{\partial e_1} D\left(\dfrac{\partial e_1}{\partial L_m}\right) + \dfrac{\partial L_2}{\partial e_2} D\left(\dfrac{\partial e_2}{\partial L_m}\right) + \ldots + \dfrac{\partial L_2}{\partial e_k} D\left(\dfrac{\partial e_k}{\partial L_m}\right) = j_2 K_m, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\dfrac{\partial L_k}{\partial e_1} D\left(\dfrac{\partial e_1}{\partial L_m}\right) + \dfrac{\partial L_k}{\partial e_2} D\left(\dfrac{\partial e_2}{\partial L_m}\right) + \ldots + \dfrac{\partial L_k}{\partial e_k} D\left(\dfrac{\partial e_k}{\partial L_m}\right) = j_k K_m.
\end{array}\right. \tag{46}$$

Put now

$$P_i = j_1 \frac{\partial \Delta}{\partial\left(\dfrac{\partial L_1}{\partial e_i}\right)} + j_2 \frac{\partial \Delta}{\partial\left(\dfrac{\partial L_2}{\partial e_i}\right)} + \ldots + j_k \frac{\partial \Delta}{\partial\left(\dfrac{\partial L_k}{\partial e_i}\right)}. \tag{47}$$

Then, evidently,

$$D\left(\frac{\partial e_i}{\partial L_m}\right) = \frac{P_i K_m}{\Delta}. \tag{48}$$

Thus the correction for each partial derivative, caused by a given transformation, is reduced to the product of three factors, one of which is a function of i, another a function of m, and the third independent of either. The elements e_i, before the performance of any transformation, may be what we choose. Delaunay chose his a, e, γ so that they were connected with L, G, H, the conjugates of the angular variables l, g, h, by the relations:

$$L = \sqrt{\mu a}, \quad G = L\sqrt{1-e^2}, \quad H = G(1-2\gamma^2).$$

Part II.

Application of Delaunay's Method to the Minor Planet of the Hecuba Type.

IX.

Delaunay's lunar theory affords a plentiful assortment of the transformations just discussed, but their application in a case of planetary motion gives rise to more complex expressions. In the lunar theory it is possible to expand all coefficients in power series of all the parameters involved; but, in a planetary theory where a, the ratio of the mean distances, is a considerable fraction, it is necessary to introduce the functions of a usually denoted by the symbol $b_s^{(i)}$, as also their derivatives with respect to a. It may therefore be profitable to give as simple an illustration as possible of these transformations where $b_s^{(i)}$ must be used.

Let Jupiter be supposed to describe a circular orbit about the Sun, while a small planet, without mass, describes an orbit in the same plane. Let the radius of Jupiter's orbit be taken as the linear unit; denote its longitude by $\varepsilon' + n't$, and the masses severally of the Sun and Jupiter by m_0 and m'. Let a, e, l, and g be the mean distance, eccentricity, mean anomaly, and longitude of the perihelion of the small planet, and r and v its radius and true anomaly. Put

$$\gamma = g - \varepsilon' - n't, \quad L = \sqrt{m_0 a}, \quad \Gamma = \sqrt{m_0 a (1 - e^2)}.$$

The function we have denoted by F will have the expression:

$$F = \frac{m_0}{2a} + n'\Gamma + m' \{ [1 - 2r \cos (v + \gamma) + r^2]^{-\frac{1}{2}} - r \cos (v + \gamma) \}, \tag{49}$$

where r and v are to be eliminated through the equations:

$$r \cos v = a (\cos u - e), \quad r \sin v = a \sqrt{1 - e^2} \sin u, \quad u - e \sin u = l.$$

The position of the small planet will be known when we know L, Γ, l, γ. The differential equations for determining the latter are:

$$\left\{ \begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l}, & \frac{dl}{dt} &= -\frac{\partial F}{\partial L}, \\ \frac{d\Gamma}{dt} &= \frac{\partial F}{\partial \gamma}, & \frac{d\gamma}{dt} &= -\frac{\partial F}{\partial \Gamma}. \end{aligned} \right. \tag{50}$$

X.

In order to give an illustration of the transformations named by Delaunay *operations*, let us select from the periodic development of F, which, from (49), plainly has the form $\Sigma A_{i,i'} \cos (il + i'\gamma)$, all the terms

having the argument $\theta = l + 2\gamma$. These will be terms of long period in case the small planet has a mean motion nearly double that of Jupiter, which case has been extensively discussed by astronomers, such a minor planet being called of the Hecuba type. Taking θ as one of the angular elements, we see that we can adopt γ as the other, and thus shall have $l = \theta - 2\gamma$. In order to obtain $[F]$ from F we have the equation:

$$(51) \qquad [F] = \frac{m_0}{2a} + n'\Gamma + \frac{1}{\pi}\int_0^\pi \frac{m' d\gamma}{\sqrt{1 - 2r\cos(v+\gamma) + r^2}},$$

remembering that r and v are now the same functions of $\theta - 2\gamma$ they were before of l. The last term of F in (49) is here omitted as it contributes nothing to $[F]$. Put

$$(52) \quad [1 - 2r\cos(v+\gamma) + r^2]^{-\frac{1}{2}} = \tfrac{1}{2}B^{(0)} + B^{(1)}\cos(v+\gamma) + B^{(2)}\cos 2(v+\gamma) + \ldots,$$

where $B^{(i)}$ is the same function of r that $b_{\frac{1}{2}}^{(i)}$ is of a. In order that this series may be convergent it is necessary that $a(1+e) < 1$. Let us put

$$(53) \qquad A^{(i)} = \frac{1}{\pi}\int_0^\pi B^{(2i)} \cos i(2v - l)\, dl.$$

Then we have

$$(54) \quad [F] = \frac{m_0}{2a} + n'\sqrt{m_0 a(1-e^2)} + m'[\tfrac{1}{2}A^{(0)} + A^{(1)}\cos\theta + A^{(2)}\cos 2\theta + \ldots].$$

The investigation will be facilitated if we now make a slight change in the dependent variables employed so that they have the following equivalents:

$$(55) \qquad \begin{cases} \Theta = \sqrt{m_0 a}, & \Gamma = \sqrt{m_0 a}\,[2 - \sqrt{1-e^2}], \\ \theta = l + 2g - 2\varepsilon' - 2n't, & \gamma = \varepsilon + n't - g. \end{cases}$$

Then the differential equations determining the formulas of transformation are:

$$(56) \qquad \begin{cases} \dfrac{d\Theta}{dt} = \dfrac{\partial[F]}{\partial\theta}, & \dfrac{d\theta}{dt} = -\dfrac{\partial[F]}{\partial\Theta}, \\ \dfrac{d\Gamma}{dt} = 0, & \dfrac{d\gamma}{dt} = -\dfrac{\partial[F]}{\partial\Gamma}. \end{cases}$$

Of these equations the integral of the third, $\Gamma =$ a constant, furnishes the relation:

$$(57) \qquad a = \alpha[2 - \sqrt{1-e^2}]^{-2} = \alpha[1 - e^2 + \tfrac{1}{2}e^4 - \tfrac{1}{4}e^6 + \tfrac{3}{32}e^8 - \ldots],$$

α being a constant. By means of this relation the variable a may be eliminated from $[F]$ which thus will contain but two variables, e and θ. The equations (56) have the canonical form, but we prefer to discard the variable Γ and to use e in its stead. Supposing then that $[F]$ is made a

function of the variables e and θ, the differential equations for the latter are:

$$
(58) \qquad \left\{
\begin{aligned}
\frac{de}{dt} &= -\frac{1}{\sqrt{m_0 a}} \frac{\sqrt{1-e^2}\,[2-\sqrt{1-e^2}]^2}{e} \frac{\partial [F]}{\partial \theta}, \\
\frac{d\theta}{dt} &= \frac{1}{\sqrt{m_0 a}} \frac{\sqrt{1-e^2}\,[2-\sqrt{1-e^2}]^2}{e} \frac{\partial [F]}{\partial e},
\end{aligned}
\right.
$$

where the factor

$$
\frac{\sqrt{1-e^2}\,[2-\sqrt{1-e^2}]^2}{e} = \frac{1}{e}[1 + \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4 - \tfrac{3}{16} e^6 - \tfrac{17}{128} e^8 - \ldots].
$$

These equations form a group to be integrated by themselves. After this integration is accomplished, γ is derived through a quadrature of the equation:

$$
\frac{d\gamma}{dt} = -\frac{\partial [F]}{\partial \Gamma}.
$$

In this equation $[F]$ is a function of Θ and Γ, but we have preferred to write it as a function of a and e; thus:

$$
\frac{\partial [F]}{\partial \Gamma} = \frac{\partial [F]}{\partial a}\frac{\partial a}{\partial \Gamma} + \frac{\partial [F]}{\partial e}\frac{\partial e}{\partial \Gamma}.
$$

But

$$
\theta = \frac{\sqrt{m_0 a}}{2-\sqrt{1-e^2}}, \qquad \Gamma = \sqrt{m_0 a},
$$

$$
a = \frac{\Gamma^2}{m_0}, \qquad \sqrt{1-e^2} = 2 - \frac{\Gamma}{\theta}.
$$

Consequently

$$
\frac{\partial a}{\partial \Gamma} = 2\sqrt{\frac{a}{m_0}}, \qquad \frac{\partial e}{\partial \Gamma} = \frac{1}{\sqrt{m_0 a}} \frac{\sqrt{1-e^2}\,[2-\sqrt{1-e^2}]}{e}.
$$

Remembering that, with our adopted linear unit, $a' = 1$, $n' = \sqrt{m_0 + m'}$, we have:

$$
\frac{1}{n'}\frac{d\gamma}{dt} = -\frac{1}{m_0\sqrt{\left(1+\frac{m'}{m_0}\right)a}} \left[2a\frac{\partial [F]}{\partial a} + \frac{\sqrt{1-e^2}\,[2-\sqrt{1-e^2}]}{e}\frac{\partial [F]}{\partial e}\right].
$$

But, adopting, as before, g for the longitude of the perihelion, this is more simply:

$$
(59) \qquad \left\{
\begin{aligned}
\frac{1}{n'}\frac{dg}{dt} &= \frac{1}{\sqrt{\left(1+\frac{m'}{m_0}\right)a}} \left[2a\frac{\partial R}{\partial a} + \frac{\sqrt{1-e^2}\,[2-\sqrt{1-e^2}]}{e}\frac{\partial R}{\partial e}\right] \\
&= \frac{2}{\sqrt{\left(1+\frac{m'}{m_0}\right)a}}\, a\frac{\partial R}{\partial a} + \frac{1}{2-\sqrt{1-e^2}}\frac{1}{n'}\frac{d\theta}{dt} \\
&\quad - \frac{1}{\sqrt{\left(1+\frac{m'}{m_0}\right)a}} [2-\sqrt{1-e^2}]^2 + \frac{2}{2-\sqrt{1-e^2}},
\end{aligned}
\right.
$$

where

$$
R = \frac{m'}{m_0}\frac{1}{\pi}\int_0^\pi \frac{d\gamma}{\sqrt{1-2r\cos(v-\gamma)+r^2}}.
$$

XI.

In an application like the present, where the periodic developments of the various quantities are always tardily convergent, it is nearly impossible to give literal expressions for the coefficients. And, even if we consent to give to each coefficient its numerical value at once, the work of multiplying such periodic series together is very embarrassing, and the process easily leads to the commission of errors. Hence we adopt the method of substituting for each quantity involved the special values of it at equal intervals in the motion of the independent variable through the semicircumference. With this method of treatment it is necessary to separate the cases of non-libration and libration.

It is always an advantage in computation to have the numbers dealt with independent of any linear and temporal units. To this end let us substitute for the independent variable t, the variable $\tau = \varepsilon' + n't$ or the longitude of Jupiter; also we put

$$W = \frac{[F]}{m_0}, \qquad \nu = \frac{m'}{m_0}.$$

The coefficients of the periodic development of W are then absolute numbers. The equations which, with (57), we shall use for the elaboration of the problem, are the three following:

$$(60)\quad \left\{\begin{aligned} &W = \text{a constant}, \\ &\frac{d\tau}{d\theta} = \sqrt{(1+\nu)a}\,\frac{e}{\sqrt{(1-e^2}\,[2-\sqrt{1-e^2}]^2}\,\frac{1}{\frac{\partial W}{\partial e}}, \\ &\frac{dg}{d\tau} = \frac{2}{\sqrt{(1+\nu)a}}\,a\frac{\partial R}{\partial a} + \frac{1}{2-\sqrt{1-e^2}}\,\frac{d\theta}{d\tau} \\ &\qquad - \frac{1}{\sqrt{(1+\nu)a}}[2-\sqrt{1-e^2}]^i + \frac{2}{2-\sqrt{1-e^2}}; \end{aligned}\right.$$

W has the expression:

$$(61)\quad \left\{\begin{aligned} W &= \frac{1}{2a}[2-\sqrt{1-e^2}]^2 + \sqrt{(1+\nu)a}\,\frac{1-e^2+2\sqrt{1-e^2}}{3+e^2} + \nu\frac{1}{\pi}\int_0^\pi \frac{d\gamma}{\Delta} \\ &= \frac{1}{2a}[1 + e^2 + \tfrac{1}{2}e^4 + \tfrac{1}{4}e^6 + \tfrac{5}{32}e^8 + \tfrac{7}{64}e^{10} + \ldots] \\ &\quad + \sqrt{(1+\nu)a}\,[1 - e^2 + \tfrac{1}{4}e^4 - \tfrac{1}{8}e^6 + \tfrac{1}{64}e^8 - \tfrac{3}{128}e^{10} + \ldots] + R. \end{aligned}\right.$$

This equation contains as variables only e and θ; hence, since e should never be negative, the dependence of the two variables on each other may be shown graphically by taking e as the radius and θ as the angle in a system of polar coordinates. If we are given a pair of simultaneous values of e and θ, it is obvious that by their aid we can determine the constant

value of W. Desiring to ascertain at what points on the axis the curve passes we make in (61) in succession $\theta = 0°$ and $\theta = 180°$ and we get two equations of the forms:

$$(62)\qquad \begin{cases} D = M_1 e + M_2 e^2 + M_3 e^3 + M_4 e^4 + M_5 e^5 + \ldots, \\ D = -M_1 e + M_2 e^2 + M_3 e^3 + M_4 e^4 + M_5 e^5 + \ldots, \end{cases}$$

where D may be regarded as the arbitrary constant and the M are constants, being functions of α and ν. These equations are transcendental in e and are such that the positive roots of the one are equivalent to the negative roots of the other. If each has a positive real root continuous with the value of e which was used for the determination of the constant value of W, the variable θ generally moves through the whole range of real values. But, if the first equation has two positive real roots and the second none, θ will librate about the value $\theta = 0°$. But, if the second has two positive real roots and the first none, θ will librate about $\theta = 180°$. It will be seen that when $D = 0$ we have the limit separating non-libration from libration.

XII.

Case I.—*Non-libration.*—Here, as θ goes through the semicircumference, it can be employed as the independent variable. Then, in the first equation of (60), we assign to θ, in succession, a series of equidistant values covering the semicircumference. (Those used in our illustrative examples are 13 in number, viz., $\theta = 0°$, $\theta = 15°$, $\theta = 30°$, ..., $\theta = 180°$.) This procedure furnishes us with a like number of equations for determining the corresponding values of e. Solving these by the tentative process we have these values of e, and can apply to them the procedure of mechanical quadratures. Thus is obtained a general expression for e as a periodic function of θ involving only cosines.

As the next step these special values of e can be substituted in the right member of the second equation of (60). To the special values thus obtained for $d\tau/d\theta$ can be applied mechanical quadratures, and the resulting periodic series, involving only cosines of integral multiples of θ, can be integrated with respect to this variable. This integral may be put in the form:

$$(63)\qquad \theta_0(t + c) = \theta + \beta_1 \sin\theta + \beta_2 \sin 2\theta + \beta_3 \sin 3\theta + \ldots.$$

Knowing θ_0 we are now in possession of the period of the inequalities we are endeavoring to derive. The left member of this equation we shall designate as the time-argument, and, for brevity, denote it as ζ. In the next place we assign to ζ a series of equidistant values going from 0° to 180°, and, by a tentative process applied to (63), arrive at the corresponding values of θ

These corresponding values of θ can be substituted for θ in the expression of e as a periodic function of θ, and thus we shall have the values of e which correspond to the equidistant values of ζ. We can now readily derive the similar values of the two quantities $e \cos\theta$ and $e \sin\theta$. To these we apply mechanical quadratures and thus obtain the periodic developments of these quantities in terms of ζ.

As the last step in this work we can, through the last equation of (60), express $dg/d\tau$ as a function of e and θ, and, by the substitution of the special values of the latter variables, obtain the special values of $dg/d\tau$ which correspond to the equidistant values of ζ. To these apply mechanical quadratures and the periodic series for $dg/d\tau$ is obtained. This being integrated we have the series for g, and the solution of the problem is completed.

XIII.

Case II.—*Libration.*—Here we are shut off from the use of θ as an independent variable on account of its not going through the semicircumference. But this difficulty is surmounted by substituting for it another variable which does move continuously from $-\infty$ to $+\infty$. In order to ascertain, in the case of libration, the limiting values of θ we have to solve the simultaneous equations:

$$W = \text{a constant}, \qquad \frac{\partial W}{\partial e} = 0,$$

the unknowns being e and θ. That is to say, a value of θ must be found which will make the first equation have two equal roots for e. This can be done by a tentative process. If we assume θ too large, generally, we shall not be able to discover real values for e from the first equation; but, if θ is taken too small, we get two values real but unequal for e. These two conditions must be brought as close as possible until we discover the point of passage from one to the other. In our illustrative example we escape the necessity of this tentative process by assuming as one of the two fundamental elements of the example not the D of (62) but the amount of libration.

The amount of libration being thus either assumed or determined, let $\varkappa$ denote the limiting value of $\sin\theta$; we then can put

$$\sin\theta = \varkappa \sin\psi; \tag{64}$$

and the motion of ψ can be regarded as extending continuously from $-\infty$ to $+\infty$. Adopting the variable ψ for replacing θ, the second equation of (60) takes the form:

$$\frac{d\tau}{d\psi} = \sqrt{(1+\nu)a}\,\frac{e}{\sqrt{1-e^2}\,[2-\sqrt{1-e^2}]^2}\,\frac{\varkappa\cos\psi}{\sqrt{1-\varkappa^2\sin^2\psi}}\,\frac{1}{\dfrac{\partial W}{\partial e}}, \tag{65}$$

where the newly introduced radical must receive the sign of $\cos\theta$. We can now make ψ play the same rôle as θ did in Case I, and there is need of no further explanations.

XIV.

We attend now to the integration of equations (60). *The operation of Delaunay's lunar theory which is numbered* 23* *has great affinity with that here detailed, and the two may be compared.* He, it is true, has six variables to our four; but, in comparing, his γ should be made to vanish and his h then becomes indeterminate.

The periodic development of the reciprocal of the distance between two planets as a function of the time has been given by Leverrier to terms of the seventh order inclusive, and those of the eighth order have afterwards been added by M. Bouquet.† We avail ourselves of this development and adopt the mode of Leverrier for noting the coefficients except in the portion which is a function of e alone. We put $A_i^{(j)} = (1/i!)\, a^i d^i b_{\frac{1}{2}}^{(j)}/da^i$, $j = 0$ in the portion factored by $\cos 0\theta$, $j = 2$ in the portion factored by $\cos\theta$, $j = 4$ in the portion factored by $\cos 2\theta$ and so on; only the numerical factors are written since the A can easily be filled in as they always commence with $A_0^{(j)}$, and the lower index always increases by a unit in each step to the right. With Leverrier we put χ for $\frac{1}{2}e$. This then is the development of a'/Δ, preserving only the terms involving the integral multiples of θ as arguments:

$$
\begin{aligned}
\frac{a'}{\Delta} = {} & \frac{1}{2}A_0^{(0)} + [A_1^{(0)} + A_2^{(0)}]\chi^2 + \frac{3}{1}[A_3^{(0)} + A_4^{(0)}]\chi^4 + \frac{4.5}{1.2}[A_5^{(0)} + A_6^{(0)}]\chi^6 + \frac{5.6.7}{1.2.3}[A_7^{(0)} + A_8^{(0)}]\chi^8 \\
& + \Big\{ -[4+1]\chi + \Big[14+\frac{5}{2}-6-3\Big]\chi^3 + \Big[-\frac{5}{3}-\frac{53}{12}+\frac{34}{3}+5-16-10\Big]\chi^5 \\
& \qquad + \Big[\frac{271}{36}-\frac{203}{144}+\frac{19}{4}-\frac{49}{8}+20+\frac{25}{2}-50-35\Big]\chi^7 \Big\}\cos\theta \\
& + \Big\{ [22+7+1]\chi^2 + \Big[-\frac{596}{3}-\frac{212}{3}+8+16+4\Big]\chi^4 + \Big[\frac{1300}{3}+\frac{743}{3}-49 \\
& \qquad -112+2+45+15\Big]\chi^6 + \Big[-\frac{8312}{45}-\frac{500212}{45}-\frac{280}{9}+\frac{1228}{5} \\
& \qquad +0-236-16+140+56\Big]\chi^8 \Big\}\cos 2\theta \\
& + \Big\{ -\Big[134+\frac{93}{2}+10+1\Big]\chi^3 + \Big[\frac{4053}{2}+\frac{6289}{8}+107-\frac{117}{2}-32-5\Big]\chi^5 \\
& \qquad + \Big[-\frac{64177}{40}-\frac{384789}{80}-\frac{12123}{20}+\frac{6249}{8}+348-\frac{189}{2}-102-21\Big]\chi^7 \Big\}\cos 3\theta
\end{aligned}
\tag{66}
$$

* Mémoires de l'Académie des Sciences, Vol. XXVIII, p. 493.

† Annales de l'Observatoire de Paris, vols. I, XIX.

$$+\left\{\left[\frac{2570}{3}+\frac{932}{3}+80+13+1\right]\chi^4+\left[-\frac{275528}{15}-\frac{109972}{15}-1676+48-176\right.\right.$$
$$\left.+54+6\right]\chi^6+\left[\frac{6259444}{45}+\frac{592976}{9}+\frac{234338}{15}-\frac{44756}{15}-\frac{10036}{3}-\frac{1432}{3}$$
$$\left.\left.+400+196+28\right]\chi^8\right\}\cos 4\theta$$

$$+\left\{-\left[\frac{33797}{6}+\frac{50345}{24}+\frac{1795}{3}+\frac{245}{2}+16+1\right]\chi^5+\left[\frac{5652235}{36}+\frac{9141589}{144}+\frac{210217}{12}\right.\right.$$
$$\left.\left.+\frac{48985}{24}-\frac{2020}{3}-\frac{775}{2}-82-7\right]\chi^7\right\}\cos 5\theta$$

(66) $$+\left\{\left[\frac{188616}{5}+\frac{71499}{5}+4357+1024+174+19+1\right]\chi^6+\left[-\frac{45378432}{35}\right.\right.$$
$$\left.\left.-\frac{3695460}{7}-\frac{803616}{5}-\frac{150828}{5}+80+2156+720+116+8\right]\chi^8\right\}\cos 6\theta$$

$$+\left\{-\frac{46064791}{2520}+\frac{70738549}{720}+\frac{1880921}{60}+\frac{193921}{24}+\frac{4844}{3}+\frac{469}{2}-22-1\right\}\chi^7\quad\cos 7\theta$$

$$+\left\{\frac{552146674}{315}+\frac{213998824}{315}+\frac{2018552}{9}+\frac{926516}{15}+\frac{41380}{3}+\frac{7192}{3}\right.$$
$$\left.+304+25+1\right\}\chi^8\quad\cos 8\theta$$

From this expression we must eliminate the variable a by means of its value in terms of e given by (57). We put

$$p=[2-\sqrt{1-e^2}]^{-2}-1.$$

Let Leverrier's coefficient of $\cos j\theta$ be denoted thus:

$$c_0 A_0^{(2j)}+c_1 A_1^{(2j)}+c_2 A_2^{(2j)}+\ldots=\Sigma c_i A_i^{(2j)},$$

where c_i is a function of e. Denoting the similar coefficient, after the variable a has been eliminated through (57), by $\Sigma f_i A_i^{(2j)}$, we evidently have

$$f_i=(1+p)^i c_i+\frac{i}{1}(1+p)^{i-1}pc_{i-1}+\frac{i(i-1)}{1.2}(1+p)^{i-2}p^2c_{i-2}+\ldots.$$

By means of this formula we obtain the following expression, in which a, the argument of the various quantities $A_i^{(2j)}$, is the constant a of (57):

$$\frac{a'}{\Delta}=\frac{1}{2}+[0-1+1]\chi^2+[0+0-8-9+3]\chi^4+[0+0+48+100+0-50+10]\chi^6$$
$$+[0-4-160-600-224+720+240-245+35]\chi^8$$

(67) $$+\left\{-[4+1]\chi+\left[14+\frac{45}{2}+2-3\right]\chi^3+\left[-\frac{5}{3}-\frac{1325}{12}-\frac{218}{3}+65+32-10\right]\chi^5\right.$$
$$\left.+\left[\frac{271}{36}+\frac{33829}{144}+\frac{6209}{12}-\frac{3025}{8}-796+\frac{105}{2}+190-35\right]\chi^7\right\}\quad\cos\theta$$

$$
\begin{aligned}
&+\Big\{\;[22+7+1]\chi^2+\Big[-\frac{596}{3}-\frac{560}{3}-56+4+4\Big]\chi^4+\Big[\frac{1300}{3}+1557+\frac{3517}{3}\\
&\qquad+248-222-35+15\Big]\chi^6+\Big[-\frac{8312}{45}-\frac{740632}{45}-\frac{95032}{9}-\frac{19112}{5}\\
&\qquad\qquad+3680+2184-496-280+56\Big]\chi^8\Big\}\cos 2\theta\\
&+\Big\{-\Big[134+\frac{93}{2}+10+1\Big]\chi^3+\Big[\frac{4053}{2}+\frac{12065}{8}+559+\frac{147}{2}-16-5\Big]\chi^5\\
&\qquad+\Big[-\frac{64177}{40}-\frac{1400349}{80}-\frac{248943}{20}-\frac{26439}{8}+612+\frac{971}{2}+18-21\Big]\chi^7\Big\}\cos 3\theta\\
&+\Big\{\;\Big[\frac{2570}{3}+\frac{932}{3}+80+13+1\Big]\chi^4+\Big[-\frac{275528}{15}-\frac{60004}{5}-\frac{14404}{3}-1068-48\\
&\qquad+34+6\Big]\chi^6+\Big[\frac{6259444}{45}+\frac{1602224}{9}+\frac{1180994}{15}+\frac{630004}{15}\\
(67)\qquad&\qquad\qquad+\frac{11372}{3}-\frac{7912}{3}-800+28+28\Big]\chi^8\Big\}\cos 4\theta\\
&+\Big\{-\Big[\frac{33797}{6}+\frac{50345}{24}+\frac{1795}{3}+\frac{245}{2}+16+1\Big]\chi^5+\Big[\frac{5652235}{36}+\frac{13594381}{144}\\
&\qquad\qquad+\frac{469037}{12}+\frac{256585}{24}+\frac{4628}{3}-\frac{95}{2}-58-7\Big]\chi^7\Big\}\cos 5\theta\\
&+\Big\{\;\Big[\frac{188616}{5}+\frac{71499}{5}+4357+1024+174+19+1\Big]\chi^6+\Big[-\frac{45378432}{35}\\
&\qquad-\frac{5152104}{7}-\frac{1549888}{5}-\frac{473688}{5}-19088-1704+240+88+8\Big]\chi^8\Big\}\cos 6\theta\\
&+\Big\{-\frac{46064791}{2520}+\frac{70738549}{720}+\frac{1880921}{60}+\frac{191863}{24}+\frac{4844}{3}+\frac{469}{2}-22-1\Big\}\chi^7\cos 7\theta\\
&+\Big\{\frac{552146674}{315}+\frac{213998824}{315}+\frac{2018552}{9}+\frac{926516}{15}+\frac{41380}{3}+\frac{7192}{3}+304+25+1\Big\}\chi^8\cos 8\theta
\end{aligned}
$$

In forming the value of $dg/d\tau$ we need to know the derivative of the foregoing expression with respect to α. By noting the equation:

$$\alpha\frac{\partial A_i^{(2j)}}{\partial\alpha}=iA_i^{(2j)}+(i+1)\,A_{i+1}^{(2j)},$$

and changing our mode of noting the coefficients so that the number first given is the coefficient of $A_1^{(2j)}$ instead of $A_0^{(2j)}$, we have:

$$
\begin{aligned}
\alpha\frac{\partial\frac{a'}{\Delta}}{\partial\alpha}&=\tfrac{1}{2}+[-1+0+3]\chi^2+[0-16-51-24+15]\chi^4+[0+96+444+400-250\\
&\qquad-240+70]\chi^6+[-4-328-2280-3296+2480+5760-35-1680+315]\chi^8\\
(68)\qquad&+\Big\{-[5+2]\chi+\Big[\frac{73}{2}+49-3-12\Big]\chi^3+\Big[-\frac{1345}{12}-\frac{2197}{6}-23+388+110-60\Big]\chi^5\\
&\qquad+\Big[\frac{34913}{144}+\frac{108337}{72}+\frac{3343}{8}-\frac{9093}{2}-\frac{7435}{2}+1455+1085-280\Big]\chi^7\Big\}\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\{ [29+16+3]\chi^2+\left[-\frac{1156}{3}-\frac{1456}{3}-156+32+20\right]\chi^4+\left[\frac{5971}{3}+\frac{16376}{3}\right.\\
&\qquad\left.+4261+104-1285-120+105\right]\chi^6+\left[-\frac{83216}{5}-\frac{270176}{5}\right.\\
&\qquad\qquad\left.-\frac{647168}{15}-\frac{2848}{5}+29320+10128-5432-1792+504\right]\chi^8\Big\}\cos 2\theta\\
&+\Big\{-\left[\frac{361}{2}+113+33+4\right]\chi^3+\left[\frac{28277}{8}+\frac{16537}{4}+\frac{3795}{2}+230-105-30\right]\chi^5\\
&\qquad+\left[-\frac{1528703}{80}-\frac{2396121}{40}-\frac{1890243}{40}-\frac{21543}{2}+\frac{11975}{2}\right.\\
&\qquad\qquad\left.+3021-21-168\right]\chi^7\Big\}\cos 3\theta\\
&+\Big\{ \left[\frac{3502}{3}+\frac{2344}{3}+279+56+5\right]\chi^4+\left[-\frac{91108}{3}-\frac{504064}{15}-17608-4464\right.\\
&\qquad\left.-70+240+42\right]\chi^6+\left[\frac{14270564}{45}+\frac{23108204}{45}+\frac{1810998}{5}+\frac{2747456}{15}\right.\\
&\qquad\qquad\left.+\frac{17300}{3}-20624-5404+448+252\right]\chi^8\Big\}\cos 4\theta\\
&+\Big\{-\left[\frac{185533}{24}+\frac{64705}{12}+\frac{4325}{2}+554+85+6\right]\chi^5+\left[\frac{36203321}{144}+\frac{19222825}{72}\right.\\
&\qquad\left.+\frac{1194659}{8}+\frac{293609}{6}+\frac{44855}{6}-633-455-56\right]\chi^7\Big\}\cos 5\theta\\
&+\Big\{ \left[52023+\frac{186568}{5}+16143+4792+965+120+7\right]\chi^6\\
&\qquad+\left[-\frac{71138952}{35}-\frac{73219472}{35}-\frac{6070728}{5}-\frac{2276512}{5}-103960-8784\right.\\
&\qquad\qquad\left.+2296+768+72\right]\chi^8\Big\}\cos 6\theta\\
&-\Big\{\frac{587299425}{5040}+\frac{93309601}{360}+\frac{4721157}{40}+\frac{230615}{6}+\frac{55475}{6}+1539+161+8\Big\}\chi^7\cos 7\theta\\
&+\Big\{\frac{766145498}{315}+\frac{569296288}{315}+\frac{12872308}{15}+\frac{4533664}{15}+\frac{242860}{3}+16208\\
&\qquad+2303+208+9\Big\}\chi^8\cos 8\theta.
\end{aligned}
\tag{68}
$$

It is desirable to have the means of verifying these truncated developments of a'/Δ derived from the work of Leverrier and M. Bouquet. In fact, by the application of the first of two following theorems, an error has been found in M. Bouquet's expression for (225); in the coefficient of K_3, $-h$ should be substituted for h. The two theorems are the following:

The coefficient of $\cos j\theta$ *in the periodic development of* a'/Δ *is the same as that of* s^j *in the expansion of the expression*

$$\sum_{i=0}^{i=\infty} A_i^{(2j)}(-\chi)^i\left(s+\frac{1}{s}\right)^i\left[1-\chi\left(s+\frac{1}{s}\right)\right]\left(\frac{s-\omega}{1-\omega s}\right)^{2j}\varepsilon^{j\chi\left(s-\frac{1}{s}\right)}$$

in a power series with reference to s.

The coefficient of $\cos j\theta$ *in the same development after a has been replaced by* $\alpha\left[2-\sqrt{1-e^2}\right]^2$ *is the same as that of* s^j *in the expansion of the expression*

$$\sum_{i=0}^{i=\infty} A_i^{(2j)}\left[p-\chi(1+p)\left(s+\frac{1}{s}\right)\right]^i\left[1-\chi\left(s+\frac{1}{s}\right)\right]\left(\frac{s-\omega}{1-\omega s}\right)^{2j}\varepsilon^{j\chi\left(s-\frac{1}{s}\right)}$$

in a power series with reference to s.

In these expressions ω stands for $e/(1+\sqrt{1-e^2})$.

XV.

The two linear elements which determine all the coefficients in the periodic developments involved in this problem may be taken to be the constant α of (57) and the constant D of (62). It is proposed to elaborate two examples illustrating the subject in hand, one exhibiting non-libration, the other libration. In both we will assign to α such a value as makes $\log\alpha = 9.8$. This value makes the period of revolution of the small planet nearly or exactly half that of Jupiter. Whether we are to have a case of non-libration or libration will then depend on the value assigned to the second constant D.

In the first place then we compute the values of such of the quantities $A_i^{(j)}$ as are needed in this investigation, correspondent to $\log\alpha = 9.8$, by procedures which it is unnecessary to detail. The results are contained in the following table:

VALUES OF LOG $A_i^{(j)}$ FOR LOG $\alpha = 9.8$.

j	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$
0	0.354 4041 774	9.845 4797 897	9.935 0116 655	9.989 1230	0.111 3716
2	9.564 3962 993	9.965 8367 1	0.002 7463 5	0.007 7852	0.121 2342
4	9.035 0709 047	9.694 9897 1	9.992 6030 6	0.098 8192	0.164 1565
6	8.555 0516 205	9.374 1611 5	9.842 7442 0	0.099 5969	0.231 3986
8	8.096 8549 86	9.031 7132 5	9.624 7440 8	0.005 5654	0.239 5776
10	7.651 0634 5	8.677 0326 9	9.367 4385 2	9.849 2650	0.178 8047
12	7.213 2942 8	8.314 4461 7	9.084 6837 2	9.649 5418	0.062 8948
14	6.781 1495	7.946 3186 8	8.784 1408 8	9.419 6593	9.905 2677
16	6.353 1544	7.574 0873	8.470 4851	9.167 1668	9.715 8220

j	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
0	0.251 2555	0.408 5603	0.576 748	0.753 37	0.937 0
2	0.257 4382	0.412 7086	0.579 698	0.755 57	0.938 5
4	0.278 2372	0.426 0622	0.588 970	0.762 42	0.943 9
6	0.329 4253	0.453 2441	0.606 143	0.774 66	0.956 8
8	0.385 1308	0.503 3035	0.637 049	0.794 30	0.977 1
10	0.398 9502	0.552 5313	0.684 255	0.826 57	1.004 5
12	0.360 4400	0.570 3163	0.729 206	0.870 86	1.029 0
14	0.272 3635	0.545 6897	0.750 571	0.909 38	1.044 5
16	0.146 2268	0.480 0088	0.721 959	0.935 78	1.045 7

Substituting these values in (67) we get

$$
\begin{aligned}
\frac{a'}{\Delta} = {} & 1.1307697497 + 0.04010033e^2 - 0.7367846e^4 + 1.17661e^6 + 0.6155e^8 \\
& + [-1.19571949 + 3.1113902\, e^2 - 2.669146\, e^4 - 1.90033e^6]e \cos\theta \\
& + [1.70905245 - 9.8883917\, e^2 + 30.18579\, e^4 - 62.2057e^6]e^2 \cos 2\theta \\
& + [-3.00445698 + 27.190861\, e^2 - 117.01214\, e^4]e^3 \cos 3\theta \\
& + [5.7966694 - 71.99282\, e^2 + 369.2943\, e^4]e^4 \cos 4\theta \\
& + [-11.800399 + 186.12652\, e^2]e^5 \cos 5\theta \\
& + [24.86635 - 475.7506\, e^2]e^6 \cos 6\theta \\
& - 52.40299e^7 \cos 7\theta \\
& + 118.0918\, e^8 \cos 8\theta.
\end{aligned} \tag{69}
$$

We adopt the mass of Jupiter so that $\nu = 1/1047.355$. Then, in the expression (61) of W, the portion which is independent of the interaction of Jupiter and the small planet, developed in powers of e^2, becomes:

$$
\begin{aligned}
& 1.58715\,39467\,862 \\
& - 0.00226\,07543\,2 \quad e^2 \\
& + 0.59490\,01358 \quad e^4 \\
& + 0.09877\,323 \quad e^6 \\
& + 0.13623\,71 \quad e^8 \\
& + 0.06804\,8 \quad e^{10}.
\end{aligned}
$$

If we omit from the expansion of W its constant term, and call D the constant of the thus modified W, as in (62), we have, as an integral of our problem,

$$
\begin{aligned}
D = {} & -0.002222246709e^2 + 0.5941966641e^4 + 0.09989664\, e^6 + 0.1368248e^8 \\
& + [-0.00114165636 + 0.0029707121e^2 - 0.002548463e^4 - 0.00181441e^6]e \cos\theta \\
& + [0.00163177954 - 0.0094412989e^2 + 0.02882097\, e^4 - 0.0593931e^6]e^2 \cos 2\theta \\
& + [-0.0028686138 + 0.025961456\, e^2 - 0.11172156\, e^4]e^3 \cos 3\theta \\
& + [0.0055345794 - 0.06873774\, e^2 + 0.3525971\, e^4]e^4 \cos 4\theta \\
& + [-0.01126686 + 0.1777110e^2]\, e^5 \cos 5\theta \\
& + [0.02374204 - 0.4542400e^2]\, e^6 \cos 6\theta \\
& - 0.0500337e^7 \cos 7\theta \\
& + 0.1127524e^8 \cos 8\theta.
\end{aligned} \tag{70}
$$

By making $\theta = 0°$ in the preceding equation, we get, as the correspondent of the first equation of (62), the following:

$$
\left.\begin{aligned}
& - 0.00114\,16563\,6\, e - 0.00059\,06875\,5\, e^2 \\
& + 0.00010\,20983\,5\, e^3 + 0.59028\,99445\, e^4 \\
& + 0.01214\,61357\, e^5 + 0.08372\,1913\, e^6 \\
& + 0.01414\,140\, e^7 + 0.08854\,10\, e^8
\end{aligned}\right\} = D. \tag{71}
$$

It will be seen by comparison of the coefficients of this equation that, unless e is very small, it will not do to regard the equation as approximately

a quadratic in e; for $e=0.1$ the term in e^4 is ten times more important than the term in e^2. The supposition that the mean motion of the small planet is nearly double that of Jupiter makes the coefficient of e^2 nearly vanish. In fact a very small change in the adopted value of α would make this coefficient 0.

What sort of a curve we shall have exhibiting graphically the connection between e and θ will depend on the value assigned to D. To bring this out in a clear manner we compute the values of the left member of the preceding equation for each 0.01 in the value of e between the limits $\pm$ 0.3, and thus have the following table:

e	D	e	D	e	D	e	D
—0.30	+0.0051 0216	—0.15	+0.0004 5648	+0.01	—0.0000 1148	+0.16	+0.0001 9224
0.29	44 8080	0.14	3 7472	0.02	2298	0.17	2 8623
0.28	39 2024	0.13	3 0675	0.03	3430	0.18	4 0095
0.27	34 1625	0.12	2 5066	0.04	4509	0.19	5 3896
0.26	29 6473	0.11	2 0477	0.05	5485	0.20	7 0298
0.25	25 6178	0.10	1 6715	0.06	6294	0.21	8 9590
0.24	22 0365	0.09	1 3659	0.07	6857	0.22	11 2071
0.23	18 8675	0.08	1 1166	0.08	7082	0.23	13 8067
0.22	16 0764	0.07	9115	0.09	6861	0.24	16 7912
0.21	13 6306	0.06	7400	0.10	6073	0.25	20 1960
0.20	11 4987	0.05	5928	0.11	4582	0.26	24 0580
0.19	9 6512	0.04	4622	0.12	—0.0000 2236	0.27	28 4159
0.18	8 0599	0.03	3420	0.13	+0.0000 1129	0.28	33 3102
0.17	6 6982	0.02	2269	0.14	5695	0.29	38 7834
—0.16	+0.0005 5411	—0.01	+0.0000 1136	+0.15	+0.0001 1657	+0.30	+0.0044 8802

As e ought always to be positive, in the first half of this table we may change the sign of e, provided we suppose that the corresponding value of D is regarded as appertaining to the special value 180° for θ, while, in the remainder of the table, this value corresponds to $\theta = 0°$.

From the course of the values of D in the table we see there is one minimum $= -0.00007082$, which occurs for $e = 0.08$ about; consequently, if D is chosen greater than this the equation (71) will have two real roots for e. If D is positive one of these roots will be negative; changing the sign of the latter it will belong to the value $\theta = 180°$; the positive root will belong to $\theta = 0°$. Thus, in this case, the motion of θ is generally through the whole semicircumference, and hence is continuous from $-\infty$ to $+\infty$. But, if D is negative, both roots will be positive, and thus belong to $\theta = 0°$. In this case, therefore, θ departs from 0° and comes back to it without having reached 180°. This is called a libration; we see that $D = 0$ marks the dividing point between continuous and libratory motion for θ. The

latter case also has the largest swing in the values of e, viz, from $e = 0$ to about $e = 0.127$. Generally, the larger D is, the smaller will be the variation in e. Thus, if $D = +0.0045$, e will vary from 0.29 to 0.30. If there is libration e cannot exceed 0.127. These remarks, however, must be understood as applying only to the values holding for $\theta = 0°$ and $\theta = 180°$. Larger values for e may obtain for values of θ lying between 0° and 180°.

XVI.

For our illustrative example, in the case of a continuous motion for θ, we assign to D the value $+0.0001$ in (70). All the coefficients of the various periodic series will now have determinate numerical values. The preceding table shows that, for this assumption, the eccentricity will have, when $\theta = 0°$, the approximate value $e = 0.1475$, and, when $\theta = 180°$, the approximate value $e = 0.0745$. In this case these are the limiting values, as e continuously diminishes while θ is passing from 0° to 180°.

Attending now to the elaboration of our selected example, in (70) we give to θ, in succession, the values 15°, 30°, 45°, 60°, 75°, 90° and get as the right member of (70), [the value $\theta = 0°$ has already been considered in (71)],

θ				
15°	$-0.00110\,27554e$	$-0.00080\,93046e^2$	$+0.00084\,1072e^3$	$+0.58878\,755e^4$
30	98 87034	140 65774	257 2712	58670 872
45	80 72730	222 24671	412 9027	58866 208
60	57 08282	303 83568	435 3970	59615 002
75	$-0.00029\,54824$	363 56296	$+0.00279\,7293$	60514 036
90	0	$-0.00385\,42466$	0	$+0.60917\,254$

θ				
15°	$+0.01297\,98e^5$	$+0.09048\,7e^6$	$-0.02180\,7e^7$	$+0.20531e^8$
30	$+0.00755\,03$	$+0.12493\,4$	$-0.11214\,3$	$+0.32869$
45	$-0.01219\,27$	$+0.16863\,4$	$-0.08332\,4$	-0.10302
60	$-0.03286\,91$	$+0.14359\,7$	$+0.17465\,3$	-0.52039
75	$-0.02990\,01$	$+0.04056\,8$	$+0.29851\,4$	$+0.30818$
90	0	$-0.02140\,4$	0	$+1.11581$

The coefficients in the second quadrant for θ are the same as in the first, but in reverse order, except, that for the odd powers of e the sign must be reversed.

Making the right members of these 13 equations equal to $+0.0001$, we solve them with reference to e as the unknown, and substitute, in succession, the values thus obtained and the corresponding value of θ

in the right member of the second equation of (60). The results obtained are the following:

θ	e	$d\tau/d\theta$
0°	0.14746 2372	18.321384
15	0.14702 7366	18.443969
30	0.14569 2650	18.827993
45	0.14335 8947	19.521595
60	0.13982 8670	20.600475
75	0.13480 9503	22.197678
90	0.12791 2165	24.508178
105	0.11869 0066	27.615097
120	0.10709 4867	30.992715
135	0.09444 8152	32.866681
150	0.08360 2031	31.947061
165	0.07679 7488	29.983003
180	0.07454 7767	29.112462

The mean of the numbers in the third column, attributing half weight to the first and last, is 25.101781; and this is the number of revolutions of Jupiter in the period of the inequalities we are investigating. If the sidereal revolution of Jupiter is put at 11.861980 Julian years, the latter period is 297.75681 such years.

From the special values given in this table we can derive the two periodic series representing them. Integrating the latter, and, for brevity, putting ζ for $\theta_0(t+c)$, we get the following expressions:

$$e = \left\{\begin{array}{l} \quad 0.11918\,891 \\ +0.03553\,171 \cos\theta \\ -0.00857\,010 \cos 2\theta \\ +0.00123\,337 \cos 3\theta \\ +0.00027\,721 \cos 4\theta \\ -0.00029\,767 \cos 5\theta \\ +0.00012\,029 \cos 6\theta \\ -0.00001\,828 \cos 7\theta \\ -0.00000\,783 \cos 8\theta \\ +0.00000\,780 \cos 9\theta \\ -0.00000\,374 \cos 10\theta \\ +0.00000\,037 \cos 11\theta \\ +0.00000\,033 \cos 12\theta \end{array}\right\}, \qquad \zeta = \theta + \left\{\begin{array}{lr} - & 60173''40 \sin\theta \\ - & 1643.74 \sin 2\theta \\ + & 4612.77 \sin 3\theta \\ - & 2132.15 \sin 4\theta \\ + & 544.20 \sin 5\theta \\ + & 6.13 \sin 6\theta \\ - & 87.51 \sin 7\theta \\ + & 52.96 \sin 8\theta \\ - & 16.90 \sin 9\theta \\ - & 0.02 \sin 10\theta \\ + & 3.89 \sin 11\theta \\ - & 1.98 \sin 12\theta \end{array}\right\}$$

The first of these is simply a transformation of the equation $W = D$ by which e is expressed in terms of θ. From the second, by attributing to ζ in succession the 13 values 0°, 15°, 30°, ..., 180°, using a tentative process, we can get the corresponding values of θ. Thence by substitution in former

results, the corresponding values of the four quantities e, $e\cos\theta$, $e\sin\theta$ and $d\tau/d\theta$ can be obtained. The results follow, the first column containing the argument:

ζ	θ	e	$e\cos\theta$	$e\sin\theta$	$d\tau/d\theta$
0°	0° 0′ 0″.00	0.14746 237	+0.14746 237	0.00000 000	18.321384
15	20 27 58.05	0.14664 872	0.13739 211	+0.05127 624	18.551258
30	40 25 23.50	0.14418 648	0.10976 569	0.09349 457	19.274084
45	59 23 15.52	0.13999 969	0.07129 163	0.12048 825	20.547277
60	76 57 25.45	0.13402 600	+0.03024 713	0.13056 830	22.453723
75	92 51 50.00	0.12634 244	—0.00631 251	0.12618 464	25.044275
90	107 3 42.77	0.11722 678	0.03439 485	0.11206 745	28.087856
105	119 48 15.87	0.10725 784	0.05331 151	0.09307 055	30.953203
120	131 35 27.90	0.09727 656	0.06457 314	0.07275 327	32.686384
135	143 3 46.20	0.08820 764	0.07050 393	0.05300 740	32.666482
150	154 48 43.11	0.08092 658	0.07323 176	0.03444 156	31.300251
165	167 9 46.91	0.07619 338	0.07428 902	+0.01692 847	29.759410
180	180 0 0.00	0.07454 777	—0.07454 777	0.00000 000	29.112462

From the numbers in the fourth and fifth columns are derived the series:

$$
e\cos\theta = \left\{\begin{array}{l}
+0.00071\,143 \\
+0.10563\,221 \cos \zeta \\
+0.03542\,782 \cos 2\zeta \\
+0.00539\,467 \cos 3\zeta \\
+0.00031\,839 \cos 4\zeta \\
-0.00002\,420 \cos 5\zeta \\
-0.00000\,130 \cos 6\zeta \\
+0.00000\,245 \cos 7\zeta \\
+0.00000\,111 \cos 8\zeta \\
+0.00000\,026 \cos 9\zeta \\
-0.00000\,044 \cos 10\zeta \\
-0.00000\,032 \cos 11\zeta \\
+0.00000\,030 \cos 12\zeta
\end{array}\right\}, \qquad
e\sin\theta = \left\{\begin{array}{l}
+0.11737\,247 \sin \zeta \\
+0.03373\,708 \sin 2\zeta \\
+0.00528\,997 \sin 3\zeta \\
+0.00035\,675 \sin 4\zeta \\
-0.00001\,633 \sin 5\zeta \\
-0.00000\,316 \sin 6\zeta \\
-0.00000\,095 \sin 7\zeta \\
-0.00000\,062 \sin 8\zeta \\
+0.00000\,041 \sin 9\zeta \\
+0.00000\,018 \sin 10\zeta \\
-0.00000\,004 \sin 11\zeta
\end{array}\right\}
$$

These forms for the integrals of our problem are to be preferred since they can also be used for the case of libration.

XVII.

To complete the solution the periodic series giving the position of the perihelion must be derived. Using logarithms instead of the actual

coefficients, the first term of the right member of the third equation of (60) has the expression:

$$
\begin{aligned}
\frac{2}{\sqrt{(1+\nu)\,a}}\, a \frac{\partial R}{\partial a} = \; & [6.9251786] + [7.126034]e^2 - [8.00771]e^4 + [7.9475]e^6 + [8.7115]e^8 \\
& + \{ -[7.9015032] + [8.284709]e^2 + [7.66255]e^4 - [8.9051]e^6 \} \, e \cos \theta \\
& + \{ \;\; [8.3084076] - [9.068286]e^2 + [9.55239]e^4 - [9.7559]e^6 \} \, e^2 \cos 2\theta \\
& + \{ -[8.707357] + [9.66685]\, e^2 - [0.2895]\, e^4 \} \, e^3 \cos 3\theta \\
& + \{ \;\; [9.104403] - [0.20025]\, e^2 + [0.9097]\, e^4 \} \, e^4 \cos 4\theta \\
& + \{ -[9.50120] + [0.7008]\, e^2 \} \, e^5 \cos 5\theta \\
& + \{ \;\; [9.89773] + [1.1811]\, e^2 \} \, e^6 \cos 6\theta \\
& - \;\; [0.2839]\, e^7 \cos 7\theta \\
& + \;\; [0.6902]\, e^8 \cos 8\theta .
\end{aligned}
$$

The remaining terms of this expression for $dg/d\tau$ can readily be derived from the values of e and $d\tau/d\theta$ correspondent to the argument ζ which have just been given. Calling these the second part of $dg/d\tau$, we have the following results:

ζ	First Part	Second Part	$dg/d\tau$
0°	+0.00004 9796	—0.00579 1193	—0.00574 1397
15	0.00005 9848	0.00572 9341	0.00566 9493
30	0.00012 1787	0.00554 3667	0.00542 1880
45	0.00023 7477	0.00509 7143	0.00485 9666
60	0.00039 1638	0.00420 4082	0.00381 2444
75	0.00059 5969	0.00270 5347	—0.00210 9378
90	0.00084 3915	—0.00024 2584	+0.00060 1331
105	0.00109 9821	+0.00326 3944	0.00436 3765
120	0.00131 9503	0.00773 7116	0.00905 6619
135	0.00146 5221	0.01284 0064	0.01430 5285
150	0.00153 5401	0.01789 1412	0.01942 6813
165	0.00156 2180	0.02178 6499	0.02334 8679
180	+0.00156 9151	+0.02328 0093	+0.02484 9244

The quantities in the last column furnish the periodic series for $dg/d\tau$. The absolute term shows that the mean motion of the perihelion of the minor planet is 0.00490 0079 times the mean motion of Jupiter. The integration of this series gives the expression for g. These two expressions follow; (g) is the arbitrary constant added to complete the integral, and, in the second term, the unit of t is a Julian year.

$$
\frac{dg}{d\tau} = \left\{
\begin{array}{lr}
+\,0.00490\,0079 & \\
-\,0.01441\,6898 & \cos \zeta \\
+\,0.00444\,6030 & \cos 2\zeta \\
-\,0.00080\,4772 & \cos 3\zeta \\
+\,0.00017\,6760 & \cos 4\zeta \\
-\,0.00006\,7470 & \cos 5\zeta \\
+\,0.00003\,1972 & \cos 6\zeta \\
-\,0.00000\,2614 & \cos 7\zeta \\
+\,0.00000\,0139 & \cos 8\zeta \\
-\,0.00000\,3981 & \cos 9\zeta \\
-\,0.00000\,1706 & \cos 10\zeta \\
+\,0.00000\,0414 & \cos 11\zeta \\
+\,0.00000\,0648 & \cos 12\zeta
\end{array}
\right\}, \qquad
g = \left\{
\begin{array}{lrr}
(g) & +\,535''.3662\,t & \\
- & 74645.14 & \sin \zeta \\
+ & 11509.92 & \sin 2\zeta \\
- & 1388.93 & \sin 3\zeta \\
+ & 228.80 & \sin 4\zeta \\
- & 69.87 & \sin 5\zeta \\
+ & 27.59 & \sin 6\zeta \\
- & 1.93 & \sin 7\zeta \\
+ & 0.09 & \sin 8\zeta \\
- & 2.27 & \sin 9\zeta \\
- & 0.88 & \sin 10\zeta \\
+ & 0.19 & \sin 11\zeta \\
+ & 0.28 & \sin 12\zeta
\end{array}
\right\}.
$$

It is of interest to know the mean motion of the small planet which is not obvious at the beginning of the solution. We have the equation:

$$\frac{d(l+g)}{d\tau} = \frac{d\theta}{d\tau} - \frac{dg}{d\tau} + 2.$$

Substituting in the right member the mean motions of θ and g, its value is found to be 2.03493 7731; then, if for Jupiter we have $\mu' = 299.''12838$, for the small planet $\mu = 608.''70762$.

XVIII.

Illustration in the Case of Libration.—In the example we have chosen to illustrate the theory, libration, when it exists, is always about the value $\theta = 0°$. In addition to the value $\log \alpha = 9.8$ let us assume that the D of (70) is to be so chosen that the half-swing of θ may be 50°. Making, therefore, $\theta = 50°$ in (70), we get the first of the following equations in e, and the second by taking the derivative of the first with respect to e:

$$\left\{\begin{array}{ll} -0.00073\,38426 & e \\ -0.00250\,58226 & e^2 \\ +0.00439\,3829 & e^3 \\ +0.58063\,532 & e^4 \\ -0.02026\,791 & e^5 \\ +0.17135\,53 & e^6 \\ -0.01446\,68 & e^7 \\ -0.32494\,7 & e^8 \end{array}\right\} = D, \qquad \left\{\begin{array}{ll} -0.00073\,38426 & \\ -0.00501\,16452 & e \\ +0.01318\,1487 & e^2 \\ +2.36254\,128 & e^3 \\ -0.10133\,955 & e^4 \\ +1.02813\,18 & e^5 \\ -0.10126\,76 & e^6 \\ -2.59957\,6 & e^7 \end{array}\right\} = 0.$$

Both of these equations should be satisfied when θ is at the limit of its swing, viz., when $\theta = \pm 50°$. The root of the second equation which is applicable to our purpose is $e = 0.07606\ 124$, and this value substituted in the first gives $D = -0.000048\ 63102$, which is the value of D which brings about a libration of 50°.

From the equation $\sin\theta = \sin 50° \sin\psi$ we obtain the following corresponding values:

ψ	θ		
0°	0°	0′	0.″00
15	11	26	8.27
30	22	31	15.64
45	32	47	51.90
60	44	33	38.75
75	47	43	35.34
90	50	0	0.00

By means of these values we determine the form of (70) corresponding to the seven values of ψ. The coefficients are given in the following table

(the small figures at the top of the columns denote the order of the final decimal):

ψ	e	e^2	e^3	e^4	e^5	e^6	e^7	e^8
0°	$-114\,16564^{10}$	$-59\,0688^{9}$	$+10\,210^{6}$	$+59028\,994^{8}$	$+1214\,61^{7}$	$+8372\,2^{6}$	$+1414^{5}$	$+8854^{5}$
15	111 89923	71 8977	54 217	58936 012	1284 01	8713 9	− 640	15961
30	105 45925	106 9471	164 931	58751 743	1188 52	10353 7	6636	30349
45	95 96627	154 8255	291 587	58665 083	+ 487 32	13434 5	12360	30000
60	85 42473	202 7039	385 516	58769 012	− 671 35	16174 8	11285	+ 4731
75	76 79586	237 7533	429 468	58965 923	1663 22	17134 1	− 4928	−22703
90	− 73 38426	−250 5823	+439 383	+59063 532	−2026 79	+17135 5	+ 1447	−32495

The expressions in this table constitute the left members of 7 equations of the 8th degree; they must be equated to the same quantity $D = -0.00004\,863102$. The two smallest real roots of each should be derived (they are those suited to our purpose). The connection of these roots with the variable ψ is settled in following way: the larger of the two roots is made to correspond to the value of ψ standing as the argument in the table, while the smaller is assigned to the value $180° - \psi$; the two roots being equal for $\psi = 90°$, the common value is assigned to that value of ψ. This arrangement is made in order that ψ and τ may augment together. These values of e together with the corresponding values of θ are, in succession, substituted in (65); thus we have the values of $d\tau/d\psi$ corresponding to equidistant values of ψ. These results are contained in the following table:

ψ	e	$d\tau/d\psi$
0°	0.10846 187	37.23986
15	0.10765 795	37.70988
30	0.10518 818	39.14539
45	0.10088 191	41.57020
60	0.09452 189	44.76280
75	0.08605 349	47.65995
90	0.07606 124	48 00164
105	0.06601 592	44.40759
120	0.05744 757	38.68685
135	0.05102 653	33.34281
150	0.04671 683	29.47279
165	0.04426 767	27.22320
180	0.04347 566	26.49213

It should be noted that, in the computation of the third column, for $\psi = 90°$, the factor $\cos\psi/(\partial W/\partial e)$ takes on the indeterminate form $0/0$;

employing the usual method of treating vanishing fractions, this factor equals $-1/[(\partial^2 W/\partial e^2)(de/d\psi)]$. If here we should use the equation $W = D$ to determine $de/d\psi$, the result would again be indeterminate. But this difficulty is avoided by employing the value of e as a periodic function of ψ given by the quantities of the second column. Thus if

$$e \quad a_0 + a_1 \cos\psi + a_2 \cos 2\psi + a_3 \cos 3\psi + \ldots,$$

then

$$\frac{de}{d\psi} = -a_1 \sin\psi - 2a_2 \sin 2\psi - 3a_3 \sin 3\psi - \ldots,$$

and, for the special value $\psi = 90^\circ$, this becomes:

$$\frac{de}{d\psi} = -a_1 + 3a_3 - 5a_5 + 7a_7 - \ldots.$$

For this special value of ψ it is found that

$$e\frac{\partial^2 W}{\partial e^2} = +0.00288\,92836, \quad \frac{de}{d\psi} = -0.03939\,373.$$

The mean of the numbers in the last column of the table, allowing half weight to the first and last, is 38.65409, which is the number of revolutions of Jupiter contained in the period of libration; thus this period is 458.5144 Julian years. From the special values of e and $d\tau/d\psi$ given in the table we derive the periodic series representing them. The latter can be integrated, and, as before, we put ζ for $\theta_0(t+c)$. Thus we get the following expressions:

$$e = \left\{\begin{array}{l} \,0.0759\,8399 \\ +\,0.0338\,8957 \cos\psi \\ -\,0.0000\,4144 \cos 2\psi \\ -\,0.0015\,2988 \cos 3\psi \\ +\,0.0000\,3031 \cos 4\psi \\ +\,0.0001\,5016 \cos 5\psi \\ -\,0.0000\,0467 \cos 6\psi \\ -\,0.0000\,1929 \cos 7\psi \\ +\,0.0000\,0062 \cos 8\psi \\ +\,0.0000\,0281 \cos 9\psi \\ -\,0.0000\,0012 \cos 10\psi \\ -\,0.0000\,0026 \cos 11\psi \\ +\,0.0000\,0008 \cos 12\psi \end{array}\right\}, \qquad \zeta = \psi + \left\{\begin{array}{l} +\,29862''.37 \sin\psi \\ -\,20922.46 \sin 2\psi \\ -\,416.31 \sin 3\psi \\ +\,1650.00 \sin 4\psi \\ +\,13.01 \sin 5\psi \\ -\,193.31 \sin 6\psi \\ -\,0.37 \sin 7\psi \\ +\,27.39 \sin 8\psi \\ +\,0.01 \sin 9\psi \\ -\,4.65 \sin 10\psi \\ -\,0.02 \sin 11\psi \\ +\,0.81 \sin 12\psi \end{array}\right\}.$$

The first of these is simply a transformation of the equation $W = D$, by which e is expressed in terms of the auxiliary variable ψ.

Attributing to ζ, in succession, the 13 values 0°, 15°, 30°, ..., 180°, by a tentative process we can get the corresponding values of ψ, as also by substitution those of e, $e \cos \theta$ and $e \sin \theta$. These results follow:

ζ	ψ			e	$e \cos \theta$	$e \sin \theta$
0°	0°	0′	0″.00	0.1084 6187	+0.1084 6187	0.0000 0000
15	15	30	0.64	0.1068 0283	0.1045 4084	+0.0218 6455
30	30	35	58.93	0.1022 4969	0.0941 5541	0.0398 7176
45	44	58	46.75	0.0956 9588	0.0804 5265	0.0518 1767
60	58	28	33.58	0.0879 7453	0.0666 2875	0.0574 4671
75	71	8	20.78	0.0797 1930	0.0549 1390	0.0577 8952
90	83	15	23.39	0.0714 4655	0.0463 7246	0.0543 5260
105	95	20	46.24	0.0636 2489	0.0411 4864	0.0485 2749
120	108	7	15.59	0.0566 9409	0.0388 6515	0.0412 7614
135	122	25	27.98	0.0510 7994	0.0389 6451	0.0330 2919
150	139	5	9.66	0.0468 5171	0.0405 2863	0.0235 0559
165	158	31	3.57	0.0443 2198	0.0425 4216	+0.0124 3393
180	180	0	0.00	0.0434 7566	+0.0434 7566	0.0000 0000

From the data of the fourth and fifth columns result the periodic series:

$$e \cos \theta = \left\{ \begin{array}{l} +\, 0.0604\,2349 \\ +\, 0.0309\,3374 \cos \zeta \\ +\, 0.0147\,2940 \cos 2\zeta \\ +\, 0.0015\,8319 \cos 3\zeta \\ +\, 0.0007\,3464 \cos 4\zeta \\ -\, 0.0000\,1013 \cos 5\zeta \\ +\, 0.0000\,6769 \cos 6\zeta \\ -\, 0.0000\,1229 \cos 7\zeta \\ +\, 0.0000\,1611 \cos 8\zeta \\ -\, 0.0000\,0669 \cos 9\zeta \\ +\, 0.0000\,0106 \cos 10\zeta \\ +\, 0.0000\,0528 \cos 11\zeta \\ -\, 0.0000\,0363 \cos 12\zeta \end{array} \right\}, \qquad e \sin \theta = \left\{ \begin{array}{l} +\, 0.0571\,8416 \sin \zeta \\ +\, 0.0093\,8541 \sin 2\zeta \\ +\, 0.0030\,1596 \sin 3\zeta \\ +\, 0.0000\,5256 \sin 4\zeta \\ +\, 0.0001\,9928 \sin 5\zeta \\ -\, 0.0000\,1597 \sin 6\zeta \\ +\, 0.0000\,1768 \sin 7\zeta \\ -\, 0.0000\,0390 \sin 8\zeta \\ +\, 0.0000\,0771 \sin 9\zeta \\ -\, 0.0000\,0714 \sin 10\zeta \\ +\, 0.0000\,0492 \sin 11\zeta \end{array} \right\}.$$

XIX.

In computing, for this case, the values of $dg/d\tau$ by the third equation of (60) we make a like division into two parts as in the former case. Sub-

stituting the values of e and θ which correspond to the values 0°, 15°, 30°, ..., 180° of ζ we get the following special values:

ζ	First Part	Second Part	$dg/d\tau$
0°	+ 0.00019 583	— 0.05008 981	— 0.04989 398
15	20 725	4843 252	4822 527
30	24 656	4385 892	4361 236
45	30 013	3722 995	3692 982
60	36 434	2938 834	2902 400
75	42 685	2098 623	2055 938
90	47 979	1247 567	1199 588
105	51 817	— 0.00410 869	— 0.00359 052
120	54 013	+ 0.00397 237	+ 0.00451 250
135	54 781	1211 642	1266 423
150	54 497	1991 948	2046 445
165	53 746	2629 903	2683 649
180	+ 0.00053 360	+ 0.02891 583	+ 0.02944 943

From the quantities in the last column we obtain the periodic series for $dg/d\tau$, and thence by integration the expression for g; these results follow:

$$\frac{dg}{d\tau}=\left\{\begin{array}{l} -0.0116\,3174 \\ -0.0373\,2443\cos\zeta \\ +0.0007\,9929\cos 2\zeta \\ -0.0020\,0735\cos 3\zeta \\ +0.0005\,0972\cos 4\zeta \\ -0.0002\,9143\cos 5\zeta \\ +0.0000\,8500\cos 6\zeta \\ -0.0000\,4117\cos 7\zeta \\ +0.0000\,1080\cos 8\zeta \\ -0.0000\,0438\cos 9\zeta \\ +0.0000\,0251\cos 10\zeta \\ -0.0000\,0293\cos 11\zeta \\ +0.0000\,0214\cos 12\zeta \end{array}\right\},\qquad g=\left\{\begin{array}{l} (g)-1270''.844\,t \\ -297586''.8\sin\zeta \\ +\ 3186.4\sin 2\zeta \\ -\ 5334.9\sin 3\zeta \\ +\ 1016.0\sin 4\zeta \\ -\ 464.7\sin 5\zeta \\ +\ 113.0\sin 6\zeta \\ -\ 46.9\sin 7\zeta \\ +\ 10.8\sin 8\zeta \\ -\ 3.9\sin 9\zeta \\ +\ 2.0\sin 10\zeta \\ -\ 2.1\sin 11\zeta \\ +\ 1.4\sin 12\zeta \end{array}\right\}.$$

The unit of t in the second expression is a Julian year.

By using the same formula as in the former case we find that the mean μ of the small planet in this case has the value 609''.47474.

XX.

In attempting to apply the preceding method to the case where $D=0$, we should find that $dg/d\tau$ became infinite at the point where $\theta=0°$ or $\theta=180°$, and, when D is quite small, we should have to deal with inconveniently large numbers. This difficulty is surmounted by computing the differentials of the two quantities $e\cos g$ and $e\sin g$ in place of that of g.

MEMOIR No. 67.

Ptolemy's Problem.

(Astronomical Journal, Vol. XXI, pp. 33–35, 1900.)

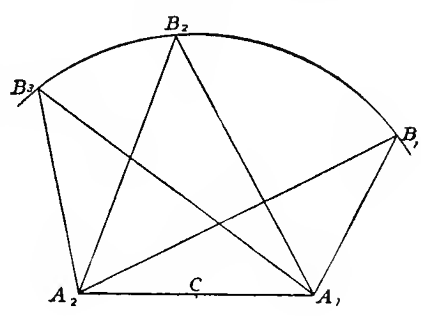
In the case of the inferior planets Ptolemy discovered the position of the line of apsides and the eccentricity of the eccentric circle by observations of greatest elongations. When two were found of the same magnitude, but situated on opposite sides of the mean Sun, the position of the line of apsides was discovered by bisecting the angle between the mean positions of the Sun. And the eccentricity was found by comparing the magnitude of the greatest elongations when the mean Sun was at either end of the line of apsides.

These devices are impracticable in the case of the superior planets, and Ptolemy had recourse to observations made when the true longitude of the planet differed by 180° from the mean longitude of the Sun. Observations fulfilling this condition cannot generally be obtained, but, by interpolation between several observations in close proximity, the desired data may be got. At these special times the Earth, planet and center of the epicycle are in a right line; thus, as we do not have to deal with distance, but only with orientation, we need not pay any attention to the epicycle. It is understood that the period of the planet is known, there are then three unknowns to be determined, viz.: the position of the line of apsides, the eccentricity of the eccentric circle, and the epoch of the mean longitude; thus three observations are necessary.

What I have ventured to call Ptolemy's Problem may be stated in a geometrical form as follows:

On a given common base to construct three triangles such that, while their vertices are equidistant from the middle point of the base, the differences of the angles at each end of the base may be equivalent to given angles.

In the figure, let A_1A_2 be the common base, C its middle point, and B_1, B_2, B_3 the vertices of the three triangles. The angles $B_1A_1B_2$, $B_1A_1B_3$, $B_1A_2B_2$, $B_1A_2B_3$ are known. Adopt A_1C as the linear unit, and let A_2 be the point from which the observations are made, while A_1 is the point about which the center of the epicycle rotates uniformly. Take C for the origin of a system of rectangular co-ordinates, CA_1 being



the positive direction of the axis of x. Let the general equations of the radii A_1B and A_2B be severally

$$y = \tan\mu\,.\,(x-1), \qquad y = \tan\upsilon\,.\,(x+1)$$

where μ and υ denote severally the mean and true anomaly measured, as with Ptolemy, from the aphelion. In order to have the rectangular co-ordinates of the point B, we solve these equations, regarding x and y as the unknowns, and get

$$x = \frac{\tan\mu + \tan\upsilon}{\tan\mu - \tan\upsilon} = \frac{\sin(\mu+\upsilon)}{\sin(\mu-\upsilon)}$$
$$y = \frac{2\tan\mu\tan\upsilon}{\tan\mu - \tan\upsilon} = \frac{\cos(\mu-\upsilon) - \cos(\mu+\upsilon)}{\sin(\mu-\upsilon)}$$

It will be more convenient to employ as the variables $\chi = \mu + \upsilon$ and $\psi = \mu - \upsilon$. Then as a constant quantity,

$$BC^2 = x^2 + y^2 = \frac{\sin^2\chi + (\cos\chi - \cos\psi)^2}{\sin^2\psi}$$

But, if $2u^{-1}$ is put for $x^2 + y^2 + 1$, this equation can be given the form

$$u - u\cos\chi\cos\psi = \sin^2\psi.$$

This equation constitutes the relation between the true and mean anomalies in Ptolemy's theory of eccentric circles. It holds for each of the three observations, and we thus have the data requisite for the determination of the three unknowns u, χ and ψ. If the latter symbols are used for the first observation we should add to them certain known arcs for the second and third observations. But it is conducive to symmetry to suppose that the known arcs to be added to χ and ψ in the several observations in their order are α_1, α_2, α_3, and again β_1, β_2, β_3. It will shorten the writing of some of the formulas if we impose upon the β the condition $\beta_1+\beta_2+\beta_3=0$. As to the α, the computation is shortened if we suppose one of them vanishes. To abbreviate we write ψ_i for $\psi + \beta_i$, then the equations for solution are

$$u - u\cos(\chi + \alpha_1)\cos\psi_1 = \sin^2\psi_1$$
$$u - u\cos(\chi + \alpha_2)\cos\psi_2 = \sin^2\psi_2$$
$$u - u\cos(\chi + \alpha_3)\cos\psi_3 = \sin^2\psi_3$$

By developing $\cos(\chi + \alpha)$ we have the modified form:

$$u - \cos\alpha_1\cos\psi_1\,.\,u\cos\chi + \sin\alpha_1\cos\psi_1\,.\,u\sin\chi = \sin^2\psi_1$$
$$u - \cos\alpha_2\cos\psi_2\,.\,u\cos\chi + \sin\alpha_2\cos\psi_2\,.\,u\sin\chi = \sin^2\psi_2$$
$$u - \cos\alpha_3\cos\psi_3\,.\,u\cos\chi + \sin\alpha_3\cos\psi_3\,.\,u\sin\chi = \sin^2\psi_3$$

Let these equations be regarded as linear, and as determining the unknowns u, $u\cos\chi$, and $u\sin\chi$. Employ Δ to denote the determinant formed from the coefficients of the equations, and S to denote summation

with respect to the cyclical permutation of the subscripts 1, 2, 3; thus, of three terms only one will be written, the remaining two being derived from this by the mentioned cyclical permutation. Then the expression for Δ is

$$\Delta = S \,.\, \sin(\alpha_1 - \alpha_2) \cos\psi_1 \cos\psi_2$$

and the values of the unknowns u, $u \cos\chi$, $u \sin\chi$ are given by the expressions

$$\Delta u = S \,.\, \sin(\alpha_2 - \alpha_3) \cos\psi_2 \cos\psi_3 \sin^2\psi_1$$
$$\Delta u \cos\chi = S \,.\, [\sin\alpha_2 \cos\psi_2 - \sin\alpha_3 \cos\psi_3] \sin^2\psi_1$$
$$\Delta u \sin\chi = S \,.\, [\cos\alpha_2 \cos\psi_2 - \cos\alpha_3 \cos\psi_3] \sin^2\psi_1$$

The substitution of these values in the identity

$$(\Delta u \cos\chi)^2 + (\Delta u \sin\chi)^2 \equiv (\Delta u)^2$$

gives an equation involving only the unknown ψ, and which serves to determine this quantity.

With advantage these expressions may be transformed; thus, in case of the second,

$$\Delta u \cos\chi = S \,.\, \sin\alpha_1 \cos\psi_1 [\sin^2\psi_3 - \sin^2\psi_2]$$

and by applying the formula

$$\sin^2 x - \sin^2 y \equiv \sin(x - y) \sin(x + y)$$

this becomes, mindful of the relation $\beta_1 + \beta_2 + \beta_3 = 0$,

$$\Delta u \cos\chi = \tfrac{1}{2} S \,.\, \sin\alpha_1 \sin(\beta_3 - \beta_2) [\sin 3\psi + \sin(\psi - 2\beta_1)]$$

In like manner

$$\Delta u \sin\chi = \tfrac{1}{2} S \,.\, \cos\alpha_1 \sin(\beta_3 - \beta_2) [\sin 3\psi + \sin(\psi - 2\beta_1)]$$
$$\begin{aligned} \Delta u = \tfrac{1}{8} S \,.\, \sin(\alpha_2 - \alpha_3) [&2 \sin(\beta_1 - \beta_2) \sin(\beta_1 - \beta_3) \\ &+ \cos(\beta_2 - \beta_3) + 2 \sin(\beta_1 - \beta_2) \sin(2\psi - \beta_2) \\ &+ 2 \sin(\beta_1 - \beta_3) \sin(2\psi - \beta_3) - \cos(4\psi + \beta_1)] \end{aligned}$$

If we put

$$A_1 \genfrac{}{}{0pt}{}{\cos}{\sin} \alpha_1 = \pm \tfrac{1}{2} S \,.\, \sin\alpha_1 \sin(\beta_3 - \beta_2) \genfrac{}{}{0pt}{}{\cos}{\sin} 2\beta_1$$
$$A_2 = \tfrac{1}{2} S \,.\, \sin\alpha_1 \sin(\beta_3 - \beta_2)$$
$$A_3 \genfrac{}{}{0pt}{}{\cos}{\sin} \alpha_2 = \pm \tfrac{1}{2} S \,.\, \cos\alpha_1 \sin(\beta_3 - \beta_2) \genfrac{}{}{0pt}{}{\cos}{\sin} 2\beta_1$$
$$A_4 = \tfrac{1}{2} S \,.\, \cos\alpha_1 \sin(\beta_3 - \beta_2)$$
$$A_5 = \tfrac{1}{8} S \,.\, \sin(\alpha_2 - \alpha_3) [2 \cos(\beta_2 - \beta_3) - \cos 3\beta_1]$$
$$A_5 \genfrac{}{}{0pt}{}{\cos}{\sin} \alpha_3 = -\tfrac{1}{4} S \,.\, \sin(\alpha_2 - \alpha_3) [\cos(\beta_2 - \beta_3) \genfrac{}{}{0pt}{}{\cos}{\sin} 2\beta_1 \pm \genfrac{}{}{0pt}{}{\cos}{\sin} \beta_1]$$
$$A_7 \genfrac{}{}{0pt}{}{\cos}{\sin} \alpha_4 = -\tfrac{1}{8} S \,.\, \sin(\alpha_2 - \alpha_3) \genfrac{}{}{0pt}{}{\cos}{\sin} \beta_1$$

we shall have

$$\Delta u \cos\chi = A_1 \sin(\psi + \alpha_1) + A_2 \sin 3\psi$$
$$\Delta u \sin\chi = A_3 \sin(\psi + \alpha_2) + A_4 \sin 3\psi$$
$$\Delta u = A_5 + A_6 \cos(2\psi + \alpha_3) + A_7 \cos(4\psi + \alpha_4)$$

From these expressions can be derived the algebraic equation on which the

solution of the problem depends. Let us adopt $\tan\psi = x$ as the unknown. Then

$$\sin\psi = x(1+x^2)^{-\frac{1}{2}}\ ,\ \cos\psi = (1+x^2)^{-\frac{1}{2}},$$
$$\sin 2\psi = 2x(1+x^2)^{-1}\ ,\ \cos 2\psi = (1-x^2)(1+x^2)^{-1}$$
$$\sin 3\psi = 3x(1+x^2)^{-\frac{1}{2}} - 4x^3(1+x^2)^{-\frac{3}{2}},$$
$$\sin 4\psi = 4x(1-x^2)(1+x^2)^{-2}\ ,$$
$$\cos 4\psi = (1-4x^2+2x^4)(1+x^2)^{-2}$$

For brevity put

$$\Delta u\,(1+x^2)^{\frac{3}{2}} = \rho$$

then

$$\rho\cos\chi = B_1(1+x^2) + B_2x + B_3x^3$$
$$\rho\sin\chi = B_4(1+x^2) + B_5x + B_6x^3$$
$$\rho(1+x^2)^2 = B_7 + B_8x + B_9x^2 + B_{10}x^3 + B_{11}x^4$$

where the coefficients B have the following values:

$$\begin{aligned}
B_1 &= -S\,.\sin\alpha_1\sin(\beta_3-\beta_2)\sin\beta_1\cos\beta_1\\
B_2 &= \ \ S\,.\sin\alpha_1\sin(\beta_3-\beta_2)\,(1+\cos^2\beta_1)\\
B_3 &= -S\,.\sin\alpha_1\sin(\beta_3-\beta_2)\sin^2\beta_1\\
B_4 &= -S\,.\cos\alpha_1\sin(\beta_3-\beta_2)\sin\beta_1\cos\beta_1\\
B_5 &= \ \ S\,.\cos\alpha_1\sin(\beta_3-\beta_2)\,(1+\cos^2\beta_1)\\
B_6 &= -S\,.\cos\alpha_1\sin(\beta_3-\beta_2)\sin^2\beta_1\\
B_7 &= \tfrac{1}{2}S\,.\sin(\alpha_2-\alpha_3)\,[\cos(\beta_2-\beta_3)+\cos\beta_1]\sin^2\beta_1\\
B_8 &= \ \ S\,.\sin(\alpha_2-\alpha_3)\,[\cos(\beta_2-\beta_3)\cos\beta_1+1]\sin\beta_1\\
B_9 &= \tfrac{1}{4}S\,.\sin(\alpha_2-\alpha_3)\,[2\cos(\beta_2-\beta_3)+2\cos\beta_1-\cos 3\beta_1]\\
B_{10} &= \tfrac{1}{2}S\,.\sin(\alpha_2-\alpha_3)\,[2\cos(\beta_2-\beta_3)\cos\beta_1-1]\sin\beta_1\\
B_{11} &= \tfrac{1}{8}S\,.\sin(\alpha_2-\alpha_3)\,[4\cos(\beta_2-\beta_3)\cos\beta_1-1-4\cos^2\beta_1]\cos\beta_1
\end{aligned}$$

The equation in x is then

$$\left\{\begin{array}{l}[B_1(1+x^2)+B_2x+B_3x^3]^2\\ +[B_4(1+x^2)+B_5x+B_6x^3]^2\end{array}\right\}(1+x^2) = [B_7+B_8x+B_9x^2+B_{10}x^3+B_{11}x^4]^2$$

and thus is of the eighth degree. We do not elaborate it further, as it is less laborious to employ the expressions for Δu, $\Delta u\cos\chi$ and $\Delta u\sin\chi$.

As an illustration take the example given by Ptolemy in the case of Mars. Here, at the three selected oppositions of Mars with the mean Sun, the observed longitudes of the planet in their order were 81°, 148° 50′ and 242° 34′. From the known period it is gathered that, exclusive of whole circumferences, motion in the mean longitude between the first and second observations was 81° 44′, and between the second and third 95° 28′. Making $\alpha_1 = 0°$ and adopting the condition $\beta_1+\beta_2+\beta_3 = 0$, these data make

$$\begin{array}{lll}
\alpha_1 = \ \ 0° & \alpha_2 = \ \ 149°\,34' & \alpha_3 = \ \ 338°\,46'\\
\beta_1 = -\ 9°\,50'\,40'' & \beta_2 = +\ \ 4°\ \ 3'\,20'' & \beta_3 = +\ \ 5°\,47'\,20''
\end{array}$$

The substitution of these values in the formulas gives (logarithms are in [ ]),

$$\Delta u\cos\chi = [9.0480661]\sin(\psi+170°\,32'\,22''.35) - [9.0482551]\sin 3\psi$$
$$\Delta u\sin\chi = [9.3825734]\sin(\psi-\ \ 8°\ \ 2'\,37''.59) + [9.3860693]\sin 3\psi$$
$$\begin{aligned}\Delta u = -\,0.08286966 &+ [8.8855859]\cos(2\psi+95°\,49'\,15''.62)\\ &+ [8.9513935]\cos(4\psi+\ \ 8°\,21'\,48''.29)\end{aligned}$$

By moving ψ from 0° to 180° we discover the following six real solutions of the problem:

	1	2	3
	° ′ ″	° ′ ″	° ′ ″
ψ	1 37 22.78	2 47 43.58	71 28 20.82
χ	132 11 58.15	283 57 57.55	294 21 58.28
log u	8.0893041	8.2967311	9.9836184
	4	5	6
ψ	85 9 41.39	85 30 8.91	96 52 54.68
χ	77 23 26.50	284 47 47.15	117 14 48.82
log u	9.9958862	0.0008790	9.9886814

The two remaining roots of the equation of the 8th degree are imaginary.

Although all the six solutions satisfy the equations, the second is the only one which fulfills all the conditions of the problem. Those not involved in the equations are the following: μ and v must together lie between 0° and 180° or between 180° and 360°, and, in the first case, μ must exceed v, and, in the second case, v must exceed μ. Every right line drawn through a point has two orientations differing 180°; in the equations this duplicity is left undecided, but, in the problem, that orientation must be chosen which is directed towards the point of intersection on the circumference.

The values of the mean and true anomalies, at the times of the several observations, given by the second solution are:

μ	v
° ′ ″	° ′ ″
318 27 30.57	325 30 26.98
40 11 30.57	33 20 26.98
135 39 30.57	127 4 26.98

Whence it follows that the longitude of the apogee is 115° 29′ 33″.01; and, from u we deduce that the eccentricity of the orbit $= \frac{1}{BC} = 0.1000026$.

Ptolemy's values of these quantities are 115° 30′ and 0.1. His procedure in treating the problem virtually consists in the assumption that the eccentricity is so small that, for a first approximation, we may put unity for cos ψ in the equation defining the connection between the two anomalies. This makes the values of the unknowns depend on equations of the first degree. The linearity of the equations is maintained in the following approximations by computing the length of certain lines in the geometrical figure from the elements of the preceding approximation. Ptolemy's method is ingenious but tedious, at least in the narration of it.

MEMOIR No. 68.

Normal Positions of Ceres.

(Astronomical Journal, Vol. XXI, pp. 51–54, 1900.)

Having been at some pains to collect the observations of Ceres for the century it has now been known, and having formed normals from that part of the material which seemed suitable for the purpose, I have concluded to publish the results apart from any comparison with a definite theory. The elaboration of the latter involves so much work, that, although something has been done, I can hardly hope to finish it; but the labor of forming the normals need not be lost.

Ceres has been observed at every opposition since its discovery; but on two occasions, which will be noticed in the list to be given, the material is so scanty and discordant that no normals were formed. Some of the observations in the decade following the discovery of the planet were unreduced, especially those made at Palermo and Milan; these have been reduced as well as the data permit. But, for the remaining material, the published reduction has been accepted without the application of any corrections. No accurate ephemeris of the planet was published for the interval 1801–1830; accordingly, approximate tables were constructed giving the heliocentric position. The theory employed was that in *A. J.*, Vol. XVI, pp. 57–62. But only the ten largest equations were tabulated, as that seemed sufficient for the purpose of normal-forming.

The following is a description of the material used in each opposition. As it was concluded to form but one normal for each of these occasions, the aim has been to limit the range of observation to 40 days. Where the observations are found in out of the way places, the place of publication is given.

1801.—For this normal we have only the observations of Piazzi at Palermo, between Jan. 1-Feb. 11. They have been reduced anew. The observation at the transit instrument on Jan. 18 is in error in some incorrigible way and is rejected.

1802.—29 observations at Palermo, Mar. 2-Apr. 19; 9 at Vienna, Mar. 3-Mar. 20; 6 at Greenwich, Mar. 6-Apr. 21; 13 at Paris, Apr. 7-Apr. 30; 27 at Seeberg, Mar. 3-Apr. 19.

1803.—7 observations at Greenwich, June 23-July 18; 17 at Paris, June 25-July 27; 13 at Seeberg, July 1-July 23; 13 at Milan, June 27-July 26; 28 at Palermo, June 17-July 23. The Milan observations are in the *Effemeridi Milano.*

1804.—5 observations at Paris, Sept. 13-Oct. 6; 19 at Seeberg, Sept. 13-Oct. 24; 12 at Milan, Sept. 19-Oct. 17; 15 at Palermo, Oct. 2-Oct. 25.

1806.—6 observations at Palermo, Jan. 8-Feb. 8; 4 at Milan, Jan. 4-Jan. 12; 2 at Ofen, Jan. 22-Jan. 29.

1807.—17 observations at Palermo, May 2-May 24; 15 at Paris, Apr. 21-May 24; 9 at Göttingen, Apr. 26-May 7; 5 at Padua, Apr. 10-Apr. 24; 15 at Milan, Apr. 19-May 13.

1808.—21 observations at Palermo, Aug. 2-Aug. 25; 1 at Göttingen, July 25; 9 at Milan, July 28-Aug. 14.

1809.—4 observations at Palermo, Nov. 11-Nov. 18; 4 at Milan, Nov. 1-Nov. 6; 8 at Paris, Oct. 23-Nov. 9.

1811.—4 observations at Greenwich, Mar. 9-Mar. 22; 8 at Seeberg, Feb. 17-Feb. 25; 8 at Milan, Feb. 13-Feb. 25; 3 at Paris, Feb. 18-Feb. 27.

1812.—6 observations at Palermo, June 8-June 19; 7 at Milan, June 6-June 13; 2 at Greenwich, June 14-June 15.

1813.—3 observations at Vienna, Sept. 18-Sept. 20; 7 at Wilna, Sept. 2-Sept. 10; 4 at Copenhagen, Sept. 8-Sept. 18. These observations are in the *Berl. Jahrbuch,* 1817, p. 146.

1814.—There has been found but one complete observation at Greenwich on Dec. 16, and a R. A. at Königsberg on Dec. 3. As the two R. A.'s disagree, no normal has been formed for this opposition.

1816.—4 observations at Paris, Apr. 19-May 1; 12 at Königsberg, Mar. 26-Apr. 18.

1817.—As but one observation was found, which was made at Greenwich July 19, it was thought not worth while to give a normal for this opposition.

1818.—5 observations at Greenwich, Oct. 7-Oct. 16; 8 at Königsberg, Oct. 3-Oct. 20.

1820.—2 observations at Mannheim, Jan. 31-Feb. 2; 3 at Berlin, Feb. 8-Feb. 14; 3 at Munich, Feb. 5-Feb. 8. The first and second sets are in the *Berl. Jahrbuch* for 1823 and 1824.

1821.—5 observations at Greenwich May 16-May 26; 5 at Paris, May 24-June 1; 9 at Königsberg, May 8-June 3.

1822.—3 observations at Paris, Aug. 17-Aug. 29; 6 at Königsberg, Aug. 18-Sept. 3.

1823.—3 observations at Greenwich, Dec. 1-Dec. 9; 5 at Paris, Nov. 10-Dec. 7.

1825.—2 observations at Greenwich, Mar. 11-Mar. 18; 7 at Paris, Mar. 17-Mar. 29; 6 at Königsberg, Mar. 8-Mar. 19; 7 at Göttingen, Mar. 9-Apr. 7.

1826.—6 observations at Greenwich, June 23-July 1; 4 at Königsberg, June 27-July 2.

1827.—8 observations at Königsberg, Sept. 20-Sept. 30.

1829.—3 observations at Göttingen, Jan. 22-Feb. 11.

1830.—8 observations at Greenwich, Apr. 19-May 3; 7 at Königsberg, Apr. 19-May 5; 9 at Göttingen, Apr. 24-May 5; 8 at Åbo, Apr. 17-May 12; 10 at Vienna, Apr. 17-May 12.

1831.—6 observations at Greenwich, July 22-Aug. 7; 3 at Cambridge, July 22-July 31; 5 at Vienna, July 20-Aug. 6.

1832.—4 observations at Königsberg, Oct. 29-Nov. 7; 6 at Altona, Oct. 21-Nov. 16; 6 at Kremsmünster, Oct. 16-Nov. 10.

1834.—4 observations at Greenwich, Feb. 7-Feb. 21; 6 at Königsberg, Feb. 6-Mar. 3; 7 at Mannheim, Jan. 31-Feb. 23; 6 at Cracow, Feb. 13-Mar. 1; 8 at Munich, Feb. 9-Feb. 20; 11 at Vienna, Feb. 9-Mar. 2.

1835.—3 observations at Greenwich, May 28-June 15; 10 at Königsberg, June 2-June 17; 10 at Kremsmünster, June 2-June 22; 8 at Vienna, June 3-June 27.

1836.—6 observations at Greenwich, Aug. 17-Sept. 30; 5 at Vienna, Sept. 9-Sept. 29; 1 at Helsingfors, Sept. 7; 3 at Cracow, Sept. 5-Sept. 16; 2 at Kremsmünster, Aug. 31-Sept. 1.

1837.—4 observations at Greenwich, Nov. 17-Jan. 4; 4 at Paris, Dec. 3-Dec. 26; 1 at Königsberg, Dec. 1; 3 at Vienna, Dec. 10-Dec. 30; 1 at Kremsmünster, Dec. 15.

1839.—7 observations at Greenwich, Mar. 27-May 2; 8 at Paris, Mar. 26-Apr. 26; 9 at Königsberg, Mar. 26-Apr. 20; 6 at Vienna, Apr. 5-Apr. 29; 4 at Kremsmünster, Mar. 24-Apr. 23.

1840.—3 observations at Greenwich, July 27-Aug. 14; 11 at Paris, July 14-Aug. 14; 3 at Vienna, July 21-Aug. 6.

1841.—6 observations at Greenwich, Oct. 12-Nov. 12; 6 at Paris, Oct. 11-Oct. 28; 1 at Königsberg, Oct. 5; 8 at Vienna, Oct. 18-Nov. 11; 7 at Kremsmünster, Oct. 11-Nov. 5.

1843.—2 observations at Greenwich, Jan. 30-Feb. 6; 2 at Paris, Jan. 19-Feb. 13; 2 at Königsberg, Jan. 13-Feb. 3; 5 at Kremsmünster, Feb. 1-Feb. 15.

1844.—7 observations at Greenwich, Apr. 28-June 1; 14 at Paris, May 1-June 1; 9 at Königsberg, May 15-June 1; 9 at Kremsmünster, May 3-June 1; 11 at Hamburg, May 7-May 30.

1845.—8 observations at Greenwich Aug. 15-Sept. 9; 5 at Paris, Aug. 22-Sept. 6; 9 at Königsberg, Aug. 11-Sept. 5; 8 at Kremsmünster, Aug. 9-Sept. 8.

1846.—4 observations at Greenwich, Nov. 3-Dec. 4; 2 at Paris, Nov. 20-Nov. 26; 5 at Bonn, Nov. 10-Dec. 1.

1848.—7 observations at Greenwich, Mar. 7-Apr. 14; 8 at Paris, Mar. 12-Apr. 14; 2 at Königsberg, Mar. 13-Mar. 20; 9 at Hamburg, Mar. 22-Apr. 4.

1849.—4 observations at Greenwich, July 13-July 26; 13 at Paris, June 18-July 13; 5 at Königsberg, July 7-July 13; 3 at Leipzig, July 11-July 15.

1850.—9 observations at Greenwich, Sept. 6-Oct. 5; 1 at Paris, Sept. 6.

1852.—6 observations at Greenwich, Dec. 10-Jan. 23; 7 at Kremsmünster, Jan. 1-Jan. 24.

1853.—10 observations at Greenwich, Apr. 7-May 25; 5 at Paris, Apr. 17-May 10; 4 at Königsberg, Apr. 22-May 17; 6 at Kremsmünster, Apr. 25-May 24; 1 at Bonn, Apr. 13.

1854.—4 observations at Greenwich, July 20-Aug. 29; 11 at Paris, Aug. 2-Aug. 30; 7 at Kremsmünster, July 24-Aug. 14; 3 at Bonn, July 25-Aug. 13.

1855.—3 observations at Greenwich, Oct. 15-Nov. 10; 5 at Paris, Oct. 19-Nov. 11; 4 at Kremsmünster, Oct. 28-Nov. 13; 2 at Bonn, Oct. 22-Nov. 2.

1857.—8 observations at Greenwich, Feb. 16-Mar. 16; 3 at Paris, Feb. 24-Mar. 11; 10 at Königsberg, Feb. 3-Mar. 1; 10 at Berlin, Jan. 31-Feb. 27; 13 at Kremsmünster, Feb. 14-Mar. 18; 11 at Bonn, Feb. 5-Feb. 25.

1858.—11 observations at Greenwich, May 18-June 18; 7 at Paris, May 19-June 11; 12 at Königsberg, May 29-June 18.

1859.—10 observations at Greenwich, Aug. 19-Sept. 19; 10 at Paris, Aug. 19-Sept. 18; 7 at Königsberg, Aug. 22-Sept. 19.

1860.—7 observations at Greenwich, Nov. 15-Dec. 19; 3 at Paris, Nov. 18-Dec. 20; 3 at Königsberg, Nov. 30-Dec. 4; 3 at Berlin, Nov. 22-Dec. 6; 2 at Kremsmünster, Dec. 8-Dec. 21.

1862.—13 observations at Greenwich, Mar. 25-May 5; 11 at Paris, Mar. 30-May 5; 3 at Berlin, Apr. 2-Apr. 9; 1 at Vienna, May 2; 3 at Copenhagen, Apr. 15-Apr. 21; 1 at Königsberg, Apr. 9; 8 at Kremsmünster, Apr. 2-May 3.

1863.—7 observations at Greenwich, July 2-Aug. 7; 13 at Paris, July 5-Aug. 3; 1 at Leiden, July 12; 2 at Kremsmünster, July 19-July 28; 1 at Berlin, July 3.

1864.—11 observations at Greenwich and Paris, Oct. 3-Nov. 3; 2 at Königsberg, Oct. 3-Oct. 11; 8 at Leiden, Oct. 2-Oct. 20; 5 at Cracow, Oct. 16-Oct. 23; 1 at Washington, Oct. 25-28.

1866.—5 observations at Greenwich and Paris, Jan. 22-Feb. 23; 10 at Leiden, Jan. 15-Feb. 15; 4 at Washington, Jan. 31-Feb. 26.

1867.—11 observations at Greenwich and Paris, May 7-June 4; 6 at Königsberg, May 8-May 31; 5 at Bonn, May 18-May 30; 2 at Leipzig, June 1-June 2; 2 at Leiden, May 17-June 1; 2 at Kremsmünster, June 5-June 6.

1868.—7 observations at Greenwich and Paris, Aug. 25-Sept. 8; 5 at Kremsmünster, Aug. 26-Sept. 4; 4 at Leiden, Aug. 8-Aug. 25; 11 at Warsaw, Aug. 10-Sept. 8; 9 (in Dec.) at Padua, Aug. 15-Sept. 5; 3 at Washington, Aug. 13-Aug. 29.

1869.—6 observations at Greenwich and Paris, Nov. 8-Dec. 4; 2 at Berlin, Nov. 12-Dec. 1; 3 at Leipzig, Nov. 12-Nov. 29; 4 at Warsaw, Nov. 23-Dec. 10.

1871.—4 observations at Greenwich, Mar. 21-Apr. 4; 5 at Kremsmünster, Mar. 13-Mar. 24; 5 at Berlin, Mar. 1-Mar. 24; 3 at Leiden, Feb. 28-Mar. 13.

1872.—11 observations at Greenwich and Paris, June 14-July 12; 3 at Königsberg, July 6-July 12; 2 at Kremsmünster, July 7-July 10; 2 at Berlin, June 20-June 29; 2 at Leipzig, June 23-July 8; 3 at Neufchâtel, June 14-June 22.

1873.—13 observations at Greenwich and Paris, Sept. 19-Oct. 17; 2 at Königsberg, Sept. 28-Oct. 13; 2 at Kremsmünster, Sept. 11-Oct. 3; 2 at Berlin, Sept. 18-Sept. 21; 1 at Vienna, Oct. 10; 3 at Leiden, Sept. 20-Sept. 27; 4 at Madrid, Sept. 24-Sept. 27.

1875.—9 observations at Greenwich and Paris, Jan. 5-Jan. 28; 5 at Washington, Dec. 11-Dec. 23.

1876.—16 observations at Greenwich and Paris, Apr. 21-May 19; 1 at Vienna, May 14; 6 at Washington, Apr. 22-May 13.

1877.—7 observations at Greenwich and Paris, July 24-Aug. 27.

1878.—7 observations at Greenwich and Paris, Oct. 25-Nov. 19; 4 at Hamburg, Nov. 6-Nov. 20.

1880.—5 observations at Greenwich, Feb. 11-Mar. 15; 4 at Königsberg, Mar. 6-Mar. 14; 5 at Pulkowa, Feb. 16-Feb. 23; 1 at Washington, Mar. 5.

1881.—11 observations at Greenwich, May 30-June 30; 1 at Hamburg, June 29; 3 at Washington, June 12-June 24.

1882.—8 observations at Greeuwich, Sept. 2-Oct. 4.

1883.—10 observations at Greenwich, Nov. 20-Dec. 15; 10 at Washington, Nov. 17-Dec. 17.

1885.—14 observations at Greenwich, Mar. 27-Apr. 25; 6 at Hamburg, Mar. 28-Apr. 28; 3 at Berlin, Apr. 24-Apr. 28.

1886.—2 observations at Greenwich, July 5-Aug. 2; 6 at Paris, July 22-Aug. 10.

1887.—6 observations at Greenwich, Sept. 29-Oct. 31; 9 at Paris, Oct. 14-Nov. 3.

1889.—8 observations at Greenwich, Jan. 8-Feb. 15.

1890.—9 observations at Greenwich, May 14-June 9; 8 at Paris, May 21-June 9.

1891.—11 observations at Paris, Aug. 24-Sept. 16.

1892.—6 observations at Greenwich, Nov. 1-Nov. 30.

1894.—10 observations at Greenwich, Feb. 20-Mar. 27; 10 at Paris, Mar. 10-Mar. 24.

1895.—6 observations at Greenwich, June 5-July 9; 9 at Toulouse, June 15-July 6.

1896.—4 observations at Greenwich, Sept. 9-Oct. 23.

1897.—9 observations at Greenwich, Nov. 30-Dec. 31.

The dates of the normals are for Greenwich mean noon, and the given values of the coordinates are *true* not *apparent*. In the column headed No. Obs. where there are two numbers, the first belongs to the R. A., and the second to the Decl.; where but one number is given this is common to both. In the preceding list the number of observations given is that of R. A. (except in one case where there was none).

Normal Positions of Ceres.

Date.			True R. A.			True Decl.			No. Obs.
			h	m	s	°	′	″	
1801	Jan.	21	3	26	31.25	+16	57	21.6	22-21
1802	Mar.	31	12	2	57.76	+17	59	48.4	84-80
1803	July	8	18	37	7.29	—28	44	23.2	78-67
1804	Oct.	1	0	37	34.42	—12	53	56.6	51-49
1806	Jan.	21	6	41	58.91	+30	34	28.2	12-11
1807	May	6	14	50	31.06	— 5	18	26.0	61
1808	Aug.	10	21	15	12.22	—29	25	56.6	31-29
1809	Nov.	3	2	45	57.21	+ 4	54	55.8	16
1811	Feb.	24	10	29	53.65	+27	0	41.2	23
1812	June	12	17	18	55.70	—23	7	54.0	15
1813	Sept.	11	23	36	0.01	—20	0	20.4	14
1816	Apr.	12	13	26	22.53	+ 7	10	41.5	16-15
1818	Oct.	13	1	41	7.26	— 4	33	4.8	13
1820	Feb.	7	8	28	46.06	+31	56	48.4	8-7
1821	May	25	15	58	28.54	—14	53	8.3	19

Normal Positions of Ceres.—Continued.

Date.		True R. A. h	m	s	True Decl. °	′	″	No. Obs.
1822	Aug. 24	22	26	18.54	—25	50	22.0	9-8
1823	Nov. 24	3	49	43.11	+13	48	45.5	8-6
1825	Mar. 19	12	2	38.57	+18	26	28.2	22
1826	June 28	18	37	13.12	—27	55	32.5	10
1827	Sept. 26	0	36	36.66	—13	10	13.4	8
1829	Jan. 30	6	24	16.94	+30	40	39.0	3
1830	May 1	14	43	30.55	— 4	9	0.1	43
1831	July 29	21	18	10.16	—28	36	19.4	14-10
1832	Oct. 31	2	40	56.18	+ 4	4	6.4	16-10
1834	Feb. 17	10	21	31.16	+27	17	46.5	42-41
1835	June 10	17	14	5.88	—22	35	32.3	31
1836	Sept. 10	23	30	5.77	—19	38	21.2	17-16
1837	Dec. 15	5	1	35.07	+22	16	8.7	13
1839	Apr. 13	13	18	1.12	+ 8	12	23.7	34
1840	Aug. 2	19	38	43.35	—31	15	29.4	15-16
1841	Oct. 26	1	23	32.81	— 5	50	54.2	28-26
1843	Feb. 5	8	18	38.23	+32	2	25.3	11
1844	May 17	15	59	4.89	—14	10	35.2	50-48
1845	Aug. 29	22	16	34.94	—26	38	14.5	30
1846	Nov. 19	3	47	36.12	+13	10	4.4	11
1848	Mar. 28	11	44	36.17	+19	59	47.3	26-22
1849	July 5	18	22	26.38	—28	11	45.6	25-23
1850	Sept. 18	0	37	56.44	—13	4	21.3	10
1852	Jan. 8	6	32	47.89	+29	9	31.1	13
1853	May 5	14	30	44.15	— 3	1	41.6	26-25
1854	Aug. 13	20	57	56.49	—30	15	24.5	25-28
1855	Nov. 4	2	30	18.54	+ 3	8	17.2	14-16
1857	Feb. 22	10	7	26.90	+28	27	6.2	55
1858	June 6	17	11	32.29	—22	2	35.3	30
1859	Sept. 1	23	31	59.71	—20	13	37.5	27
1860	Dec. 4	5	4	30.93	+21	15	0.0	18-17
1862	Apr. 21	13	5	28.44	+ 9	5	26.2	40-37
1863	July 16	19	48	49.23	—30	4	5.7	24
1864	Oct. 16	1	26	23.54	— 6	8	42.6	27-28
1866	Feb. 4	8	10	44.46	+32	5	59.1	19-21
1867	May 26	15	45	39.55	—13	43	22.6	28
1868	Aug. 27	22	12	40.98	—26	50	32.8	30-39
1869	Nov. 23	3	37	5.90	+12	35	19.5	15-16
1871	Mar. 19	11	43	0.10	+20	30	36.0	17
1872	June 30	18	20	11.97	—27	44	34.7	23-22
1873	Sept. 30	0	23	7.31	—14	34	29.6	27
1875	Jan. 5	6	25	8.51	+28	40	38.6	14

Normal Positions of Ceres.—Concluded.

Date.		True R. A.			True Decl.			No. Obs.
		h	m	s	°	′	″	
1876	May 6	14	21	47.04	— 2	2	46.2	23
1877	Aug. 17	20	48	16.32	—30	44	40.7	7
1878	Nov. 7	2	22	8.73	+ 2	25	48.7	11-10
1880	Feb. 28	9	53	47.79	+29	23	14.7	15
1881	June 16	16	54	4.83	—21	55	43.7	15-16
1882	Sept. 23	23	7	52.33	—22	28	41.0	8-7
1883	Dec. 4	4	58	28.24	+20	45	39.7	20-13
1885	Apr. 14	13	3	26.43	+ 9	53	57.6	23
1886	July 30	19	27	9.79	—31	2	52.7	8
1887	Oct. 22	1	16	15.96	— 7	4	20.9	15-14
1889	Feb. 4	8	0	40.55	+32	13	15.1	8
1890	May 27	15	35	47.70	—12	51	32.2	17-18
1891	Sept. 6	21	57	44.06	—27	52	42.0	11
1892	Nov. 18	3	33	18.75	+11	45	45.3	6
1894	Mar. 15	11	33	59.87	+21	29	40.2	20-21
1895	June 22	18	20	58.40	—27	0	17.7	15-14
1896	Sept. 26	0	20	25.27	—14	58	1.5	4-5
1897	Dec. 19	6	29	58.43	+27	2	21.4	9

MEMOIR No. 69.

Secular Perturbations of the Planets.

(American Journal of Mathematics, Vol. XXIII, pp. 317–336, 1901.)

Gauss first clearly indicated the rôle elliptic functions play in this subject.* Halphen has since presented the investigation in a very elegant manner.† The modifications made by the latter in the procedure of Gauss are chiefly the transference of the origin of rectangular coordinates to the attracted planet, and, instead of the differential of the eccentric anomaly, the adoption of the element of area described by the radius of the disturbing planet expressed in terms of the differentials of the rectangular coordinates. He also appeals to the qualities of the cone formed by the orbit of the attracting planet as contour of base and the position of the attracted planet as vertex; this improvement, however, had been previously indicated by Bour.‡ A remarkable degree of elegance is attained by these changes; but it seems to me that additional statements are needed to show the connection with the astronomical problem which originally suggested the investigation; for Halphen, like Gauss, treats only the attraction of a certain form of ring. This, of course, is to ignore the second integration which the problem demands. Perhaps, therefore, I shall be pardoned if I here attempt to supply the mentioned lack.

In the fashion of Halphen we take the attracted planet as the origin of rectangular coordinates, but the orientation of the axes is, for the present, left indeterminate. The coordinates of the attracting planet we denote by x, y, z; and the coordinates of the Sun, which are the negatives of those of the attracted planet referred to the Sun, will be x_0, y_0, z_0. Let ρ be the distance of the attracting planet from the origin, so that $\rho^2 = x^2 + y^2 + z^2$; and let g denote the planet's mean anomaly. Then the secular perturbations of the attracted planet depend on the three definite integrals

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{x}{\rho^3}\,dg\,,\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{y}{\rho^3}\,dg\,,\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{z}{\rho^3}\,dg\,.$$

* Gauss, Werke, vol. III, pp. 331–355.

† G. H. Halphen, Traité des Fonctions Eliptiques et de leurs Applications, Tom. II, pp. 310–328.

‡ Journal de l'École Polytechnique, Cahier XXXVI, pp. 59–84.

But while g is passing from 0 to 2π, the area described by the radius of the planet augments from 0 to πab, if a and b are severally the major and minor semi-axes. Thus, if σ denote this varying area, the preceding integrals may be written

$$\frac{1}{\pi ab}\int_0^{\pi ab}\frac{x}{\rho^3}\,d\sigma,\quad \frac{1}{\pi ab}\int_0^{\pi ab}\frac{y}{\rho^3}\,d\sigma,\quad \frac{1}{\pi ab}\int_0^{\pi ab}\frac{z}{\rho^3}\,d\sigma.$$

The tetrahedron, with $d\sigma$ as base and the origin as vertex, has, for volume one-sixth of the following expression:

$$6V = x_0(ydz - zdy) + y_0(zdx - xdz) + z_0(xdy - ydx).$$

But, if h denote the perpendicular from the origin on the plane of the orbit of the attracting planet, we also have $3V = hd\sigma$. Hence

$$d\sigma = \frac{x_0}{2h}(ydz - zdy) + \frac{y_0}{2h}(zdx - xdz) + \frac{z_0}{2h}(xdy - ydx).$$

Then, if we derive the quantities P_x, P_y, P_z, etc., from integrating the expressions

$$dP_x = \tfrac{1}{2}\frac{x(ydz - zdy)}{\rho^3},\qquad dP_y = \tfrac{1}{2}\frac{y(ydz - zdy)}{\rho^3},\qquad dP_z = \tfrac{1}{2}\frac{z(ydz - zdy)}{\rho^3},$$

$$dQ_x = \tfrac{1}{2}\frac{x(zdx - xdz)}{\rho^3},\qquad dQ_y = \tfrac{1}{2}\frac{y(zdx - xdz)}{\rho^3},\qquad dQ_z = \tfrac{1}{2}\frac{z(zdx - xdz)}{\rho^3},$$

$$dR_x = \tfrac{1}{2}\frac{x(xdy - ydx)}{\rho^3},\qquad dR_y = \tfrac{1}{2}\frac{y(xdy - ydx)}{\rho^3},\qquad dR_z = \tfrac{1}{2}\frac{z(xdy - ydx)}{\rho^3},$$

around the whole orbit, the integrals above, which we will denote by X, Y, Z, will be given by the following expressions:

$$X = \frac{1}{\pi abh}(x_0P_x + y_0Q_x + z_0R_x),$$

$$Y = \frac{1}{\pi abh}(x_0P_y + y_0Q_y + z_0R_y),$$

$$Z = \frac{1}{\pi abh}(x_0P_z + y_0Q_z + z_0R_z).$$

The nine quantities involved in these expressions and obtained through integration are homogeneous and of the dimension zero with respect to the linear unit. Moreover, if, in the cone formed by the orbit of the attracting planet as directrix and the origin as vertex, the plane of the base is shifted in any manner whatever, these nine quantities remain unchanged. For example, consider the expressions

$$\frac{x(ydz - zdy)}{\rho^3},\qquad \frac{y(ydz - zdy)}{\rho^3},\qquad \frac{z(ydz - zdy)}{\rho^3}.$$

If we put

$$x = \rho \sin \theta, \quad y = \rho \cos \theta \cos \lambda, \quad z = \rho \cos \theta \sin \lambda,$$

they are transformed into

$$\sin \theta \cos^2 \theta \, d\lambda, \quad \cos^3 \theta \cos \lambda \, d\lambda, \quad \cos^3 \theta \sin \lambda \, d\lambda.$$

Let the equation of the cone be

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy = 0,$$

or

$$A \tan^2 \theta + B \cos^2 \lambda + C \sin^2 \lambda + D \sin \lambda \cos \lambda + E \tan \theta \sin \lambda + F \tan \theta \cos \lambda = 0.$$

Since, from this equation, θ is obtainable as a function of λ, it is evident that the integrals of the preceding differential expressions, extended to the whole course of variation of λ, depend solely on the elements of the cone and are altogether independent of the plane section called the base.

A simple addition of the differentials shows that

$$P_x + Q_y + R_z = 0.$$

Also we have

$$d(Q_z - R_y) = \tfrac{1}{2} d\left(\frac{x}{\rho}\right), \quad d(R_x - P_z) = \tfrac{1}{2} d\left(\frac{y}{\rho}\right), \quad d(P_y - Q_x) = \tfrac{1}{2} d\left(\frac{z}{\rho}\right).$$

But the second members of these equations, integrated along the orbit to the point of beginning, give zero as the result: hence

$$Q_z = R_y, \quad R_x = P_z, \quad P_y = Q_x.$$

Thus

$$X = \frac{1}{\pi abh}(x_0 P_x + y_0 Q_x + z_0 P_z),$$

$$Y = \frac{1}{\pi abh}(x_0 Q_x + y_0 Q_y + z_0 R_y),$$

$$Z = \frac{1}{\pi abh}(x_0 P_z + y_0 R_y + z_0 R_z).$$

And, if we put

$$\Phi = \frac{1}{2\pi abh}[x_0^2 P_x + y_0^2 Q_y + z_0^2 R_z + 2x_0 y_0 Q_x + 2y_0 z_0 R_y + 2z_0 x_0 P_z],$$

we shall have

$$X = \frac{\partial \Phi}{\partial x_0}, \quad Y = \frac{\partial \Phi}{\partial y_0}, \quad Z = \frac{\partial \Phi}{\partial z_0}.$$

The orientation of the axes of coordinates which serve to define the variables x, y, z, x_0, y_0, z_0 has been left undetermined; but now suppose that the axes of symmetry of the cone are employed for this purpose. Then the equation of the cone takes the form

$$\frac{x^2}{G_x} + \frac{y^2}{G_y} + \frac{z^2}{G_z} = 0,$$

G_x, G_y, G_z being constants of which two are of one sign and the other of

the opposite sign. Plainly, if this equation is satisfied by the set of values x, y, z, it is satisfied by any of the eight sets $\pm x$, $\pm y$, $\pm z$. Consequently, each positive element of the six quantities Q_z, R_y, R_x, P_z, P_y, Q_x is accompanied by a corresponding negative element. Thus, in this case, these quantities vanish. With this selection of axes we, therefore, have

$$X = \frac{x_0}{\pi abh} P_x, \quad Y = \frac{y_0}{\pi abh} Q_y, \quad Z = \frac{z_0}{\pi abh} R_z.$$

The naming of the coordinates is, of course, arbitrary, but, to settle the choice, we suppose that G_x, G_y, G_z are in the order of algebraic magnitude, the first and second being negative, while the last is positive. The equation of the cone appears to involve three variables, but, as we may divide the left member by the square of any one of them, it is, in reality, a relation between two variables; thus, but one variable can be regarded as independent. The equation is then satisfied if we make

$$x = \varepsilon \sqrt{-G_x} \cos T, \quad y = \varepsilon \sqrt{-G_y} \sin T, \quad z = \varepsilon \sqrt{G_z},$$

where ε is an indeterminate which disappears when the substitution is made in dP_x, dQ_y, dR_z, and T is the new variable introduced by Gauss and may be regarded as indicating the position of the planet in its orbit; its function, in this respect, being precisely similar to those fulfilled by the mean, eccentric and true anomalies, and it may thus, with propriety, be designated as a *perspective anomaly*. When T goes from 0 to 2π, the planet makes a complete circuit of its orbit.

The substitution made, we have

$$\begin{aligned} ydz - zdy &= -\varepsilon^2 \sqrt{-G_y G_z} \cos T dT, \\ zdx - xdz &= -\varepsilon^2 \sqrt{-G_x G_z} \sin T dT, \\ xdy - ydx &= \quad \varepsilon^2 \sqrt{G_x G_y}\, dT, \\ \rho^2 &= \quad \varepsilon^2 [G_z - G_y \sin^2 T - G_x \cos^2 T]. \end{aligned}$$

The quantities G_x, G_y, G_z are usually determined in such a way that $G_x G_y G_z = a^2 b^2 h^2$; also we may introduce k the modulus of the elliptic integrals involved and m such that

$$k^2 = \frac{G_y - G_x}{G_z - G_x}, \quad m = \sqrt{G_z - G_x}.$$

Then our integrals take the forms

$$X = -\frac{x_0}{m^3}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^2 T dT}{(1 - k^2 \sin^2 T)^{\frac{3}{2}}}, \qquad Y = -\frac{y_0}{m^3}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 T dT}{(1 - k^2 \sin^2 T)^{\frac{3}{2}}},$$

$$Z = \frac{z_0}{m^3}\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{dT}{(1 - k^2 \sin^2 T)^{\frac{3}{2}}}.$$

The methods of evaluating these definite integrals severally proposed

by Legendre, Gauss and Jacobi have all about the same degree of rapid convergence, but that of the last is to be preferred because it expresses the values explicitly in terms of a parameter q called the *nome*. Putting $k = \sin\theta$, q can be derived from the equation

$$\frac{q + q^9 + q^{25} + \ldots.}{1 + 2(q^4 + q^{16} + q^{36} + \ldots.)} = \left[\frac{\sin\frac{\theta}{2}}{1 + \sqrt{\cos\theta}}\right]^2.$$

The solution is most readily accomplished by the method of tentation. Then, if we adopt two functions of q, K and L such that

$$K = \frac{4}{\cos^2\theta\,(1 + \sqrt{\cos\theta})^2}[1 + 2(q^4 + q^{16} + \ldots.)]^2,$$

$$L = \frac{(1 + \sqrt{\cos\theta})^3}{\sin^2\theta\cos\frac{5}{2}\theta}\,\frac{q - 4q^4 + 9q^9 - 16q^{16} + \ldots.}{[1 + 2(q^4 + q^{16} + \ldots.)]^3},$$

we have

$$X = -\frac{x_0}{m^3}L\cos^2\theta, \qquad Y = \frac{y_0}{m^3}(L - K), \qquad Z = \frac{z_0}{m^3}(K - L\sin^2\theta).*$$

It will be seen from these expressions that, if X, Y, Z are regarded as the components along the axes of coordinates of a force acting on the attracted planet, one elliptic integral suffices for determining the orientation of the resultant, but that an additional one is required if the magnitude of the latter is to be found. It will be an advantage, therefore, if instead of tabulating K and L as functions of k, q or θ, we take two other quantities M and $\varkappa$, such that

$$M = K - L\sin^2\theta, \quad \sin^2\varkappa = \frac{L\cos^2\theta}{K - L\sin^2\theta}, \quad \cos^2\varkappa = \frac{K - L}{K - L\sin^2\theta}.$$

Then we shall have

$$X = -\frac{M}{m^3}\sin^2\varkappa\,.\,x_0, \qquad Y = -\frac{M}{m^3}\cos^2\varkappa\,.\,y_0, \qquad Z = \frac{M}{m^3}z_0.$$

Let R denote the magnitude of the resultant, H and Λ severally the latitude and longitude of the point in the heavens towards which it is directed; the circles of reference being the principal axes of the sphero-conic traced in the heavens by the frequently mentioned cone. Also let r_0 denote the distance of the Sun from the attracted planet and η_0, λ_0 severally its latitude and longitude referred to the same circles. Then our equations will stand

$$R\cos H\sin\Lambda = -\frac{M}{m^3}\sin^2\varkappa\,.\,r_0\cos\eta_0\sin\lambda_0,$$

$$R\sin H \quad = -\frac{M}{m^3}\cos^2\varkappa\,.\,r_0\sin\eta_0,$$

$$R\cos H\cos\Lambda = \frac{M}{m^3}\,.\,r_0\cos\eta_0\cos\lambda_0.$$

---

* For the proof of these formulas, reference may be made to Bertrand, Calcul Intégral, Liv. III, Chap. VII.

If we put

$$N = \frac{M}{m^3} r_0 \cos \eta_0,$$

we shall have

$$\tan \Lambda = -\sin^2 x \tan \lambda_0, \quad \tan H = -\frac{\cos^2 x \tan \eta_0 \cos \Lambda}{\cos \lambda_0}, \quad R = \frac{N \cos \lambda_0}{\cos H \cos \Lambda}.$$

As, except in very particular cases, it is not easy to select at the outset the axes of symmetry of the cone for the adopted axes of coordinates, we must find the position of these axes in reference to another system which is known. The equation of the cone having a very complicated expression when the axes of coordinates are quite general, we select a particular system such that the treatment may be easy as possible. Let the axis of x have the orientation of the line going from the centre of the ellipse described by the attracting planet to its perihelion, that of y the orientation of the line going from the centre to the point where the eccentric anomaly is 90° and that of z the orientation of the line going from the centre to the north pole of the plane of the orbit. Let the coordinates of the centre of the ellipse be, in their order, A, B, C; we prefer these designations although, with e as the eccentricity and the previous notation, they have the equivalents

$$A = x_0 - ae, \qquad B = y_0, \qquad C = z_0.$$

Then the equations of the orbit of the attracting planet are

$$\left(\frac{x-A}{a}\right)^2 + \left(\frac{y-B}{b}\right)^2 = 1, \quad z = C.$$

In order to have the equation of the cone so frequently mentioned it is only necessary to multiply the several terms of the first equation by factors selected from the equivalent quantities $1 = \frac{z}{C} = \frac{z^2}{C^2}$, in such a way that they may all become homogeneous and of two dimensions in x, y, z. Thus the equation of the cone may be written in the shape

$$\frac{z^2}{C^2} - \left[\frac{x - \frac{A}{C} z}{a}\right]^2 - \left[\frac{y - \frac{B}{C} z}{b}\right]^2 = 0.$$

This equation is not, in general, referred to the axes of symmetry, hence we proceed to make the linear transformation of variables which will bring this about. Let X, Y, Z denote the rectangular coordinates referred to the symmetric axes of the cone, and write the formulas of transformation thus:

$$\begin{aligned}
\mathrm{X} &= \alpha x + \beta y + \gamma z, & \alpha^2 + \beta^2 + \gamma^2 &= 1, & \alpha\alpha' + \beta\beta' + \gamma\gamma' &= 0, \\
\mathrm{Y} &= \alpha' x + \beta' y + \gamma' z, & \alpha'^2 + \beta'^2 + \gamma'^2 &= 1, & \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' &= 0, \\
\mathrm{Z} &= \alpha'' x + \beta'' y + \gamma'' z, & \alpha''^2 + \beta''^2 + \gamma''^2 &= 1, & \alpha''\alpha + \beta''\beta + \gamma''\gamma &= 0,
\end{aligned}$$

the inverse of which are

$$x = \alpha X + \alpha' Y + \alpha'' Z, \quad \alpha^2 + \alpha'^2 + \alpha''^2 = 1, \quad \alpha\beta + \alpha'\beta' + \alpha'\beta'' = 0,$$
$$y = \beta X + \beta' Y + \beta'' Z, \quad \beta^2 + \beta'^2 + \beta''^2 = 1, \quad \beta\gamma + \beta'\gamma' + \beta''\gamma'' = 0,$$
$$z = \gamma X + \gamma' Y + \gamma'' Z, \quad \gamma^2 + \gamma'^2 + \gamma''^2 = 1, \quad \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' = 0.$$

The equation of the cone, after substition of the new variables, should take the form

$$\frac{X^2}{G_x} + \frac{Y^2}{G_y} + \frac{Z^2}{G_z} = 0,$$

where G_x, G_y, G_z are constants and functions of the quantities A, B, C, a, b. They are the roots of a certain cubic which may be obtained in the following way: Let $V = 0$ denote the first form of the equation of the cone, and set

$$\frac{x}{G} = \tfrac{1}{2}\frac{\partial V}{\partial x}, \quad \frac{y}{G} = \tfrac{1}{2}\frac{\partial V}{\partial y}, \quad \frac{z}{G} = \tfrac{1}{2}\frac{\partial V}{\partial z}.$$

From these equations, which are linear in x, y, z, eliminate these variables; the result is a cubic in G whose roots are the values of G_x, G_y, G_z. In the special form of V with which we have to deal, these equations, after a slight modification, can be thus stated:

$$\frac{x}{G} = \frac{A}{C}\frac{z}{G + a^2}, \quad \frac{y}{G} = \frac{B}{C}\frac{z}{G + b^2}, \quad \frac{z}{G} = \frac{z}{C^2} - \frac{A}{C}\frac{x}{G} - \frac{B}{C}\frac{y}{G}.$$

The elimination of x, y, z from these equations gives

$$\frac{A^2}{G + a^2} + \frac{B^2}{G + b^2} + \frac{C^2}{G} = 1.$$

It is known that the roots of this equation in G are all real, two of them being negative and one positive. The mode of assignment of these roots as values of G_x, G_y, G_z has already been described. Thus we are enabled to discover the values of the latter without reference to the nine coefficients α, β, γ, etc. But in our further progress we shall need to know the latter. They are readily found from the first and second equations of the penultimate group combined with the conditions $\alpha^2 + \beta^2 + \gamma^2 = 1$, etc. In the first we make $x = \alpha$, $y = \beta$, $z = \gamma$ and set G_x for G; again make $x = \alpha'$, $y = \beta'$, $z = \gamma'$ and set G_y for G; lastly make $x = \alpha''$, $y = \beta''$, $z = \gamma''$ and set G_z for G. Thus we have the equations

$$\alpha = \frac{A}{C}\frac{G_x}{G_x + a^2}\gamma, \quad \beta = \frac{B}{C}\frac{G_x}{G_x + b^2}\gamma, \quad \gamma^2 = \frac{1}{1 + \frac{A^2}{C^2}\left(\frac{G_x}{G_x + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_x}{G_x + b^2}\right)^2},$$

$$\alpha' = \frac{A}{C}\frac{G_y}{G_y + a^2}\gamma', \quad \beta' = \frac{B}{C}\frac{G_y}{G_y + b^2}\gamma', \quad \gamma'^2 = \frac{1}{1 + \frac{A^2}{C^2}\left(\frac{G_y}{G_y + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_y}{G_y + b^2}\right)^2},$$

$$\alpha'' = \frac{A}{C}\frac{G_z}{G_z + a^2}\gamma'', \quad \beta'' = \frac{B}{C}\frac{G_z}{G_z + b^2}\gamma'', \quad \gamma''^2 = \frac{1}{1 + \frac{A^2}{C^2}\left(\frac{G_z}{G_z + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_z}{G_z + b^2}\right)^2}.$$

These equations furnish the values of the nine coefficients of the transformation. The signs of the three γ's are inderterminate: one may take them as positive.

The cubic in G furnishes us with the relations

$$\frac{A^2}{G_x + a^2} + \frac{B^2}{G_x + b^2} + \frac{C^2}{G_x} = 1,$$
$$\frac{A^2}{G_y + a^2} + \frac{B^2}{G_y + b^2} + \frac{C^2}{G_y} = 1,$$
$$\frac{A^2}{G_z + a^2} + \frac{B^2}{G_z + b^2} + \frac{C^2}{G_z} = 1.$$

Regarding A^2, B^2, C^2 as the unknowns in these equations, their solution gives

$$A^2 = \frac{(G_x + a^2)(G_y + a^2)(G_z + a^2)}{a^2(a^2 - b^2)},$$
$$B^2 = \frac{(G_x + b^2)(G_y + b^2)(G_z + b^2)}{b^2(b^2 - a^2)},$$
$$C^2 = \frac{G_x G_y G_z}{a^2 b^2}.$$

By means of these values we can eliminate A, B, C from the foregoing expressions for the coefficients of transformation, and thus obtain values depending only on the five quantities G_x, G_y, G_z, a, b. We have thus

$$\alpha^2 = \frac{1}{a^2(a^2 - b^2)} \frac{G_x(G_x + b^2)(G_y + a^2)(G_z + a^2)}{(G_x - G_y)(G_x - G_z)},$$
$$\beta^2 = \frac{1}{b^2(b^2 - a^2)} \frac{G_x(G_x + a^2)(G_y + b^2)(G_z + b^2)}{(G_x - G_y)(G_x - G_z)},$$
$$\gamma^2 = \frac{1}{a^2 b^2} \frac{G_y G_z (G_x + a^2)(G_x + b^2)}{(G_x - G_y)(G_x - G_z)}.$$

The values of α'^2, β'^2, γ'^2 and again of α''^2, β''^2, γ''^2, are obtained from the preceding by simply making a cyclical permutation of the subscripts attached to the G; firstly from x, y, z into y, z, x, secondly from x, y, z into z, x, y. On account of the divisor $a^2 - b^2$, small when the eccentricity of the planet's orbit is small, some of these formulas labor under a disadvantage.

In computing the values of the definite integrals

$$\frac{1}{\pi ab}\int \frac{x}{\rho^3} d\sigma, \quad \frac{1}{\pi ab}\int \frac{y}{\rho^3} d\sigma, \quad \frac{1}{\pi ab}\int \frac{z}{\rho^3} d\sigma,$$

where x, y, z are referred to the system of axes determined by those of the ellipse described by the attracting planet, we may suppose that the four quantities x_0, y_0, z_0, h are simply constants and unaffected by the transformation we have just made to pass from the system of coordinates x, y, z to

that of x, y, z. If, for the sake of discrimination, we designate the components along the mentioned axes as X', Y', Z', reserving X, Y, Z for the components along the symmetric axes, we evidently have

$$X' = \alpha X + \alpha' Y + \alpha'' Z, \qquad Y' = \beta X + \beta' Y + \beta'' Z, \qquad Z' = \gamma X + \gamma' Y + \gamma'' Z.$$

The components of the right members are determined the moment we have solved the cubic in G, for which the coordinates A, B, C furnish the requisite data. We have only to make

$$x_0 = A + ae, \qquad y_0 = B, \qquad z_0 = C, \qquad h = \sqrt{C^2},$$

where h is always to be taken positively. Substituting for $\alpha, \alpha', \alpha'', \beta, \beta', \beta''$ their values in terms of $\gamma, \gamma', \gamma''$, if we put

$$U_x = -\frac{M}{m^3} \sin^2 x \, \frac{\gamma^2 G_x^2}{C^2}\left[1 + \frac{aeA}{G_x + a^2}\right],$$

$$U_y = -\frac{M}{m^3} \cos^2 x \, \frac{\gamma'^2 G_y^2}{C^2}\left[1 + \frac{aeA}{G_y + a^2}\right],$$

$$U_z = \quad \frac{M}{m^3} \frac{\gamma''^2 G_z^2}{C^2}\left[1 + \frac{aeA}{G_z + a^2}\right],$$

we shall have

$$X' = A\left[\frac{U_x}{G_x + a^2} + \frac{U_y}{G_y + a^2} + \frac{U_z}{G_z + a^2}\right],$$

$$Y' = B\left[\frac{U_x}{G_x + b^2} + \frac{U_y}{G_y + b^2} + \frac{U_z}{G_z + b^2}\right],$$

$$Z' = C\left[\frac{U_x}{G_x} + \frac{U_y}{G_y} + \frac{U_z}{G_z}\right].$$

By substituting in U_x, U_y, U_z, the last values we have given for $\gamma, \gamma', \gamma''$, and bearing in mind that

$$G_z - G_x = m^2, \qquad G_y - G_x = m^2 \sin^2\theta, \qquad G_z - G_y = m^2 \cos^2\theta,$$

the expressions for the first mentioned quantities become

$$U_x = -\frac{M \sin^2 x}{m^7 \sin^2\theta} G_x (G_x + a^2 + aeA)(G_x + b^2),$$

$$U_y = \quad \frac{M \cos^2 x}{m^7 \sin^2\theta \cos^2\theta} G_y (G_y + a^2 + aeA)(G_y + b^2),$$

$$U_z = \quad \frac{M}{m^7 \cos^2\theta} G_z (G_z + a^2 + aeA)(G_z + b^2).*$$

Let it be required to find the component of this attracting force directed towards the centre of the ellipse described by the attracting planet. Putting $r^2 = A^2 + B^2 + C^2$, we ought to multiply the components given above severally by the factors $\frac{A}{r}$, $\frac{B}{r}$, $\frac{C}{r}$, and take the sum. Which, if we do,

* In the original memoir serious errors exist in the two groups of equations for the U.

and have regard to the relations, G_x, G_y, G_z satisfy as being the roots of the cubic in G, and calling this component Δ, we have

$$r\Delta = U_x + U_y + U_z.$$

But the component R_0 directed towards the Sun will be more important. Putting r_0 for the radius vector of the attracted planet, we have

$$r_0^2 = (A + ae)^2 + B^2 + C^2.$$

Then the components X, Y, Z ought to be multiplied severally by $\dfrac{A + ae}{r_0}$, $\dfrac{B}{r_0}$, $\dfrac{C}{r_0}$, and thus is obtained

$$r_0 R_0 = U_x + U_y + U_z + aeX'.$$

In fact, by multiplying X', Y', Z' severally by the three systems of three multipliers

$$\begin{array}{ccc} \dfrac{A + ae}{r_0}, & \dfrac{B}{r_0}, & \dfrac{C}{r_0}, \\ -\dfrac{B}{\sqrt{r_0^2 - C^2}}, & \dfrac{A + ae}{\sqrt{r_0^2 - C^2}}, & 0, \\ -\dfrac{(A + ae)\, C}{r_0 \sqrt{r_0^2 - C^2}}, & -\dfrac{BC}{r_0 \sqrt{r_0^2 - C^2}}, & \dfrac{\sqrt{r_0^2 - C^2}}{r_0}, \end{array}$$

we shall arrive at a set of components, of which the first, already given, is directed toward the Sun, and the second is perpendicular thereto and lying in the plane of the attracting planet's orbit, while the third is perpendicular to both the preceding. If we call these new components X'', Y'', Z'', and if the angle between the planes of the orbits of the two planets be denoted by I, we shall have

$$R_0 = X'', \qquad S_0 = Y'' \cos I - Z'' \sin I, \qquad W_0 = Y'' \sin I + Z'' \cos I,$$

where R_0 is the component towards the Sun, S_0 the component perpendicular thereto and lying in the plane of the attracted planet's orbit, and W_0 the component perpendicular to that plane. But, if we make use of a rectangular set of elements instead of a polar, it is more commodious to refer the components to the plane and line of nodes of the attracted planet on the attracting planet's orbit. Let this ascending node be distant an arc $= \omega$ from the perihelion of the latter. Then the desired components X''', Y''', Z''' will be

$$\begin{aligned} X''' &= X' \cos \omega - Y' \sin \omega, \\ Y''' &= X' \cos I \sin \omega + Y' \cos I \cos \omega - Z' \sin I, \\ Z''' &= X' \sin I \sin \omega + Y' \sin I \cos \omega + Z' \cos I. \end{aligned}$$

It is often interesting to know the position of the great circles forming the principal axes of the sphero-conic in reference to the great circle marked out in the heavens by the plane of the attracting planet's orbit. If we call Ω the longitude of the ascending node of one of the planes of symmetry of

the cone on the plane xy, and i the inclination (always between 0° and 180°), and τ the angular distance of the centre of the sphero-conic from the node measured in the direction of increasing longitudes, the four following equations can be used for the determination of these quantities:

$$\tan i \sin \Omega = + \frac{A}{C}\frac{G_y}{G_y + a^2}, \quad \tan i \cos \Omega = - \frac{B}{C}\frac{G_y}{G_y + b^2},$$

$$\sin i \sin \tau = - \left[1 + \frac{A^2}{C^2}\left(\frac{G_z}{G_z + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_z}{G_z + b^2}\right)^2\right]^{-\frac{1}{2}},$$

$$\sin i \cos \tau = - \left[1 + \frac{A^2}{C^2}\left(\frac{G_z}{G_z + a^2}\right)^2 + \frac{B^2}{C^2}\left(\frac{G_z}{G_z + b^2}\right)^2\right]^{-\frac{1}{2}},$$

where the signs of the two radicals may be taken positive or negative, thus corresponding to the four intersections of the two great circles with the one great circle.

To find the two semi-axes of the sphero-conic as measured by the arcs they subtend in the heavens, take the equation of the cone

$$\frac{\mathrm{X}^2}{G_x} + \frac{\mathrm{Y}^2}{G_y} + \frac{\mathrm{Z}^2}{G_z} = 0;$$

the section of the cone by the plane $\mathrm{Z} = \sqrt{G_z}$ gives the ellipse whose equation is

$$-\frac{\mathrm{X}^2}{G_x} - \frac{\mathrm{Y}^2}{G_y} = 1.$$

The semi-axis of this in the direction of the axis of X is $\sqrt{-G_x}$ and in the direction of the axis of Y, $\sqrt{-G_y}$. The greatest latitude η_0 of the planet moving on the sphero-conic and the greatest longitude λ_0 will be given by the equations

$$\tan \eta_0 = \sqrt{-\frac{G_y}{G_z}}, \quad \tan \lambda_0 = \sqrt{-\frac{G_x}{G_z}}.$$

The general equation connecting the variables η and λ will be obtained if, in the equation of the cone, we make

$$\mathrm{X} = \rho \cos \eta \sin \lambda, \quad \mathrm{Y} = \rho \sin \eta, \quad \mathrm{Z} = \rho \cos \eta \cos \lambda,$$

and thus is

$$\frac{\sin^2 \lambda}{G_x} + \frac{\tan^2 \eta}{G_y} + \frac{\cos^2 \lambda}{G_z} = 0.$$

These variables η and λ are expressed in terms of the perspective anomaly T as follows:

$$\cos \eta \cos \lambda = \varepsilon \sqrt{G_z} \quad \cos \eta \sin \lambda = \varepsilon \sqrt{-G_x} \cos T, \quad \sin \eta = \varepsilon \sqrt{-G_y} \sin T.$$

We see that when $T = 0$, also $\eta = 0$ and $\tan \lambda = \sqrt{-\frac{G_x}{G_z}}$; and when $T = \frac{\pi}{2}$, $\tan \eta = \sqrt{-\frac{G_y}{G_z}}$, and $\lambda = 0$; when $T = \pi$, $\eta = 0$, $\tan \lambda = -\sqrt{-\frac{G_x}{G_z}}$.

On the Solution of the Cubic in G and the Argument to be Employed in Tabulating the Elliptic Integrals.

A few words may be added in reference to the solution of the cubic in G,

$$\frac{A^2}{G+a^2}+\frac{B^2}{G+b^2}+\frac{C^2}{G}=1.$$

Using r^2 for $A^2+B^2+C^2$, this equation expanded is

$$G^3-[\mathrm{r}^2-(a^2+b^2)]\,G^2+[a^2A^2+b^2B^2-(a^2+b^2)\,\mathrm{r}^2+a^2b^2]\,G-a^2b^2C^2=0.$$

It will be noticed that in obtaining the values of k, the modulus of the elliptic integrals involved, and of m, we do not need all the roots of this equation, but only their differences. Hence, it will be advantageous to put

$$G=J+\tfrac{1}{3}[\mathrm{r}^2-(a^2+b^2)],$$
$$\mathrm{a}=a^2+\tfrac{1}{3}[\mathrm{r}^2-(a^2+b^2)],\quad \mathrm{b}=b^2+\tfrac{1}{3}[\mathrm{r}^2-(a^2+b^2)],\quad \mathrm{c}=\tfrac{1}{3}[\mathrm{r}^2-(a^2+b^2)].$$

The cubic will then take the form

$$\frac{A^2}{J+\mathrm{a}}+\frac{B^2}{J+\mathrm{b}}+\frac{C^2}{J+\mathrm{c}}=1,$$

where J will serve us equally well as G; but here we have

$$\mathrm{a}+\mathrm{b}+\mathrm{c}=A^2+B^2+C^2,$$

and the developed equation in J will be

$$J^3+[\mathrm{a}A^2+\mathrm{b}B^2+\mathrm{c}C^2-\mathrm{a}^2-\mathrm{b}^2-\mathrm{c}^2-\mathrm{ab}-\mathrm{bc}-\mathrm{ca}]\,J-[\mathrm{bc}A^2+\mathrm{ca}B^2+\mathrm{ab}C^2-\mathrm{abc}]=0$$

By the elimination of c and C and the partial reintroduction of a^2 and b^2, the shorter form is obtained,

$$J^3-[\mathrm{a}^2+\mathrm{ab}+\mathrm{b}^2-a^2A^2-b^2B^2]\,J+\mathrm{a}\,(b^2B^2-\mathrm{ab})+\mathrm{b}\,(a^2A^2-\mathrm{ab})=0.$$

Employing the well-known trigonometric process for the solution of the cubic having all its roots real, we drive ψ (between the limits $\pm 90°$) from

$$\sin 3\psi=\frac{\sqrt{27}}{2}\,\frac{\mathrm{a}\,(b^2B^2-\mathrm{ab})+\mathrm{b}\,(a^2A^2-\mathrm{ab})}{[\mathrm{a}^2+\mathrm{ab}+\mathrm{b}^2-a^2A^2-b^2B^2]^{\frac{3}{2}}}.$$

The roots of the cubic are then

$$J_x=-\frac{2}{\sqrt{3}}[\mathrm{a}^2+\mathrm{ab}+\mathrm{b}^2-a^2A^2-b^2B^2]^{\frac{1}{2}}\sin(60°+\psi),$$
$$J_y=\frac{2}{\sqrt{3}}[\mathrm{a}^2+\mathrm{ab}+\mathrm{b}^2-a^2A^2-b^2B^2]^{\frac{1}{2}}\sin\psi,$$
$$J_z=\frac{2}{\sqrt{3}}[\mathrm{a}^2+\mathrm{ab}+\mathrm{b}^2-a^2A^2-b^2B^2]^{\frac{1}{2}}\sin(60°-\psi).$$

To render these equations simpler, we will put

$$\mathrm{a}+\mathrm{b}=2n\cos\nu,\qquad \sqrt{3}\,(\mathrm{a}-\mathrm{b})=2n\sin\nu,\qquad a^2A^2+b^2B^2-2\mathrm{ab}=n^2\mu\cos\xi,$$
$$\frac{1}{\sqrt{3}}(a^2A^2-b^2B^2)=n^2\mu\sin\xi.$$

With these modifications we have

$$\sin 3\psi = \frac{\sqrt{27}}{2}\frac{\mu \cos(\nu + \xi)}{(1 - \mu \cos \xi)^{\frac{3}{2}}},$$

$$J_x = -\frac{2}{\sqrt{3}} n (1 - \mu \cos \xi)^{\frac{1}{2}} \sin (60^\circ + \psi),$$

$$J_y = \frac{2}{\sqrt{3}} n (1 - \mu \cos \xi)^{\frac{1}{2}} \sin \psi,$$

$$J_z = \frac{2}{\sqrt{3}} n (1 - \mu \cos \xi)^{\frac{1}{2}} \sin (60^\circ - \psi).$$

The modulus of the elliptic integrals involved, k, will be given by the equation

$$k^2 = \sin^2 \theta = \frac{J_y - J_z}{J_x - J_z} = \frac{\cos(60^\circ - \psi)}{\cos \psi},$$

and the quantity we have designated by m, by the formula

$$m^2 = J_z - J_x = 2n(1 - \mu \cos \xi)^{\frac{1}{2}} \cos \psi.$$

The two elliptic integrals $M \sin^2 x$, $M \cos^2 x$ are functions of ψ; consequently, they may be tabulated with the argument ψ or any function of ψ as, for instance, with $\sin 3\psi$, that is, with the absolute discriminant of the cubic. We have $\cos^2 \theta = \dfrac{\sin(30^0 - \psi)}{\cos \psi}$, and, if we put x^4 for the second member of this, we have

$$\left[\frac{\sin \frac{\theta}{2}}{1 + \sqrt{\cos \theta}}\right]^2 = \frac{1}{2}\frac{1 - x}{1 + x} = \frac{q + q^9 + q^{25} + \ldots.}{1 + 2(q^4 + q^{16} + q^{36} + \ldots.)}.$$

The value of the nome q can be derived from the infinite series (three terms suffice except in very unusual cases),

$$q = \frac{1}{2}\frac{1 - x}{1 + x} + \frac{1}{16}\left(\frac{1 - x}{1 + x}\right)^5 + \frac{15}{512}\left(\frac{1 - x}{1 + x}\right)^9 + \ldots.$$

As G_x, G_y, G_z enter into many of the preceding formulas, it may be useful to note that

$$\begin{aligned} G_x &= J_x + \mathrm{c}, & G_y &= J_y + \mathrm{c}, & G_z &= J_z + \mathrm{c}, \\ G_x + a^2 &= J_x + \mathrm{a}, & G_y + a^2 &= J_y + \mathrm{a}, & G_z + a^2 &= J_z + \mathrm{a}, \\ G_x + b^2 &= J_x + \mathrm{b}, & G_y + b^2 &= J_y + \mathrm{b}, & G_z + b^2 &= J_z + \mathrm{b}. \end{aligned}$$

Variation of the Elements of a Planet through Perturbation.

The elements we select for use are defined thus: x, y, z, denoting the rectangular coordinates of a planet referred to the Sun as origin, adopt the elements c_x, c_y, c_z, f_x, f_y, f_z, such that

$$c_x = \frac{y dz - z dy}{dt}, \qquad c_y = \frac{z dx - x dz}{dt}, \qquad c_z = \frac{x dy - y dx}{dt},$$

$$f_x = \frac{c_z dy - c_y dz}{dt} - \frac{\mu x}{r}, \quad f_y = \frac{c_x dz - c_z dx}{dt} - \frac{\mu y}{r}, \quad f_z = \frac{c_y dx - c_x dy}{dt} - \frac{\mu z}{r},$$

where we note that μ is the sum of the masses of the Sun and planet, and r is the radius vector of the latter. These six elements are not independent but satisfy the relation

$$c_x f_x + c_y f_y + c_z f_z = 0.$$

The additional element needed to complete the number six is the element everywhere attached by addition to the time. With these constants the two equations of the path of the planet in space are

$$c_x x + c_y y + c_z z = 0, \qquad \mu r + f_x x + f_y y + f_z z = c_x^2 + c_y^2 + c_z^2 = k^2.$$

To understand the correlation of the different terms of these equations it must be borne in mind that μ is a constant of three dimensions in reference to the linear unit. Consequently, the c are of two dimensions and the f of three dimensions in reference to this unit.

The second equation belongs to a quadric surface not, in general, referred to its axes of symmetry. Removing the radical from it, it becomes

$$(\mu^2 - f_x^2)\,x^2 + (\mu^2 - f_y^2)\,y^2 + (\mu^2 - f_z^2)\,z^2 - 2 f_x f_y\, xy - 2 f_y f_z\, yz - 2 f_z f_x\, zx \\ + 2k^2 f_x x + 2k^2 f_y y + 2k^2 f_z z - k^4 = 0.$$

The cubic, which must be solved in order that this may be referred to its axes of symmetry, is

$$\chi^3 - [3\mu^2 - (f_x^2 + f_y^2 + f_z^2)]\,\chi^2 + [(\mu^2 - f_x^2)(\mu^2 - f_y^2) + (\mu^2 - f_y^2)(\mu^2 - f_z^2) \\ + (\mu^2 - f_z^2)(\mu^2 - f_x^2) - (f_x^2 f_y^2 + f_y^2 f_z^2 + f_z^2 f_x^2)]\,\chi - \mu^4[\mu^2 - (f_x^2 + f_y^2 + f_z^2)] = 0.$$

But, if e denotes the eccentricity, we have $f_x^2 + f_y^2 + f_z^2 = \mu^2 e^2$; thus the preceding equation reduces to

$$\chi^3 - \mu^2(3 - e^2)\,\chi^2 + \mu^4(3 - 2e^2)\,\chi - \mu^6(1 - e^2) = 0.$$

Put $\chi = \mu^2 x$, then

$$x^3 - (3 - e^2)\,x^2 + (3 - 2e^2)\,x - (1 - e^2) = 0,$$

or

$$(x - 1)^3 + e^2(x - 1)^2 = 0.$$

The three roots are $x = 1$, $x = 1$, $x = 1 - e^2$. The second equation of the orbit of the planet is, therefore, a quadric of revolution about the major axis, in the present investigation an ellipsoid, as we suppose $e^2 < 1$. The expression $c_x^2 + c_y^2 + c_z^2$ is invariant, as also is $f_x^2 + f_y^2 + f_z^2$, both being independent of the orientation of the axes of coordinates. The system of principal axes will be arrived at if we suppose $f_y = 0$, $f_z = 0$, which imply also $c_x = 0$, as is shown by the relation $c_x f_x + c_y f_y + c_z f_z = 0$, unless we have $f_x = 0$, when, the quadric being a sphere, all systems of axes are principal. The equations of the orbit then take the form

$$c_y y + c_z z = 0, \qquad \mu r + \mu e x = k^2,$$

where $k^2 = c_y^2 + c_z^2$. The radical removed, the second equation becomes

$$\mu^2(1-e^2)x^2 + \mu^2 y^2 + \mu^2 z^2 + 2k^2\mu e x - k^4 = 0.$$

But, a being the semi-axis major, $k^2 = \mu a(1-e^2)$, therefore, the preceding equation takes the form

$$\frac{x^2}{a^2} + \frac{y^2+z^2}{a^2(1-e^2)} + 2\frac{e}{a}x - (1-e^2) = 0,$$

or

$$\frac{(x+ae)^2}{a^2} + \frac{y^2+z^2}{a^2(1-e^2)} = 1.$$

The adoption of the attracted planet instead of the centre of the Sun as the origin of coordinates renders it necessary, while employing the elements of the planet, to substitute $-x_0$, $-y_0$, $-z_0$ for x, y, z. Thus, with this notation, the equations of the Sun's path in space are

$$c_x x_0 + c_y y_0 + c_z z_0 = 0, \qquad \mu r_0 - f_x x_0 - f_y y_0 - f_z z_0 = k^2.$$

The values of the differentials with respect to the time of the constants c and f must now be stated. Denoting the mass of the attracting planet by m', the perturbative function R for secular perturbations is $\frac{m'}{\rho}$, and we have

$$\frac{dc_x}{dt} = y_0\frac{\partial R}{\partial z} - z_0\frac{\partial R}{\partial y} = \frac{m'}{\rho^3}(z_0 y - y_0 z),$$

$$\frac{dc_y}{dt} = z_0\frac{\partial R}{\partial x} - x_0\frac{\partial R}{\partial z} = \frac{m'}{\rho^3}(x_0 z - z_0 x),$$

$$\frac{dc_z}{dt} = x_0\frac{\partial R}{\partial y} - y_0\frac{\partial R}{\partial x} = \frac{m'}{\rho^3}(y_0 x - x_0 y),$$

$$\frac{df_x}{dt} = \frac{dz_0}{dt}\frac{dc_y}{dt} - \frac{dy_0}{dt}\frac{dc_z}{dt} + \frac{m'}{\rho^3}(c_z y - c_y z),$$

$$\frac{df_y}{dt} = \frac{dx_0}{dt}\frac{dc_z}{dt} - \frac{dz_0}{dt}\frac{dc_x}{dt} + \frac{m'}{\rho^3}(c_x z - c\,x),$$

$$\frac{df_z}{dt} = \frac{dy_0}{dt}\frac{dc_x}{dt} - \frac{dx_0}{dt}\frac{dc_y}{dt} + \frac{m'}{\rho^3}(c_y x - c_x y).*$$

There is still one element to be added to the preceding to complete the number of six independent constants; this is the mean longitude at epoch. The well-known equation for its variation shows that a portion is immediately derivable from the motion of the perihelion, and another from that of the node; what remains (calling the element l) is given by the equation

$$\frac{dl}{dt} = -2\frac{an}{\mu}\left[x_0\frac{\partial R}{\partial x} + y_0\frac{\partial R}{\partial y} + z_0\frac{\partial R}{\partial z}\right] = 2\,an\frac{m'}{\mu}\left[x_0\frac{x}{\rho^3} + y_0\frac{y}{\rho^3} + z_0\frac{z}{\rho^3}\right],$$

where n denotes the mean motion of the attracted planet.

* For these formulas consult Laplace, Mécanique Céleste, Tom. I, Liv. II, Art. 64.

The system just given is the most general as respects the orientation of the axes of coordinates. Let us now specialize by taking, for the plane of xy, the plane of the orbit of the attracted planet. This makes $c_x = 0$, $c_y = 0$, $c_z = k$ and the equation between the c and the f shows that, in consequence, we have $f_z = 0$. Then our equations become

$$\begin{aligned}
\frac{dc_x}{dt} &= -\frac{m'}{\rho^3}\,y_0 z\,, & \frac{df_x}{dt} &= -\frac{dy_0}{dt}\frac{dk}{dt} + k\frac{m'}{\rho^3}\,y\,,\\
\frac{dc_y}{dt} &= \frac{m'}{\rho^3}\,x_0 z\,, & \frac{df_y}{dt} &= \frac{dx_0}{dt}\frac{dk}{dt} - k\frac{m'}{\rho^3}\,x\,;\\
\frac{dk}{dt} &= \frac{m'}{\rho^3}(y_0 x - x_0 y)\,, & \frac{df_z}{dt} &= \frac{dy_0}{dt}\frac{dc_x}{dt} - \frac{dx_0}{dt}\frac{dc_y}{dt}\,.
\end{aligned}$$

But, our specialization, applied to the definitions of the f, gives

$$-k\frac{dx_0}{dt} = \mu\frac{y_0}{r_0} - f_y\,, \qquad k\frac{dy_0}{dt} = \mu\frac{x_0}{r_0} - f_x\,.$$

These values enable us to eliminate the differentials of x_0, y_0 from the preceding equations. By differentiating the equation between the c and f we get

$$f_x\frac{dc_x}{dt} + f_y\frac{dc_y}{dt} + k\frac{df_z}{dt} = 0\,.$$

The substitution of preceding values in this renders it an identity, hence the equation for $\frac{df_z}{dt}$ is superfluous. Moreover, after use in substitution, the equation for $\frac{dk}{dt}$ is no longer needed, since the secular motion of the semi-axes a vanishes. Thus the equations to be employed are reduced to the five following:

$$\begin{aligned}
\frac{dc_x}{dt} &= -m'y_0\frac{z}{\rho^3}\,,\\
\frac{dc_y}{dt} &= m'x_0\frac{z}{\rho^3}\,,\\
\frac{df_x}{dt} &= \frac{m'}{k}\left(f_x - \mu\frac{x_0}{r_0}\right)\left(y_0\frac{x}{\rho^3} - x_0\frac{y}{\rho^3}\right) + m'k\frac{y}{\rho^3}\,,\\
\frac{df_y}{dt} &= \frac{m'}{k}\left(f_y - \mu\frac{y_0}{r_0}\right)\left(y_0\frac{x}{\rho^3} - x_0\frac{y}{\rho^3}\right) - m'k\frac{x}{\rho^3}\,,\\
\frac{dl}{dl} &= 2an\frac{m'}{\mu}\left(x_0\frac{x}{\rho^3} + y_0\frac{y}{\rho^3}\right).
\end{aligned}$$

If χ denote the longitude of the perihelion of the attracted planet, we have $f_x = \mu e \cos \chi$, $f_y = \mu e \sin \chi$; also, if i is the inclination and ☊ the longitude of the ascending node, $c_x = k \sin i \sin ☊$, $c_y = -k \sin i \cos ☊$; whence

$$\frac{d(\sin i \sin ☊)}{dt} = -\frac{m'}{k} y_0 \frac{z}{\rho^3},$$

$$\frac{d(\sin i \cos ☊)}{dt} = -\frac{m'}{k} x_0 \frac{z}{\rho^3},$$

$$\frac{d(e \sin \chi)}{dt} = \frac{m'}{k}\left(e \sin \chi - \frac{y_0}{r_0}\right)\left(y_0 \frac{x}{\rho^3} - x_0 \frac{y}{\rho^3}\right) - \frac{m'}{\mu} k \frac{x}{\rho^3},$$

$$\frac{d(e \cos \chi)}{dt} = \frac{m'}{k}\left(e \cos \chi - \frac{x_0}{r_0}\right)\left(y_0 \frac{x}{\rho^3} - \bar{x}_0 \frac{y}{\rho^3}\right) + \frac{m'}{\mu} k \frac{y}{\rho^3}.$$

The point of departure for longitudes is still undetermined; but, if this is taken to be the ascending node of the attracted planet's orbit on that of the attracting planet, we can employ the X''', Y''', Z''' we have previously defined. Thus, using brackets with subscript 0 to denote integration along the orbit of the attracting planet we have the equations

$$\left[\frac{d(\sin i \sin ☊)}{dt}\right]_0 = -\frac{m'}{k} y_0 Z''',$$

$$\left[\frac{d(\sin i \cos ☊)}{dt}\right]_0 = -\frac{m'}{k} x_0 Z''',$$

$$\left[\frac{d(e \sin \chi)}{dt}\right]_0 = \frac{m'}{k}\left(e \sin \chi - \frac{y_0}{r_0}\right)\left(y_0 X''' - x_0 Y'''\right) - \frac{m'}{\mu} k X''',$$

$$\left[\frac{d(e \cos \chi)}{dt}\right]_0 = \frac{m'}{k}\left(e \cos \chi - \frac{x_0}{r_0}\right)\left(y_0 X''' - x_0 Y'''\right) + \frac{m'}{\mu} k Y''',$$

$$\left[\frac{dl}{dt}\right]_0 = 2an\frac{m'}{\mu}\left(x_0 X''' + y_0 Y'''\right).$$

The integration round the orbit of the disturbed planet has still to be executed in order to arrive at the secular motion of the elements. For this we are confined to the use of mechanical quadratures. Here we may use either of the three anomalies, or any variable which will show the position of the planet on its orbit, as the independent variable.

MEMOIR No. 70.

On the Use of the Sphero-Conic in Astronomy.*

(Astronomical Journal, vol. XXII, pp. 53–56, 1901.)

If a planet circulating about the Sun in an elliptic orbit is viewed from a fixed point its apparent path as projected on the celestial sphere is a curve named a sphero-conic. This curve is divided into symmetrical quadrants by two great circles at right angles to each other, and intersecting at a point called the center. The subject is especially interesting as showing what would be the apparent course of any planet as viewed from the Earth if the latter were stopped at any point of its orbit. The two envelopes of all these curves, when the Earth is made to take all positions in its orbit, evidently embrace between them the zodiac of the planet. A valuable application is also found in the secular perturbations of one planet by another.

The paths of the superior major planets, as seen from any point of the Earth's orbit, do not greatly deviate from a great circle; but those of Mercury and Venus take a spindle-shaped form which is of great interest. Hence, we propose to illustrate the matter in the case of Venus, as it would be viewed from the point of the Earth's orbit where the eccentric anomaly is 90°. The necessary values of the elements of the problem are taken from Prof. Newcomb's Tables of the Sun and Venus for the epoch 1900. The distance of the Sun from the point of view is unity in this case, but we may always adopt this distance as the linear unit. The solar eccentricity being 0.01675104, and the longitude of the perigee 281° 13′ 15″.0, the Sun's longitude as seen from this point is 12° 10′ 50″.31. Let u_0 denote the angular distance of the Sun from the ascending node of its orbit on that of Venus, ω the angular distance of this node from the perihelion of Venus, i the inclination of the orbits, a the semi-axis major, and e the eccentricity of Venus. The problem involves no more than these five quantities, which have the following values:

$$u_0 = 116^\circ\, 24'\, 3''.58\,, \qquad \omega = 125^\circ\, 36'\, 56''.93\,, \qquad i = 3^\circ\, 23'\, 37''.07\,,$$
$$\log a = 9.85933781\,, \qquad e = 0.00682069\,.$$

* This article is intended as supplementary to one in the American Journal of Mathematics, vol. XXIII, p. 317.

If we imagine a system of rectangular axes having its origin at the point of view, the axis of x being directed towards the perihelion of Venus, the axis of y to a point in the plane of the orbit a quadrant in advance of the perihelion, and the axis of z towards the north pole of the orbit, the coordinates of the Sun will have the following expressions:

$$x_0 = \cos\omega \cos u_0 - \cos i \sin\omega \sin u_0$$
$$y_0 = \sin\omega \cos u_0 + \cos i \cos\omega \sin u_0$$
$$z_0 = \sin i \sin u_0.$$

By employing the auxiliary quantities k, k', K, K', determined from the equations

$$k \cos K = \cos\omega, \qquad k' \cos K' = \cos i \cos\omega,$$
$$k \sin K = \cos i \sin\omega, \qquad k' \sin K' = \sin\omega,$$

the expressions for x_0 and y_0 become

$$x_0 = k \cos(u_0 + K), \qquad y_0 = k' \sin(u_0 + K').$$

In our example

$$\log k = 9.9994966, \qquad \log k' = 9.9997418,$$
$$K = 125^\circ\ 39'\ 48''.37, \qquad K' = 125^\circ\ 34'\ 5''.60.$$

The formulas give these values of the coordinates of the Sun:

$$x_0 = -0.4679356, \qquad y_0 = -0.8821706, \qquad z_0 = +0.05302156.$$

Let $b = a\sqrt{1-e^2}$ denote the semi-axis minor of Venus. The elements of the sphero-conic constituting the apparent orbit of Venus in the heavens are found through the solution of a certain cubic. Let A, B, C denote the rectangular coordinates of the center of the elliptic orbit of Venus; then

$$A = x_0 - ae, \qquad B = y_0, \qquad C = z_0,$$

also let r denote the distance of this center from the origin, so that

$$r^2 = A^2 + B^2 + C^2.$$

Then the cubic will be thus expressed:

$$G^3 + (a^2 + b^2 - r^2)\, G^2 - [b^2(r^2 - a^2) + a^2(e^2 B^2 + C^2)]\, G - a^2 b^2 C^2 = 0.$$

Accurate computation of the coefficients is facilitated by using the equations

$$a^2 + b^2 - r^2 = 2b^2 - 1 + 2ae\,x_0, \qquad r^2 - a^2 = 1 - b^2 - 2ae\,x_0.$$

In the present example we have

$$G^3 + 0.04175306\, G^2 - 0.2533680\, G - 0.0007695486 = 0.$$

The roots of this cubic are always real; we will name them in the order of their algebraic magnitude, G_x, G_y, G_z. The value of G_x is nearly $-a^2$;

in fact, neglecting quantities of the fourth order with respect to e and i, the expression for G_x is

$$G_x = -a^2\left(1 - e^2\frac{A^2}{r^2}\right).$$

In our example this gives a sufficiently accurate value, the error probably not exceeding a unit in the 8th decimal; thus $G_x = -0.52320407$. The quadratic containing the remaining roots is

$$G^2 - 0.4814501\,G - 0.001470838 = 0.$$

Thus,

$$G_y = -0.00303587 \text{ and } G_z = +0.48448688.$$

The equation of the cone having its vertex at the origin and the orbit of Venus as directrix, when referred to the axes of symmetry, has the form

$$\frac{\mathrm{X}^2}{G_x} + \frac{\mathrm{Y}^2}{G_y} + \frac{\mathrm{Z}^2}{G_z} = 0$$

the axis of Z being in the body of the cone, that of X in the direction of longitude, and that of Y in the direction of latitude. Now let η denote latitude measured from the major-axis of the sphero-conic, and λ longitude measured from the minor-axis. Then, making

$$\mathrm{X} = \rho\cos\eta\sin\lambda, \qquad \mathrm{Y} = \rho\sin\eta, \qquad \mathrm{Z} = \rho\cos\eta\cos\lambda$$

we obtain, as the equation connecting the variables η and λ in the sphero-conic

$$\frac{\sin^2\lambda}{G_x} + \frac{\tan^2\eta}{G_y} + \frac{\cos^2\lambda}{G_z} = 0$$

The greatest longitude λ_0 and the greatest latitude η_0 of the planet moving on the sphero-conic will be given by the equations

$$\tan\lambda_0 = \sqrt{-\frac{G_x}{G_z}}, \qquad \tan\eta_0 = \sqrt{-\frac{G_y}{G_z}}$$

The equation of the sphero-conic can be put in the form

$$\tan^2\eta = \frac{\tan^2\eta_0}{\sin^2\lambda_0}\sin(\lambda_0 + \lambda)\sin(\lambda_0 - \lambda)$$

In the present example

$$\eta_0 = 4°\ 31'\ 33''.75, \qquad \lambda_0 = 46°\ 6'\ 3''.50.$$

As it is interesting we will derive the equation of the stereographic projection of this curve. Taking the radius of the projected sphere as unity, and placing the pole of projection at the center of the sphero-conic, the equations connecting the variables λ and η with the projected coordinates x and y are

$$\tan\lambda = \frac{2x}{1 - x^2 - y^2}, \qquad \sin\eta = \frac{2y}{1 + x^2 + y^2}$$

The inverse of these formulas, which may be used for plotting the projection of the curve, is

$$x = \frac{\sin\lambda\cos\eta}{1+\cos\lambda\cos\eta}, \qquad y = \frac{\sin\eta}{1+\cos\lambda\cos\eta}$$

Thus the equation of the projected sphero-conic is

$$\frac{1}{G_x}\frac{4x^2}{(1-x^2-y^2)^2+4x^2} + \frac{1}{G_y}\frac{4y^2}{(1+x^2+y^2)^2-4y^2} + \frac{1}{G_z}\frac{(1-x^2-y^2)^2}{(1-x^2-y^2)^2+4x^2} = 0$$

Taking two constants α and β this may be put in the form

$$\frac{y^2}{(1+x^2+y^2)^2-4y^2} = \frac{\alpha(1-x^2-y^2)^2+\beta x^2}{(1-x^2-y^2)^2+4x^2}$$

The projected curve is therefore an octic.

To have the position of the sphero-conic, let Ω be the longitude of the ascending node of the major-axis of that curve on the orbit of the planet measured from the perihelion of the latter, and ι the inclination (always between 0° and 180°), and τ the angular distance of the center of the sphero-conic from the node measured in the direction of increasing longitudes; the following equations can be used for the determination of these quantities:

$$\tan\iota\sin\Omega = \frac{A}{C}\frac{G_y}{G_y+a^2}, \qquad \tan\iota\cos\Omega = -\frac{B}{C}\frac{G_y}{G_y+b^2}, \qquad \sin^2\theta = \frac{G_y-G_x}{G_z-G_x},$$

$$\tan^2\tau = -\tan^2\theta\tan^2\lambda_0\frac{(G_z+a^2)(G_z+b^2)}{(G_x+a^2)(G_x+b^2)}$$

where θ and τ are taken in the first quadrant. In our example we get

$$\iota = 6°\ 17'\ 14''.56, \qquad \Omega = 151°\ 48'\ 30''.62$$
$$\theta = 45°\ 55'\ 41''.70, \qquad \tau = 89°\ 59'\ 58''.07$$

That τ should so nearly be equal to a quadrant is due to the smallness of the eccentricity of Venus.

In order to have the position of the ecliptic referred to the axes of the sphero-conic, let ψ denote the distance of the ascending node of the ecliptic on the major-axis of the sphero-conic from the similar point of the ecliptic on the orbit of Venus, and χ the distance of the same point from the ascending node of the major-axis on the orbit of Venus, and let I denote the inclination of the ecliptic to the mentioned axis. Then these quantities are determined by the equations:

$$\begin{aligned}
\sin I\cos\psi &= \cos\iota\sin i - \sin\iota\cos i\cos(\Omega-\omega)\\
\sin I\sin\psi &= -\sin\iota\sin(\Omega-\omega)\\
\sin I\cos\chi &= -\sin\iota\cos i + \cos\iota\sin i\cos(\Omega-\omega)\\
\sin I\sin\chi &= -\sin i\sin(\Omega-\omega)
\end{aligned}$$

$$\upsilon = u_0 - \psi, \qquad \chi - \tau = \sigma$$

In the present example we get

$$I = 3^\circ\ 34'\ 12''.94, \qquad \psi = 230^\circ\ 55'\ 7''.07, \qquad \chi = 204^\circ\ 48'\ 29''.24,$$
$$\upsilon = 245^\circ\ 28'\ 56''.51, \qquad \sigma = 114^\circ\ 48'\ 31''.17.$$

We can now get the coordinates of the Sun as referred to the axes of the sphero-conic; calling the longitude λ_0 and the latitude η_0, we have

$$\tan(\lambda_0 - \sigma) = \cos I \tan \upsilon, \qquad \sin \eta_0 = \sin I \sin \upsilon$$

where $\lambda_0 - \sigma$ is taken in the same semicircle as υ. In the example

$$\lambda_0 = 0^\circ\ 14'\ 56''.29, \qquad \eta_0 = -3^\circ\ 14'\ 52''.73$$

We now make application to the question of the secular perturbations of the Earth by Venus. For brevity put $m^2 = G_z - G_x$. In the example $\log m = 0.0016637$. Derive the *nome* q from

$$\frac{q + q^9 + q^{25} + \ldots}{1 + 2(q^4 + q^{16} + q^{36} + \ldots)} = \left(\frac{\sin \frac{1}{2}\theta}{1 + \sqrt{\cos\theta}}\right)^2$$

or from the equation

$$\cos\theta = x^2, \qquad q = \frac{1}{2}\frac{1-x}{1+x} + 2\left(\frac{1}{2}\frac{1-x}{1+x}\right)^5 + 30\left(\frac{1}{2}\frac{1-x}{1+x}\right)^9 + \ldots$$

We obtain $\log q = 8.6556780$. Next compute K and L from the equations

$$K = \frac{4}{\cos^2\theta\,(1 + \sqrt{\cos\theta})^2}[1 + 2(q^4 + q^{16} + \ldots)]^2$$

$$L = \frac{1 + \sqrt{\cos\theta}}{4\cos^2\frac{\theta}{2}\cos^{\frac{3}{2}}\theta}\,\frac{1 - 4q^3 + 9q^8 - 16q^{15} + \ldots}{[1 + 2(q^4 + q^{16} + \ldots)]^2}$$

The results in the present example are

$$\log K = 0.3906003, \qquad \log L = 0.1270540$$

The components of the action of the ring formed on the orbit of Venus on the origin are

$$X = -\frac{L\cos^2\theta}{m^3}\cos\eta_0 \sin\lambda_0 \qquad = -\frac{M}{m^3}\sin^2 x \cos\eta_0\sin\lambda_0$$

$$Y = -\frac{K - L}{m^3}\sin\eta_0 \qquad = -\frac{M}{m^3}\cos^2 x \sin\eta_0$$

$$Z = \frac{K - L\sin^2\theta}{m^3}\cos\eta_0\cos\lambda_0 = \frac{M}{m^3}\cos\eta_0\cos\lambda_0$$

Let R denote the magnitude of the resultant of these components, Λ and H severally the longitude and latitude of the point in the heavens towards which it is directed, the circles of reference being the axes of the sphero-conic. We have the equations

$$R\cos H \sin\Lambda = X, \qquad R\sin H = Y, \qquad R\cos H\cos\Lambda = Z$$

The numerical results are

$$R = 1.7446004, \qquad \Lambda = 359° \, 54' \, 31''.10, \qquad H = +2° \, 3' \, 26''.84$$

In order to have the components of the force referred to the ecliptic, it is necessary to convert Λ and H to a longitude and latitude referred to that plane. Let α and δ severally denote this longitude counted from the desceuding node of the major-axis of the sphero-conic on the ecliptic and the latitude. Then the equations are

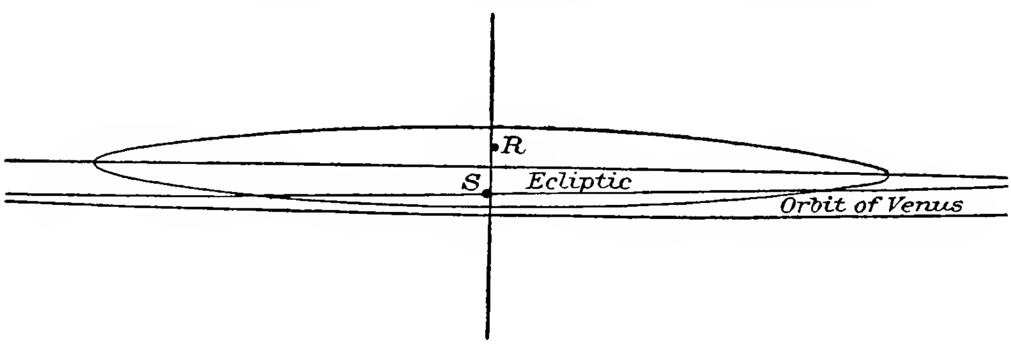
$$\begin{aligned} \sin\delta &= \cos I \sin H - \sin I \cos H \sin(\Lambda - \sigma) \\ \cos\delta \sin\alpha &= \sin I \sin H + \cos I \cos H \sin(\Lambda - \sigma) \\ \cos\delta \cos\alpha &= \cos H \cos(\Lambda - \sigma) \end{aligned}$$

Then, if R_0 denote the component directed towards the Sun, S_0 the component perpendicular to this and lying in the plane of the ecliptic, and W_0 the component directed to the north pole of that plane, we shall have

$$R_0 = R \cos\delta \cos(\alpha - \upsilon), \qquad S_0 = R \cos\delta \sin(\alpha - \upsilon), \qquad W_0 = R \sin\delta$$

The numerical results are

$$\alpha = 245° \, 0' \, 11''.53, \qquad \alpha - \upsilon = 359° \, 31' \, 15''.02, \qquad \delta = +5° \, 17' \, 41''.12,$$
$$R_0 = +1.7370960, \qquad S_0 = -0.0145277, \qquad W_0 = +0.1609908.$$



The diagram exhibits the form of the sphero-conic in the example. The projection used is the stereographic, the pole being at the center of the sphero-conic. The radius of the sphere projected is five inches. The point R is that towards which the resultant of the attraction of the elliptic ring is directed, and the point S shows the position of the Sun.

In the application to secular perturbations almost the sole aim is to obtain the values of R_0, S_0, W_0; but the preceding method involves the computation of many auxiliary quantities, hence it may be well to substitute the following, in which λ_0, η_0, Λ, H, α, δ, are not used, but the desired values are expressed in terms of the quantities which precede. We have

$$R_0 = \frac{M}{m^3}\left[\cos^2\eta_0 \cos^2\lambda_0 - \sin^2\varkappa \cos^2\eta_0 \sin^2\lambda_0 - \cos^2\varkappa \sin^2\eta_0\right]$$

But

$$\begin{aligned}\cos\eta_0\cos\lambda_0 &= \cos\sigma\cos\upsilon - \cos I\sin\sigma\sin\upsilon\\ \cos\eta_0\sin\lambda_0 &= \sin\sigma\cos\upsilon + \cos I\cos\sigma\sin\upsilon\\ \sin\eta_0 &= \sin I\sin\upsilon\end{aligned}$$

By substituting these values in the preceding equation, and putting

$$\begin{aligned}N\cos\nu &= (1+\sin^2\varkappa)(1-\tfrac{1}{2}\sin^2 I)\cos 2\sigma + \tfrac{3}{2}\cos^2\varkappa\sin^2 I\\ N\sin\nu &= (1+\sin^2\varkappa)\cos I\sin 2\sigma\end{aligned}$$

the expression for R_0 becomes

$$R_0 = \frac{M}{m^3}\left[\cos^2\sigma - \sin^2\varkappa\sin^2\sigma - N\sin\upsilon\sin(\upsilon+\nu)\right]$$

There is no need for the direct elaboration of the values of S_0 and W_0, since, provided the quantities υ and I are left evident in the expression for R_0, they are determined by the partial differential equations

$$S_0 = \tfrac{1}{2}\frac{\partial R_0}{\partial\upsilon}, \qquad W_0 = \frac{1}{2\sin\upsilon}\frac{\partial R_0}{\partial I}$$

Thus

$$S_0 = -\tfrac{1}{2}\frac{M}{m^3}N\sin(2\upsilon+\nu)$$

Moreover, if we put

$$\begin{aligned}P\cos\pi &= [(1+\sin^2\varkappa)\cos 2\sigma - 3\cos^2\varkappa]\cos I\\ P\sin\pi &= (1+\sin^2\varkappa)\sin 2\sigma\end{aligned}$$

we shall have

$$W_0 = \tfrac{1}{2}\frac{M}{m^3}P\sin I\sin(\upsilon+\pi)$$

In making use of the values of R_0, S_0, W_0 as obtained for different points in the orbit of the attracted planet it will be necessary to reduce them to a common linear unit; when we bear in mind that they are of the dimension -2 in reference to this unit, the procedure to be followed will be obvious.

As the example from Venus is abnormally simple in some respects on account of the smallness of the eccentricity of that planet, another may be given of the orbit of Mercury seen from the same point. Here the elements have the values

$$u_0 = 145°\ 2'\ 4''.91, \qquad \omega = 151°\ 14'\ 46''.49, \qquad i = 7°\ 0'\ 10''.37,$$
$$\log a = 9.5878217, \qquad e = 0.20561421.$$

As before, these are from Prof. Newcomb's Tables, and for the epoch 1900. From these elements the coordinates of the Sun are

$$x_0 = +\,0.4448295, \qquad y_0 = -\,0.8928855, \qquad z_0 = +\,0.06986952$$

and the cubic in G is

$$G^3 - 0.6421687\,G^2 - 0.1185352\,G - 0.0001049789 = 0$$

By the approximate formula $G_x = -0.1489421$, and this needs only the small correction of 7 units in the last decimal, so that $G_x = -0.1489428$. The quadratic containing the remaining roots is

$$G^2 - 0.7911115\ G - 0.0007048269 = 0$$

Whence $$G_y = -0.0008899314, \qquad G_z = +0.7920014$$

The dimensions of the sphero-conic, in this case, are

$$\lambda_0 = 23^\circ\ 26'\ 39''.57, \qquad \eta_0 = 1^\circ\ 55'\ 11''.59$$

Its position in the heavens is established by the values

$$\iota = 4^\circ\ 53'\ 41''.32, \qquad \Omega = 201^\circ\ 23'\ 16''.94, \qquad \tau = 89^\circ\ 16'\ 43''.53$$

It will be seen that, in spite of the great eccentricity of Mercury, τ does not differ much from a quadrant.

MEMOIR No. 71.

Illustrations of Periodic Solutions in the Problem of Three Bodies.

(Astronomical Journal, vol. XXII, pp. 93–97, 117–121, 1902.)

(FIRST ARTICLE.)

Whenever a system of moving bodies retakes after a certain lapse of time the same position relatively to the bodies themselves, the solution of the differential equations to which this case corresponds is called a periodic solution. Periodic solutions may be broadly divided into two classes. The first class contains those cases in which there has taken place a rotation of the whole system in the interval between the mentioned positions. The second those in which no such rotation has occurred, but the longitudes of the bodies, as well as their distances, have returned to the same values. In the illustrations to be given here we will confine ourselves to the latter class.

For expressing the differential equations of motion of two planets about their central body we will employ variables similar to those of Delaunay in the Lunar Theory, but (with Poincaré) we will denote the energy of the system by F. Then these equations may be given the form

$$\begin{aligned}
\frac{dL}{dt} &= \frac{\partial F}{\partial l}, & \frac{dG}{dt} &= \frac{\partial F}{\partial g}, & \frac{dH}{dt} &= \frac{\partial F}{\partial h} \\
\frac{dl}{dt} &= -\frac{\partial F}{\partial L}, & \frac{dg}{dt} &= -\frac{\partial F}{\partial G}, & \frac{dh}{dt} &= -\frac{\partial F}{\partial H} \\
\frac{dL'}{dt} &= \frac{\partial F}{\partial l'}, & \frac{dG'}{dt} &= \frac{\partial F}{\partial g'}, & \frac{dH'}{dt} &= \frac{\partial F}{\partial h'} \\
\frac{dl'}{dt} &= -\frac{\partial F}{\partial L'}, & \frac{dg'}{dt} &= -\frac{\partial F}{\partial G'}, & \frac{dh'}{dt} &= -\frac{\partial F}{\partial H'}
\end{aligned}$$

In these formulae l and l' may be regarded as similar in signification to the two mean anomalies of the planets, while g and g' are the angular distances of their perihelia from their nodes, and h and h' are the longitudes of the latter. For L, G, H, L', G', H' we have the following equivalents:

$$\begin{aligned}
L &= M\sqrt{a}, & G &= M\sqrt{a}\sqrt{1-e^2}, & H &= M\sqrt{a}\sqrt{1-e^2}\cos i \\
L' &= M'\sqrt{a'}, & G' &= M'\sqrt{a'}\sqrt{1-e'^2}, & H' &= M'\sqrt{a'}\sqrt{1-e'^2}\cos i'
\end{aligned}$$

where M and M' are certain functions of the masses and the other letters have the customary signification.

Ω being the potential function

$$F = \frac{\mu}{L^2} + \frac{\mu'}{L'^2} + \Omega$$

where again μ and μ' are functions of the masses. Ω is developable in an infinite periodic series of cosines whose general argument is a linear function of the angular elements, l, g, h, l', g', h'; that is, we have

$$\Omega = \Sigma . C \cos (il + i'g + i''h + i'''l' + i^{\text{iv}}g' + i^{\text{v}}h)$$

where the i are positive or negative integers, and C involves only L, G, H, L', G', H'.

Now we may imagine that by a series of operations of Delaunay we may remove from F all the sensible terms whose arguments have a sensible motion. The forms of F and the differential equations are not changed by this procedure. It is evident that, at this stage of the investigations, we may suppose our work of integration finished, provided that the conditions, to be mentioned shortly, are fulfilled. For the sake of distinction, let us say that F has now become F_0, and suppose that L, G, H, L', G', H' are now constants. Then the six equations

$$\frac{\partial F_0}{\partial l} = 0, \quad \frac{\partial F_0}{\partial g} = 0, \quad \frac{\partial F_0}{\partial h} = 0, \quad \frac{\partial F_0}{\partial l'} = 0, \quad \frac{\partial F_0}{\partial g'} = 0, \quad \frac{\partial F_0}{\partial h'} = 0$$

ought to be satisfied. Let us suppose, in addition, that the perihelia and nodes are stationary; this gives the four equations

$$\frac{\partial F_0}{\partial G} = 0, \quad \frac{\partial F_0}{\partial H} = 0, \quad \frac{\partial F_0}{\partial G'} = 0, \quad \frac{\partial F_0}{\partial H'} = 0.$$

The two remaining equations may be written

$$\frac{\partial F_0}{\partial L} = -n, \quad \frac{\partial F_0}{\partial L'} = -n'$$

where n and n', being constants, are the mean motions of the planets. Let us next suppose that n and n' are in the proportion of the integers k and k'. Then, as an eleventh equation, we may write

$$k' \frac{\partial F_0}{\partial L} - k \frac{\partial F_0}{\partial L'} = 0.$$

The last supposition is evidently necessary in order that a periodic solution may exist. Although k and k' may be taken at will, all cases will be arrived at by adopting numbers prime to each other. Thus we have eleven equations to determine twelve arbitrary constants; thus one is left indeterminate. This is as it should be, since the origin from which longitudes are measured may be taken at will.

For the group of six equations, first given, we may substitute others which are simpler. j being a positive integer, it is evident the form of F_0 is

$$F_0 = \Sigma . C \cos [j(k'l - kl') + i'g + i''h + i^{\text{iv}}g' + i^{\text{v}}h']$$

When this expression is partially differentiated with respect to any one of the elements l, g, h, l', g', h', we have a series of sines. The six equations are then satisfied if we make

$$g_0 = 0^\circ \text{ or } 180^\circ, \qquad h_0 = 0^\circ \text{ or } 180^\circ, \qquad g_0' = 0^\circ \text{ or } 180^\circ, \qquad h_0' = 0^\circ \text{ or } 180^\circ,$$
$$k'l_0 - kl_0' = 0^\circ \text{ or } 180^\circ.$$

These substitutions can be made in F_0, by which it is reduced to a function of L, G, H, L', G', H'. Using the same symbol to denote the function after this change, the equations, which remain to be satisfied, are

$$k'\frac{\partial F_0}{\partial L} - k\frac{\partial F_0}{\partial L'} = 0, \qquad \frac{\partial F_0}{\partial G} = 0, \qquad \frac{\partial F_0}{\partial H} = 0, \qquad \frac{\partial F_0}{\partial G'} = 0, \qquad \frac{\partial F_0}{\partial H'} = 0.$$

As these equations are five in number, they leave one of the six elements L, G, H, L', G', H' undetermined. Then we can take either L or L' at will, and the rest follow. Hence, provided the equations last given have suitable roots, the differential equations we stopped with will be satisfied.

The previous reasoning has been conducted on the supposition that we are dealing with three bodies; but it is plain that, however many bodies there may be in the system, similar propositions are true. The whole gist of the matter being that the initial values of the elements must be so adjusted that the secular perturbations vanish.

In the preceding we have argued as if the l, g, h, l', g', h' of Delaunay were six independent angular elements in the Problem of Three Bodies. But, as there are only four, always two less than three times the number of planets, it is necessary to modify the reasoning. Let us denote the independent angular elements by l_1, l_2, l_3, l_4, of which the last may be the single element having a finite motion. Also let their conjugate linear elements be severally L_1, L_2, L_3, L_4. Then

$$F = \Sigma . C \cos [i_1 l_1 + i_2 l_2 + i_3 l_3 + i_4 l_4]$$

From this, by a series of Delaunay transformations, will be removed all the terms involving l_4. Then we shall have

$$F_0 = \Sigma . C \cos [i_1 l_1 + i_2 l_2 + i_3 l_3].$$

After this L_4 will be a constant; so that the coefficients C can be regarded as involving only the three variables L_1, L_2, L_3. Then, to bring about a periodic solution, it is necessary, in the first place, to make

$$l_1 = 0^\circ \text{ or } 180^\circ, \qquad l_2 = 0^\circ \text{ or } 180^\circ, \qquad l_3 = 0^\circ \text{ or } 180^\circ.$$

These substitutions made in F_0 reduce it to a function of the three quantities L_1, L_2, L_3, still regarded as variable. But, in the second place, the equations

$$\frac{\partial F_0}{\partial L_1}=0, \quad \frac{\partial F_0}{\partial L_2}=0, \quad \frac{\partial F_0}{\partial L_3}=0,$$

must be satisfied. Thus, if we suppose that the constant element L_4 is selected at random, the values of the other linear elements become determinate; and the motion of the conjugate element l_4 is given by the equation

$$\frac{dl_4}{dt}=-\frac{\partial F_0}{\partial L_4}=\text{a constant}.$$

It will be noticed that the six conditions, just set down, are precisely those required to make the function F_0 at a standstill in reference to the six variables L_1, L_2, L_3, l_1, l_2, l_3.

In appearance, eight different periodic solutions ought to belong to each assigned value of L_4; but these are often not all distinct. Also we may be restricted to a limited range of values for L_4, if real solutions are to be obtained.

Our illustrations will be selected from the coplanar case of the Problem of Three Bodies; and thus one equation will drop out from each of the two groups of three conditions. In this case, if we extend somewhat the signification of the phase "line of apsides," a little consideration will show that the two remaining conditions, having reference to the angular elements, are tantamount to the statement that, in order for the existence of a periodic solution, the lines of apsides of the two planets must coincide, and of symmetrical conjunctions and oppositions there must be two upon this line in each period of the solution.

Moreover, we shall suppose that Jupiter is one of the planets and a minor planet the other. The latter being supposed to have an evanescent mass, the motion of Jupiter is Keplerian with elements not admitting adjustment. Hence one more of the second group of three conditions will fall out, and there is no opportunity for the selection of any linear element. Here all that distinguishes one periodic solution from another is the value of the rational ratio $\frac{k}{k'}$.

If we adopt the L and G of Delaunay for the minor planet, their values in this case must satisfy the equations

$$\frac{\partial F_0}{\partial G}=0, \quad \frac{\partial F_0}{\partial L}=-n$$

where n is the mean motion of the minor planet, deduced at once from the ratio $\frac{k}{k'}$ and the known n' of Jupiter.

F expanded in periodic series has the form

$$F = \Sigma\, C \cos\,[il + i'l' + j\gamma]$$

where i is a positive integer, but i' and j may be either positive or negative integers, and γ denotes the angular distance of the perihelion of the small planet from that of Jupiter. The coefficients C are of the form

$$C = A_0 e^i + A_1 e^{i+2} + A_2 e^{i+4} + \ldots.$$

the A being constants. After all the terms having arguments with sensible motions have been removed from F it is plain we may write

$$F_0 = \Sigma\, C_{i,j} \cos\,[(i+j)\, l_1 + j\gamma]$$

where $C_{i,j}$ has the same quality as C before, and l_1 denotes the argument whose motion vanishes to make the periodic solution. Then there appear to be four different solutions for each selection of the argument l_1 as follow:

Solution I $\quad \gamma = 0°, \quad l_1 = 0°, \quad k' \dfrac{\partial \Sigma C_{i,j}}{\partial L} + kn' = 0, \quad \dfrac{\partial \Sigma C_{i,j}}{\partial G} = 0$

Solution II $\quad \gamma = 180°, \quad l_1 = 0°, \quad k' \dfrac{\partial \Sigma (-1)^j C_{i,j}}{\partial L} + kn' = 0, \quad \dfrac{\partial \Sigma (-1)^j C_{i,j}}{\partial G} = 0$

Solution III $\quad \gamma = 0°, \quad l_1 = 180°, \quad k' \dfrac{\partial \Sigma (-1)^{i+j} C_{i,j}}{\partial L} + kn' = 0, \quad \dfrac{\partial \Sigma (-1)^{i+j} C_{i,j}}{\partial G} = 0$

Solution IV $\quad \gamma = 180°, \quad l_1 = 180°, \quad k' \dfrac{\partial \Sigma (-1)^i C_{i,j}}{\partial L} + kn' = 0, \quad \dfrac{\partial \Sigma (-1)^i C_{i,j}}{\partial G} = 0$

But if we put

$$\Sigma\,.\, C_{i,j}\,(i \text{ and } j \text{ both even}) = D_1, \qquad \Sigma\,.\, C_{i,j}\,(i \text{ even}, j \text{ odd}) = D_2$$
$$\Sigma\,.\, C_{i,j}\,(i \text{ odd}, j \text{ even}) = D_3, \qquad \Sigma\,.\, C_{i,j}\,(i \text{ and } j \text{ both odd}) = D_4$$

the third and fourth equations of each Solution become

Solution I $\quad k' \dfrac{\partial (D_1 + D_2 + D_3 + D_4)}{\partial L} + kn' = 0, \quad \dfrac{\partial (D_1 + D_2 + D_3 + D_4)}{\partial G} = 0$

Solution II $\quad k' \dfrac{\partial (D_1 - D_2 + D_3 - D_4)}{\partial L} + kn' = 0, \quad \dfrac{\partial (D_1 - D_2 + D_3 - D_4)}{\partial G} = 0$

Solution III $\quad k' \dfrac{\partial (D_1 - D_2 - D_3 + D_4)}{\partial L} + kn' = 0, \quad \dfrac{\partial (D_1 - D_2 - D_3 + D_4)}{\partial G} = 0$

Solution IV $\quad k' \dfrac{\partial (D_1 + D_2 - D_3 - D_4)}{\partial L} + kn' = 0, \quad \dfrac{\partial (D_1 + D_2 - D_3 - D_4)}{\partial G} = 0$

When i is even $C_{i,j}$ involves only even powers of e. But when i is odd $C_{i,j}$ involves only odd powers of the same. Hence D_1 and D_2 contain only even powers of e, while D_3 and D_4 contain only odd powers of the same. Consequently Solution IV is not distinct from I, nor III from II. There is

then no need to set $l_1 = 180°$, as all distinct solutions may be got from the assumption $l_1 = 0°$. As there are no arbitrary constants which may be taken at random in this matter, all that distinguishes one periodic solution from another are the values assigned to k and k', and the two values 0° or 180° which may be assigned to γ.

The elucidation of this subject in the fashion of synthetical geometry is worthy of attention. In the diagram let S denote the position of the Sun adopted as the origin of a system of rectangular coördinates, J the position of Jupiter, and let SJ the radius of Jupiter be the axis of x. Give the system of axes a variable rotation in the plane of Jupiter's orbit, so that the axis of x may continuously pass through the position of Jupiter. Then this planet will appear to oscillate on the axis of x between J, which we take to be the position of the planet at perihelion, and J' its position at aphelion. Suppose that, when Jupiter is at J, the minor planet crosses orthogonally the axis of x at the point P, or, in other words, that there is then a symmetrical conjunction. Now let an odd number of semi-revolutions of Jupiter elapse, Jupiter will then be at J' in aphelion, Next, let the radius SP of the minor planet and its velocity at the point P relative to the moving system of axes be so adjusted that the planet accomplishes an odd number of synodic semi-revolutions, and crosses orthogonally the axis of x at a point P' on the farther side of S, or, in other words, that there is now symmetrical opposition. Again, let the same odd number of semi-revolutions of Jupiter elapse. This planet is then at J, or, again in perihelion. On account of the symmetry of the motion of Jupiter in these two equal intervals of time, it is plain the minor planet will, from the point of symmetrical opposition, repeat its path, but in reverse order with respect to time. Thus it will arrive again at the point P, and cross orthogonally the axis of x, and a complete period of its motion will be accomplished. The axis of x will divide what may be called the synodic orbit into symmetrical halves.

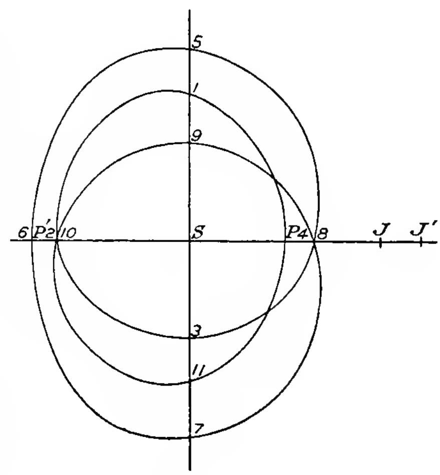


FIG. 1.

It is plain that we may, in the foregoing, interchange the points J and J'; that is, make the symmetrical conjunction occur when Jupiter is in aphelion, and the symmetrical opposition when it is in perihelion; thus the number

of periodic solutions obtained will be doubled. If we have elaborated the first group of solutions in leaving e' the eccentricity of Jupiter's orbit indeterminate, it is plain the second group will result by simply changing the sign of e'.

The adjustment, spoken of above, is not proved to be possible for all values of the odd number of semi-revolutions mentioned; that is, in some cases we might find imaginary or unsuitable roots for the equations involved. But it is certain, however, that some periodic solutions of the character described have a real existence.

Let $2i + 1$ denote the odd number of revolutions of Jupiter in the period of the solution, and $2j + 1$ the odd number of synodic revolutions of the minor planet in the same time; then

$$\frac{n}{n'} = \frac{2i + 2j + 2}{2i + 1}.$$

It is of sufficient interest to give a table to double entry for the values of this ratio corresponding to the smaller values of i and j.

i	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
0	2	4	6	8	10	12
1	$\frac{4}{3}$	2	$\frac{8}{3}$	$\frac{10}{3}$	4	$\frac{14}{3}$
2	$\frac{6}{5}$	$\frac{8}{5}$	2	$\frac{12}{5}$	$\frac{14}{5}$	$\frac{16}{5}$
3	$\frac{8}{7}$	$\frac{10}{7}$	$\frac{12}{7}$	2	$\frac{16}{7}$	$\frac{18}{7}$
4	$\frac{10}{9}$	$\frac{4}{3}$	$\frac{14}{9}$	$\frac{16}{9}$	2	$\frac{20}{9}$
5	$\frac{12}{11}$	$\frac{14}{11}$	$\frac{16}{11}$	$\frac{18}{11}$	$\frac{20}{11}$	2

The diagram is meant to illustrate the case where $j = 1$, that is, where there are three synodic revolutions of the minor planet in the period of the solution. It will be seen that the synodic orbit of the planet has, in this case, two multiple points situated on the line of symmetrical conjunctions and oppositions; the first (marked 2) lies between S and P', the second (marked 4) lies beyond the point P. As the curve has not been plotted from calculation, being intended to illustrate only the general appearance of things, this order may need to be reversed. The order of motion of the minor planet is readily followed in the diagram by the numerals attached at the completion of each quadrant of synodic movement, viz., P, 1, 2, 3, 4 . . . . 10, 11, P.

It will be noticed that the table just given does not contain all possible rational quantities, and thus the field for periodic solutions is not yet exhausted. This is because we have limited the exposition to the case where the two orthogonal crossings of the line of syzygies lie on opposite

sides of S, or where one is a conjunction and the other an opposition. But both may lie on the same side of S. Fig. 2 is designed to exemplify the later case.

Suppose that when Jupiter is in perihelion at J the minor planet is in symmetrical conjunction at P. After an odd number of semi-revolutions of Jupiter this planet will be in aphelion at J'. The radius SP of the minor planet and its velocity at P relative to the moving system of axes may be so adjusted that the planet accomplishes in the interval an integral number of synodic revolutions, and crosses orthogonally the axis of x at a point P' on the same side of S as P, and there is again symmetrical conjunction. Next, let the same odd number of semi-revolutions of Jupiter elapse. The planet is then at J and in perihelion. On account of the symmetry of the motion of Jupiter in these equal intervals, it is plain the minor planet will, from the second symmetrical conjunction, repeat its path, but in reverse order with regard to time. Thus it will arrive again at P, and intersect the axis of x orthogonally, and a complete period of its motion will have been gone through.

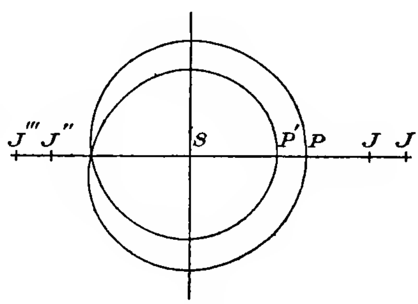


Fig. 2.

The figure represents the case where there are two synodic revolutions in the period of the solution. There is but one multiple point. If the points J and J' are rotated through a semi-circle, so that they fall into the positions J'' and J''', the figure will represent the case where there are two symmetrical oppositions.

Let $2i + 1$ denote the odd number of revolutions of Jupiter in the period of the solution, and $2j$ the even number of synodic revolutions of the minor planet in the same time. Then we shall have $\frac{n}{n'} = \frac{2i + 2j + 1}{2i + 1}$ and the following table :

i	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
0	1	3	5	7	9	11
1	1	$\frac{5}{3}$	$\frac{7}{3}$	3	$\frac{11}{3}$	$\frac{13}{3}$
2	1	$\frac{7}{5}$	$\frac{9}{5}$	$\frac{11}{5}$	$\frac{13}{5}$	3
3	1	$\frac{9}{7}$	$\frac{11}{7}$	$\frac{13}{7}$	$\frac{15}{7}$	$\frac{17}{7}$
4	1	$\frac{11}{9}$	$\frac{13}{9}$	$\frac{5}{3}$	$\frac{17}{9}$	$\frac{19}{9}$
5	1	$\frac{13}{11}$	$\frac{15}{11}$	$\frac{17}{11}$	$\frac{19}{11}$	$\frac{21}{11}$

Let us grant that in illustration we limit ourselves to cases where the two planets do not approach each other very nearly. Then at least a rough idea may be obtained of the course of the synodic orbit in neglecting the square of the disturbing force. Thus, no Delaunay transformations are necessary.

From the point of view of calculation all periodic solutions may be divided into two classes: first, those where $k - k'$ is a large integer (say 8 or larger); second, those where the same difference is a small integer (ranging from 1 to 7). Dealing with the first class, we may throw out from F all the periodic terms, since in F_0 they would be factored by powers of e higher than e^7, and retain only the so-called secular terms. With the second class, however, it will be necessary to retain, in addition, all the sensible terms involving the general argument $i\,(k'l - kl') + j\gamma$, whose motion vanishes for the periodic solution treated; for here these terms are secular to the same title as those mentioned for the first class.

With the limitations we have imposed, the construction of a periodic solution of the first class is a quite simple affair.

Let a' denote the semi-axis major of Jupiter, Δ the distance between the two planets, ω and ω' the longitudes of their perihelia, and R the secular portion of the periodic development of $\frac{a'}{\Delta}$. Neglecting the simple powers of e' because they disappear in the partial differentiation with respect to e, and carrying the approximation to the degree of neglecting terms of the 8th order with reference to the eccentricities, it is well known that

$$\begin{aligned} R = {} & A_1 e^2 + A_2 e^4 + A_3 e'^2 e^2 + A_4 e^6 + A_5 e'^2 e^4 + A_6 e'^4 e^2 \\ & - (A_7 e'e + A_8 e'e^3 + A_9 e'^3 e + A_{10} e'e^5 + A_{11} e'^3 e^3 + A_{12} e'^5 e) \cos (\omega - \omega') \\ & + (A_{13} e'^2 e^2 + A_{14} e'^2 e^4 + A_{15} e'^4 e^2) \cos 2\,(\omega - \omega') - A_{16} e'^3 e^3 \cos 3\,(\omega - \omega') \end{aligned}$$

where the A are positive constants and functions of α the ratio of the mean distances. Make in this expression $\omega - \omega' = 0^\circ$ or 180°, and take the partial derivative with reference to e. Thus

$$\begin{aligned} \frac{\partial R}{\partial e} = {} & \mp (A_7 e' + A_9 e'^3 + A_{12} e'^5) + 2\,[A_1 + (A_3 + A_{13})\, e'^2 + (A_5 + A_{15})\, e'^4]\, e \\ & \mp 3\,[A_8 e' + (A_{11} + A_{16}) e'^3]\, e^2 + 4\,[A_2 + (A_5 + A_{14})\, e'^2]\, e^3 \pm 5\, A_{10} e' e^4 + 6\, A_4 e^5 \end{aligned}$$

where the upper sign belongs to the value 0°, and the lower sign to the value 180° for $\omega - \omega'$. This expression must vanish in order that the line of apsides of the minor planet may have no secular motion. This affords the condition necessary to the determination of e. It is seen at once that if e is to have a positive value, the upper of the ambiguous signs must be

taken. The assumption of the lower sign leads to a value of e, the negative of the former; which would mean that the ellipse must be revolved through a semi-circle. Thus the latter assumption is not really distinct from the former.

Making $e' = 0.04825336$, the equation in e has been computed for every 0.01 in the value of α from $\alpha = 0.01$ up to $\alpha = 0.70$, and the value of e obtained therefrom. The following table contains these values, but only for every 0.02 in the value of α.

α	e	α	e	α	e
0.02	0.0012091	0.26	0.0155796	0.50	0.0291772
0.04	0.0024178	0.28	0.0167534	0.52	0.0302466
0.06	0.0036258	0.30	0.0179216	0.54	0.0313029
0.08	0.0048326	0.32	0.0190838	0.56	0.0323453
0.10	0.0060379	0.34	0.0202392	0.58	0.0333726
0.12	0.0072414	0.36	0.0213875	0.60	0.0343837
0.14	0.0084426	0.38	0.0225281	0.62	0.0353776
0.16	0.0096411	0.40	0.0236605	0.64	0.0363529
0.18	0.0108366	0.42	0.0247841	0.66	0.0373080
0.20	0.0120286	0.44	0.0258982	0.68	0.0382412
0.22	0.0132167	0.46	0.0270022	0.70	0.0391503
0.24	0.0144005	0.48	0.0280955		

Choosing $\frac{k}{k'} = \frac{n}{n'}$, we obtain α from the equation

$$\log \alpha = 9.9998618 - \frac{2}{3} \log \frac{n}{n'}$$

and, entering the table with this argument, have the value which must be attributed to the eccentricity of the minor planet in order that the line of apsides may have no secular motion.

The treatment of the case where $k - k'$ is a small integer is reserved for a second article.

MEMOIR No. 72.

Illustrations of Periodic Solutions in the Problem of Three Bodies.

(SECOND ARTICLE.)

The examples we now propose to treat differ from those of the first article in that, $k - k'$ being a small integer, it is necessary to annex to the usually considered non-periodic portion of the perturbative function the terms which become non-periodic through the commensurability of the mean motions of the planets.

As the first example take the case where the mean motion of the minor planet is three times that of Jupiter, that is, make $k = 3$, $k' = 1$. In the development of the perturbative function we propose to stop with terms of the order of the fourth power of eccentricities. Then, making use of Leverrier's elaboration with his notation (χ denotes half an eccentricity),* the new terms we have need of (they are sums of coefficients only) are:

$$\begin{aligned} R = {} & (172)^3\chi^2 + (173)^3\chi^4 + (174)^3\chi'^2\chi^2 + (182)^2\chi'\chi + (183)^2\chi'\chi^3 + (184)^2\chi'^3\chi \\ & + (193)^1\chi'^2\chi^2 + (206)^0\chi'^3\chi + (336)^6\chi^4 + (340)^5\chi'\chi^3 + (344)^4\chi'^2\chi^2 + (348)^3\chi'^3\chi \end{aligned}$$

After substituting for the symbolic coefficients their expressions in terms of Leverrier's $A_j^{(i)}$, the preceding expression becomes:

$$\begin{aligned} R = {} & [\tfrac{21}{2}+5+1]^3\chi^2+[-31-\tfrac{61}{3}+5+12+4]^3\chi^4+[-378-149+31+48+12]^3\chi'^2\chi^2 \\ & -[20+10+2]^2\chi'\chi+[70+29-37-42-12]^2\chi'\chi^3+[262+83-59-54-12]^2\chi'^3\chi \\ & +[-34+7+59+48+12]^1\chi'^2\chi^2-[0+13+27+18+4]^0\chi'^3\chi+[157+\tfrac{277}{3}+37+9+1]^6\chi^4 \\ & -[\tfrac{2090}{3}+\tfrac{1234}{3}+162+38+4]^5\chi'\chi^3+[1144+683+265+60+6]^4\chi'^2\chi^2 \\ & -[816+500+192+42+4]^3\chi'^3\chi \end{aligned}$$

where the coefficients of the A are written alone, j being always 0 for the first term within each pair of brackets and augmenting by a unit in each step to the right, and the common value of i for all the terms being written above and to the right of each pair.

The argument for the A is $\alpha = 0.4805969$ and, with the assistance of Runkle's tables, the needed values have been obtained and their logarithms inserted in the following table (it should be noted that, on account of the

* Annales de l'Observatoire de Paris, Tom. I, pp. 284–291.

reaction term of the perturbative function, α has been subtracted from $A_0^{(1)}$ and $A_1^{(1)}$):

LOGARITHMS OF $A_j^{(i)}$.

i	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$
0	0.3290769	9.4911841	9.4124837	9.1660038	9.0416585
1	8.6890960	9.2152952	9.3334831	9.2075472	9.0337108
2	9.2854829	9.6363530	9.5165204	9.1946468	9.0600964
3	8.8905316	9.4032790	9.4925919	9.3071685	9.0701913
4	8.515813	9.1456982	9.3790588	9.3422364	9.1445298
5	8.152840	8.8746742	9.2172230	9.3020362	9.1988251
6	7.79756	8.595126	9.0253814	9.2090472	9.2024474

Employing the same value, as before, for the eccentricity of Jupiter, we obtain, by the help of the preceding data, for the second class of considered terms, the expression

$$\frac{\partial R}{\partial e} = \mp 0.1082128 + 1.250172\, e \mp 0.630954\, e^2 + 1.765393\, e^3$$

the upper sign belonging to the case where the perihelia coincide, the lower to the case where they are opposed.

From the part of the perturbative function which is usually denominated secular, we have

$$\frac{\partial R}{\partial e} = \mp 0.0080600 + 0.287698\, e \mp 0.046723\, e^2 + 0.202990\, e^3$$

The sum of the two expressions is

$$\frac{\partial R}{\partial e} = \mp 0.1162728 + 1.537870\, e \mp 0.677677\, e^2 + 1.968383\, e^3$$

The right member of this, equated to zero (the upper sign being taken), gives $e = 0.077565$.

But, on account of the neglect of higher powers than the fourth of the eccentricities, this result is likely to be error to a considerable amount. Therefore a second determination of the amount of motion of the line of apsides of the minor planet has been made by mechanical quadratures for two different values of e. The following formulas may be used:

$$\varepsilon - e \sin \varepsilon = 3l' \qquad \varepsilon' - e' \sin \varepsilon' = l'$$

$$r \cos f = a\,(\cos \varepsilon - e) \qquad r' \cos f' = a'\,(\cos \varepsilon' - e')$$

$$r \sin f = a \sqrt{1-e^2}\, \sin \varepsilon \qquad r' \sin f' = a' \sqrt{1-e'^2}\, \sin \varepsilon'$$

$$\Delta^2 = [r - r' \cos (f-f')]^2 + [r' \sin (f-f')]^2$$

$$\mathfrak{R} = a'^2 \left[\frac{1}{\Delta^3} - \frac{1}{r'^3}\right] r' \cos (f-f') - \frac{a'^2 r}{\Delta^3}$$

$$\mathfrak{S} = -a'^2 \left[\frac{1}{\Delta^3} - \frac{1}{r'^3}\right] r' \sin (f-f')$$

Then it is necesary to compute the definite integral

$$\frac{1}{\pi}\int^{\pi}\left[-\mathfrak{R}\cos f+\mathcal{S}\left(1+\frac{r}{p}\right)\sin f\right]dl'$$

This we do for the two values $e = 0.077565$ and $e = 0.078565$. The values of the quantity under the sign of integration for every 5° of the mean anomaly of Jupiter are given in the following table :

l'	$e = 0.077565$	$e = 0.078565$	l'	$e = 0.077565$	$e = 0.078565$
°			°		
0	—2.764426	—2.757124	95	—0.156316	—0.157956
5	2.835408	2.828391	100	—0.053241	—0.054595
10	2.825459	2.817919	105	+0.018234	+0.017387
15	2.493183	2.484721	110	+0.032642	+0.032315
20	1.904867	1.895961	115	—0.015514	—0.015502
25	1.247910	1.239685	120	0.102780	0.102618
30	0.665311	0.658612	125	0.178702	0.178315
35	—0.227037	—0.222175	130	0.180609	0.179565
40	+0.050129	+0.053225	135	—0.054306	—0.052646
45	0.180203	0.181698	140	+0.225246	+0.229136
50	0.192504	0.193372	145	0.652027	0.657593
55	0.122146	0.122512	150	1.183608	1.190446
60	+0.005293	+0.004863	155	1.749316	1.756153
65	—0.126795	—0.126801	160	2.253334	2.260756
70	0.244125	0.244309	165	2.598633	2.605567
75	0.325167	0.325641	170	2.726679	2.733158
80	0.355710	0.356586	175	2.689420	2.695806
85	0.331234	0.332528	180	+2.644934	+2.651387
90	—0.258528	—0.260128			

The sums of the numbers respectively in the second and third columns (the values appertaining to 0° and 180° receiving only half weight) are

$$+\ 0.037466 \text{ and } +\ 0.146495$$

Interpolating, we find that the value of e, proper to make the definite integral vanish, is $e = 0.07722136$.

We now construct to scale and by points, Fig. 3, exhibiting the synodic orbit of the minor planet; where, in explanation, it is necessary only to say that when Jupiter is severally at J and J' the planet is severally at P and P'. On comparison of Figs. 2 and 3, it will be noticed that, in the latter, P' lying between P and J instead of between P and S, two additional multiple points lying off the line of syzygies are necessitated. The latter figure probably shows what usually occurs in periodic solutions of this type.

Our second example will be the minor planet of the Hecuba type, where $k = 2$ and $k' = 1$. Then, with similar conditions and notation as in the first example, the terms of the perturbative function, made secular by the commensurability of the mean motions, are:

$$\begin{aligned} R = {} & (11)^2 \chi + (12)^2 \chi^3 + (13)^2 \chi'^2 \chi + (42)^1 \chi' \chi^2 + (89)^0 \chi'^2 \chi \\ & + (21)^4 \chi^2 + (22)^4 \chi^4 + (23)^4 \chi'^2 \chi^2 + (51)^3 \chi' \chi + (52)^3 \chi' \chi^3 \\ & + (53)^3 \chi'^3 \chi + (84)^2 \chi'^2 \chi^2 + (118)^1 \chi'^3 \chi + (27)^6 \chi^3 + (60)^5 \chi' \chi^2 \\ & + (89)^4 \chi'^2 \chi + (33)^8 \chi^4 + (69)^7 \chi' \chi^3 + (98)^6 \chi'^2 \chi^2 + (118)^5 \chi'^3 \chi \end{aligned}$$

Or, in the more explicit form,

$$\begin{aligned} R = {} & -[4 + 1]^2 \chi + [14 + \tfrac{5}{2} - 6 - 3]^2 \chi^3 + [64 + 6 - 16 - 6]^2 \chi'^2 \chi + [-12 + 4 + 14 + 6]^1 \chi' \chi^2 \\ & - [0 + 5 + 8 + 3]^0 \chi'^2 \chi + [22 + 7 + 1]^4 \chi^2 + [-\tfrac{596}{3} - \tfrac{212}{3} + 8 + 16 + 4]^4 \chi^4 \\ & + [-1408 - 390 + 42 + 60 + 12]^4 \chi'^2 \chi^2 - [42 + 14 + 2]^3 \chi' \chi \\ & + [462 + 138 - 54 - 54 - 12]^3 \chi' \chi^3 + [1074 + 270 - 78 - 66 - 12]^3 \chi'^3 \chi \\ & + [-304 - 60 + 84 + 60 + 12]^2 \chi'^2 \chi^2 + [\tfrac{142}{3} - \tfrac{14}{3} - 38 - 22 - 4]^1 \chi'^3 \chi \\ & - [134 + \tfrac{93}{2} + 10 + 1]^6 \chi^3 + [\tfrac{825}{2} + \tfrac{291}{2} + 31 + 3]^5 \chi' \chi^2 - [416 + 151 + 32 + 3]^4 \chi'^2 \chi \\ & + [\tfrac{2570}{3} + \tfrac{932}{3} + 80 + 13 + 1]^8 \chi^4 - [3640 + \tfrac{3992}{3} + 340 + 54 + 4]^7 \chi' \chi^3 \\ & + [5757 + 2131 + 541 + 84 + 6]^6 \chi'^2 \chi^2 - [\tfrac{12010}{3} + \tfrac{4534}{3} + 382 + 58 + 4]^5 \chi'^3 \chi \end{aligned}$$

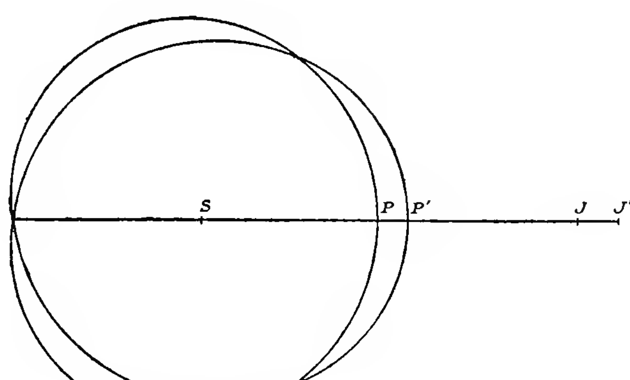


FIG. 3.

In the Hecuba type of minor planet $\alpha = 0.6297651$. With this argument we take from Runkle's tables the values of Leverrier's $A_j^{(i)}$ and their logarithms are given in the following table (α has been subtracted in the cases of $A_0^{(1)}$ and $A_1^{(1)}$):

LOGARITHMS OF $A_j^{(i)}$.

i	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
0	. . . .	9.8426431	9.9305853	9.9823155	0.1024322
1	9.1028882	9.6771459	9.9034748	9.9959539	0.1028525
2	9.5623275	9.9632340	9.9976231	0.0010594	0.1129766
3	9.2871510	9.8370606	0.0184658	0.0512301	0.1246387
4	9.0313180	9.6909111	9.9878184	0.0925462	0.1555446
5	8.786771	9.5334010	9.9236912	0.1062645	0.1934997
6	8.54962	9.368508	9.8366581	0.0921560	0.2229760
7	8.31774	9.198419	9.7331207	0.0544375	0.2359934
8	8.08977	9.02446	9.6171825	9.9974755	0.2305597

With the same value of Jupiter's eccentricity as before, we get, for the case where the perihelia coincide,

$$\frac{\partial R}{\partial e} = -1.482354 + 5.143972\,e - 13.880955\,e^2 + 17.1062\,e^3$$

and for the case where they are opposed,

$$\frac{\partial R}{\partial e} = -0.989656 + 2.193098\,e - 5.076561\,e^2 + 17.1062\,e^3$$

For the part of the perturbative function, usually denominated secular, we have, in the first case,

$$\frac{\partial R}{\partial e} = -0.028184 + 0.795798\,e - 0.34150\,e^2 + 1.8647\,e^3$$

and, in the second case,

$$\frac{\partial R}{\partial e} = +0.028184 + 0.795798\,e + 0.34150\,e^2 + 1.8647\,e^3$$

By the addition of the two portions, severally for each case, we obtain

$$\frac{\partial R}{\partial e} = -1.510538 + 5.939770\,e - 14.22245\,e^2 + 18.9709\,e^3 = 0$$

$$\frac{\partial R}{\partial e} = -0.961472 + 2.988896\,e - 4.73506\,e^2 + 18.9709\,e^3 = 0$$

The solution of these equations gives the value of e, for which, in each case, the secular motion of the line of apsides vanishes. The roots are, severally, $e = 0.45$ and $e = 0.30$. But a comparison of the values of the last with the first terms shows that these values are probably very wide of the mark. We must resort to mechanical quadratures. The formulas to be employed here are the same as in the first example, except that we substitute

$$\varepsilon - e \sin \varepsilon = 2\,l'$$

So little is known of the secular motion of the line of apsides in the difficult case of the planet of the Hecuba type that I feel justified in giving some details of the calculations I have made. Limiting ourselves at first to the case where the perihelia coincide, and where the two planets are in symmetrical conjunction when they occupy these positions, the quantity under the integral sign has the following values for the specified values of e

l' °	$e=0$	$e=0.12$	$e=0.14$	$e=0.2056$	$e=0.28$	$e=0.7$
0	—8.530590	—5.217868	—4.777629	—4.006547	—2.923007	—0.614550
10	8.461098	5.215575	4.957969	3.891521	2.927314	+0.229027
20	7.634184	4.496858	4.116017	3.049454	2.085997	1.089245
30	6.004867	3.231570	2.889654	1.927968	1.074403	1.286962
40	4.214893	2.062105	1.792201	1.062899	0.429075	1.207302
50	2.675585	1.221548	1.043364	0.552893	0.134531	0.976214
60	1.546403	0.712261	0.609033	0.323279	0.077717	0.653932
70	0.843500	0.468127	0.418578	0.337349	0.155464	+0.283321
80	0.513848	0.410583	0.395853	0.351480	0.306481	—0.098206
90	0.463977	0.460630	0.478429	0.481446	0.482434	0.455054
100	0.582306	0.612484	0.617623	0.634250	0.653194	0.751549
110	0.760216	0.771618	0.773134	0.784790	0.800732	0.949993
120	0.905745	0.914425	0.914456	0.913984	0.915747	1.010152
130	0.967521	1.003429	1.004982	1.002797	0.992655	0.896834
140	0.937333	1.008472	1.015565	1.022202	1.021184	0.611639
150	0.832348	0.916754	0.928795	0.959964	0.976675	0.272769
160	0.700922	0.752902	0.753064	0.793750	0.826882	0.138557
170	0.595117	0.589793	0.589561	0.590287	0.595183	0.354465
180	—0.554843	—0.520554	—0.514359	—0.620608	—0.466615	—0.256730

It thus appears that the line of apsides continually retrogrades for all values of e below 0.28, and it is not until $e=0.7$ that the advancing seriously begins to counterbalance the retrogradation. For $e=0$, the value of the definite integral is — 2.287926, while, for $e=0.7$, the value is — 0.013825. However, much precision cannot be attributed to the latter, as 18 points on the semi-circumference are insufficient for anything but a rude approximation.

It seemed likely that the definite integral would vanish for a value of e in the neighborhood of 0.72; hence another computation was made for $e=0.72$, doubling the number of points on the semi-circumference, with the following result:

l'	$e=0.72$	l'	$e=0.72$	l'	$e=0.72$	l'	$e=0.72$
0°	—0.559176	50°	+1.012882	95°	—0.614318	140°	—0.573594
5	—0.403687	55	0.856245	100	0.757217	145	0.370910
10	+0.379255	60	0.682514	105	0.874217	150	0.179530
15	0.899073	65	0.496646	110	0.961737	155	0.044604
20	1.175027	70	0.303366	115	1.011920	160	0.010630
25	1.305755	75	+0.107266	120	1.019747	165	0.098793
30	1.345358	80	—0.087135	125	0.981329	170	0.274107
35	1.322269	85	0.276233	130	0.892546	175	0.379638
40	1.252687	90	—0.453425	135	—0.753961	180	—0.243223
45	+1.147068						

These numbers make the value of the definite integral positive, viz., $+0.024026$. By interpolation between the results for $e=0.7$ and $e=0.72$, it is concluded that $e=0.7073$ would make the definite integral vanish; thus we adopt $e=\sin 45^\circ$, and with this Fig. 4, exhibiting the synodic orbit, has been constructed. But to establish the matter more firmly, the value of the definite integral was computed for $e=\sin 45^\circ$, this time making ε the independent variable instead of l'. In this method it is only necessary to multiply the former expression for the quantity under the integral sign by $\frac{r}{a}$. This method undoubtedly has advantages over the method with l' as independent variable. Details of the result are

ε	$e=\sin 45^\circ$	ε	$e=\sin 45^\circ$	ε	$e=\sin 45^\circ$	ε	$e=\sin 45^\circ$
0°	—0.17489	100°	+1.46880	190°	—1.21180	280°	—0.08244
10	0.19121	110	1.59050	200	1.50660	290	0.11638
20	0.21489	120	1.59338	210	1.61982	300	0.16077
30	0.18993	130	1.45842	220	1.53837	310	0.18694
40	—0.07980	140	1.18491	230	1.28591	320	0.18122
50	+0.11845	150	0.78321	240	0.92743	330	0.15205
60	0.37672	160	+0.28601	250	0.55782	340	0.11487
70	0.68259	170	—0.25444	260	0.27157	350	0.08509
80	0.98628	180	—0.77550	270	—0.11727	360	—0.07512
90	+1.25652						

These numbers make the value of the definite integral -0.004481, which seems to show that the desired value of e somewhat exceeds $\sin 45^\circ$.

To see whether the other three arrangements of the elements could bring about periodic solutions in the case of the Hecuba type of minor planet, the value of the definite integral has been computed, in the first instance, for $e=0.7$, but with Jupiter in aphelion instead of perihelion at the time of symmetrical conjunction, and, in the second instance. for

$e = 0.2056$, but with the aphelion of the minor planet in conjunction with the perihelion of Jupiter, or, which amounts to the same thing, the computation is made with $e = -0.2056$. The details of these calculations are thus shown:

l'	$e = 0.7$	$e = -0.2056$	l'	$e = 0.7$	$e = -0.2056$
°			°		
0	—0.444167	+25.81939	100	—0.592627	+0.54894
10	+0.210407	24.77975	110	0.923640	0.73260
20	0.906778	21.62186	120	1.168929	0.81629
30	1.106777	16.83959	130	1.263650	0.80290
40	1.096926	11.92495	140	1.143922	0.74732
50	0.967681	7.57606	150	0.817554	0.68782
60	0.752428	4.33151	160	0.511273	0.63966
70	0.469796	2.00845	170	0.549153	0.61324
80	+0.136597	0.78334	180	—0.335407	+0.60391
90	—0.226700	+ 0.41958			

The value of the definite integral, in the first instance, is -0.107769, which, as it is more decidedly negative than when Jupiter was in perihelion, leads us to think that a larger value of e than sin 45° is necessary for a periodic solution in this arrangement than in the case we have worked out, and, perhaps, it may not exist. In the second instance, the line of apsides continually advances, and plainly it does so all the way from $e = 0$ up to the point where the minor planet passes through the perihelion position of Jupiter, when the motion becomes infinite; hence no periodic solution is to be looked for in this direction.

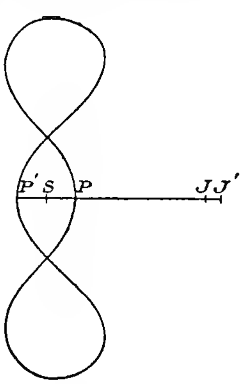


FIG. 4.

The figure shows that, although the minor planet crosses the orbit of Jupiter four times every synodic revolution, it keeps well out of the way of the latter planet. Thus the periodic perturbations must be quite small, probably no coefficient of any periodic term in the longitude exceeding 200″, with correspondingly small inequalities in the radius. The circumstance that the two planets change the order of their distance from the Sun is no bar to the representation of the coordinates of the minor planet by periodic series. The elaboration of the latter for the periodic solution, just treated, is far easier than in the general case involving Lindstedt's series. But, as Nature affords us no example of this periodic solution, perhaps the labor would be unwarranted. However, hints might be suggested in its course, profitable for the more complicated case.

MEMOIR No. 73.

On the Application of Delaunay Transformations to the Elaboration of the Secular Perturbations of the Solar System.

(Astronomical Journal, Vol. XXII, pp. 183–189, 1902.)

When it is desired to develop the coordinates or the Keplerian elements of a system of planets in infinite series, and there is no objection to the appearance of powers of the time in the expressions of the coefficients, the procedure to be followed is, in general, immediately apparent. But, when t is to be kept within the functional signs sine or cosine, the course to be adopted is not so clear. The difficulty is especially present when the problem is to determine the secular values of the elements apart as far as possible from the periodic terms. On this point the reader may be referred to Leverrier's statement of the course he pursued in the elaboration of his theory of Jupiter and Saturn (*Annales de l' Observatoire de Paris*, Tom. X, pp. 99-103), and to the *Nachtrag* of Prof. Paul Harzer's prize memoir (*Die Säcularen Veränderungen der Bahnen der grossen Planeten*). As a practical matter, if it is desired to go beyond terms of the first order with respect to planetary masses, it is impossible to get the secular perturbations without at the same time consenting to the derivation of the periodic perturbations. However, it is to be noted that the latter need be obtained only to terms one order lower than the last order to be retained in the formation of the differential equations determining the secular values of the elements. Thus, if it is proposed to neglect all terms of the third order in the formation of the mentioned equations, only the first power of the planetary masses need be considered in determining the periodic terms of the elements.

Delaunay's transformations, in his treatment of the Lunar Theory, extended so that they become applicable to planetary motions, seem eminently suited to remove whatever obscurities there may be in the processes heretofore adopted. However, it would be extremely inconvenient, not to say, impossible, to apply Delaunay's method to a group of differential equations expressed in terms of the coordinates or elements of the planets. It appears essential that we should make a linear and orthogonal transforma-

tion in the rectangular coordinates. This transformation, first indicated by Jacobi* in the case of two planets, was afterwards extended by Radau† to any number.

Denote the mass of the central body by m_0, and the masses of the planets, in an order which is at our choice, by m_1, m_2, etc.; moreover, put

$$\mu_i = m_0 + m_1 + m_2 + \ldots + m_i, \qquad \varkappa_i = \frac{m_i}{\mu_i}$$

Then the type of representation of the rectangular coordinates of the i^{th} planet relative to the central body, in this linear and orthogonal transformation, is

$$x_i + \varkappa_{i-1}\, x_{i-1} + \varkappa_{i-2}\, x_{i-2} + \ldots + \varkappa_1\, x_1$$

The differential equations these variables satisfy are of the type

$$\mu_{i-1}\, \varkappa^i \frac{d^2 x_i}{dt^2} = \frac{\partial \Omega}{\partial x_i}$$

where Ω denotes the sum of the products of every two masses of the system divided by their distance, a relation we will write thus:

$$\Omega = m_0 \sum \frac{m_i}{\Delta_{0,}} + \sum \frac{m_i m_j}{\Delta_{i,j}}$$

In order to pass from equations in terms of rectangular coordinates to those in terms of Keplerian elements it is necessary to choose a simplified form of Ω defining these elements. Calling this form Ω_0 we suppose

$$\Omega_0 = m_0 \sum \frac{m_i}{r_i}$$

where $r_i^2 = x_i^2 + y_i^2 + z_i^2$. If Ω_0 is substituted for Ω in the differential equations, and the members are divided by $\mu_{i-1}\varkappa_i$, we get a system of equations of which the type is

$$\frac{d^2 x_i}{dt^2} + m_0 \frac{\mu_i}{\mu^i_{-1}} \frac{x_i}{r_i^3} = 0$$

Let a_i be the semi-axis major, e_i the eccentricity, ϕ_i the inclination, l_i the mean anomaly, g_i the angular distance of the perihelion from the node and h_i the longitude of the node, of a planet whose rectangular coordinates are determined by equations whose type has just been written. Then the type of linear elements severally conjugate to the angular elements l_i, g_i, h_i, in a canonical system is

$$L_i = m_i \sqrt{m_0 \frac{\mu_{i-1}}{\mu_i} a_i}, \qquad G_i = L_i \sqrt{1 - e_i^2}, \qquad H_i = G_i \cos \phi_i$$

* *Sur l'élimination des nœuds dans le problème des trois corps.*

† *Sur une transformation des équations différentielles de la dynamique.*

Also construct a function

$$F = m_0 \sum \frac{m_i}{2a_i} + m_0 \sum m_i \left(\frac{1}{\Delta_{0,i}} - \frac{1}{r_i}\right) + \sum \frac{m_i m_j}{\Delta_{i,j}}$$

(This F is the negative of Poincaré's F.) Then, if the a and the planetary coordinates in the right member are replaced by the canonical elements L, G, H, l, g, h, the differential equations determining the latter are of the type:

$$\frac{dL_i}{dt} = \frac{\partial F}{\partial l_i}, \qquad \frac{dl_i}{dt} = -\frac{\partial F}{\partial L_i}$$

$$\frac{dG_i}{dt} = \frac{\partial F}{\partial g_i}, \qquad \frac{dg_i}{dt} = -\frac{\partial F}{\partial G_i}$$

$$\frac{dH_i}{dt} = \frac{\partial F}{\partial h_i}. \qquad \frac{dh_i}{dt} = -\frac{\partial F}{\partial H_i}$$

These are the differential equations required for the application of the Delaunay method to the motion of a planetary system.

It is now necessary to rigorously define how perturbations are to be divided into the two classes of terms called secular and periodic. When F is developed into an infinite periodic series, the arguments of the several terms are linear functions with integral coefficients of the linear elements l_i, g_i, h_i; consequently there are some terms whose arguments do not involve any of the l_i. These terms are denominated *secular*, while the others, in which some of the l_i are present, are denominated *periodic*. It is here assumed that the mean motions of the l_i are incommensurable; this, however, is only to insure mathematical rigor in the statements; for, if the integers expressing the ratios of the motions of the l_i are quite large, the statements are still true in a *practical sense*. To illustrate, suppose that the ratio of the mean revolutions of two planets is as 60 to 149 (which is nearly the case with Jupiter and Saturn) we should have to go to terms of the 89th order with respect to eccentricities and inclinations before anything contravening our statements was met with. Let us now suppose that, while we have been finding the formulas of transformation for the purpose of removing from F its periodic terms, we have made the substitutions also in the original Keplerian elements, precisely as Delaunay does in the three polar coordinates of the moon. After all the periodic terms have been removed from F, it is obvious that the Keplerian elements will be expressed by a series of terms in which some involve the l_i in their arguments and others do not. The first, taken together, will constitute the *periodic* perturbations of the elements, while the second, in like manner, constitute the *secular* perturbations of the same.

The circumstance that an infinite number of transformations has to be made to completely free F from its periodic terms is no valid reason for declining to accept this definition of the distinction between *secular* and *periodic* perturbations. In practice we confine ourselves to a moderate number of "Operations." Thus Delaunay, in his treatment of the Lunar Theory, found that about 500 of these transformations reduced F sensibly to a non-periodic term.

It would now seem that the application of the proposed method to determining the secular values of the Keplerian elements of the eight major planets of the solar system involves an amount of labor not to be thought of, since there are 48 Keplerian elements in addition to the function F, in all of which the transformations of every operation have to be made. But it can be shown that, for practical purposes, the transformations may be limited to F alone.

The demonstration of this may be made to depend on several theorems. The first is:

THEOREM I.— *When we have obtained the* secular *values of one set of Keplerian elements we can derive thence the* secular *values of any other set, provided we are willing to neglect terms of two dimensions with respect to planetary masses.*

For the secular terms which arise from the inter-multiplication of the periodic terms with themselves or with other periodic terms are necessarily of two dimensions with respect to planetary masses. For instance, if we are in possession of expressions for the elements $e \cos (h+g)$ and $e \sin (h+g)$ of the following form:

$$e \cos (h + g) = S + P, \qquad e \sin (h + g) = S' + P'$$

where S and S' are the secular portions and P and P' the periodic portions, it may be desired to get the secular value of e. It is obvious that, to the degree of approximation proposed, it is given by the equation

$$e = \sqrt{S^2 + S'^2}$$

although the rigorous value is the secular portion of

$$\sqrt{(S + P)^2 + (S' + P')^2}$$

for the former differs from the latter only by a quantity of the order of P^2 or P'^2.

In our linear transformation of rectangular coordinates, we can imagine that x_i, y_i, z_i, are the rectangular coordinates of a hypothetical planet, which we may designate as belonging to the i^{th} planet. Then this has its instantaneous Keplerian elements as well as the real planet to which it belongs.

We may inquire how the secular values of the elements of the two planets compare with each other; and thus is found the following:

THEOREM II. — *Provided we neglect quantities of two dimensions with respect to planetary masses, the* secular *values of the elements of each actual planet are the same as those of the corresponding elements of its belonging hypothetical planet.*

In proving this theorem we may always neglect the squares and products of the constants we have denoted by $\varkappa_i$, and thus may reduce the relations existing between the rectangular coordinates of the real and hypothetical planets to an expression involving a single $\varkappa$. Thus, while X, Y, Z denote the coordinates of the considered actual planet, let x, y, z denote those of its attached hypothetical planet, and x', y', z' those of another hypothetical planet; then we may set the equations:

$$\mathrm{X} = x + \varkappa x', \qquad \mathrm{Y} = y + \varkappa y', \qquad \mathrm{Z} = z + \varkappa z'$$

where $\varkappa$ is any constant of the order of planetary masses. In the first place we suppose that the two hypothetical planets are governed in their motions by the laws of Kepler. Thus k and k' being two constants and $r^2 = x^2 + y^2 + z^2$ and $r'^2 = x'^2 + y'^2 + z'^2$, we have six differential equations, of which the type is:

$$\frac{d^2x}{dt^2} + k\frac{x}{r^3} = 0, \qquad \frac{d^2x'}{dt^2} + k'\frac{x'}{r'^3} = 0$$

If we multiply the second equation by $\varkappa$ and add the product to the first, the result is the type equation:

$$\frac{d^2\mathrm{X}}{dt^2} + k\frac{x}{r^3} + \varkappa k'\frac{x'}{r'^3} = 0$$

Eliminating x, y, z from these equations by means of the values

$$x = \mathrm{X} - \varkappa x', \qquad y = \mathrm{Y} - \varkappa y', \qquad z = \mathrm{Z} - \varkappa z'$$

writing r^2 for $\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2$, and retaining only the first power of $\varkappa$, we have three equations of which the type is:

$$\frac{d^2\mathrm{X}}{dt^2} + k\frac{\mathrm{X}}{\mathrm{r}^3} + \varkappa\left[3k\frac{\mathrm{X}}{\mathrm{r}^3}\frac{\mathrm{X}x' + \mathrm{Y}y' + \mathrm{Z}z'}{\mathrm{r}^2} - k\frac{x'}{\mathrm{r}^3} + k'\frac{x'}{r'^3}\right] = 0$$

These three differential equations admit a perturbative function; for if we put

$$R = \varkappa\left[k\frac{\mathrm{X}x' + \mathrm{Y}y' + \mathrm{Z}z'}{\mathrm{r}^3} - k'\frac{\mathrm{X}x' + \mathrm{Y}y' + \mathrm{Z}z'}{r'^3}\right]$$

they take a form of which the type is

$$\frac{d^2\mathrm{X}}{dt^2} + k\frac{\mathrm{X}}{\mathrm{r}^3} = \frac{\partial R}{\partial \mathrm{X}}$$

But, on scrutinizing the form of R, we see that in its periodic development it has no secular portion, since the first part can have no term independent of the mean anomaly of the actual planet, and the second part no term independent of the mean anomaly of the second hypothetical planet. Hence, the theorem is true when the planets concerned are supposed to suffer no perturbations. But, it is true even when we consider perturbations; for here it is sufficient to limit ourselves to periodic perturbations of the first order, and these can affect the secular values of the elements concerned by quantities which are of the second order.

In applying the procedure of Delaunay to the elaboration of our problem it will be found that the use of the linear variables L, G, H, which are conjugate to the angular variables, l, g, h, is awkward and it will be advisable to imitate Delaunay's example in substituting others in their place. Then, in order to form the expressions for the differentials of the angular elements, it will be necessary for us to know the partial derivatives of each of the new set of variables with respect to each of the former set, but expressed in terms of the new set. In regard to this matter we have:

THEOREM III.—*Provided we neglect terms of three dimensions with respect to planetary masses in the formation of the differential equations for determining the secular values of the elements, the just-mentioned partial derivatives maintain the same expressions throughout all the transformations made to free F from its periodic terms.*

To prove this, let us suppose that a transformation is made to remove from F the periodic term having θ as argument, θ involving at least one of the angular variables l; we know, that L being one of the linear elements, the following formulas of transformation exist:

$$L = L_0 + L_1 \cos\theta + L_2 \cos 2\theta + \ldots$$
$$\theta = \theta_0 (t + c) + \theta_1 \sin [\theta_0 (t + c)] + \theta_2 \sin 2 [\theta_0 (t + c)] + \ldots$$

Then the new linear variable L, conjugate to the new angular variable l, is equivalent to the former L augmented by the expression:

$$\tfrac{1}{2} (\theta_1 L_1 + 2\theta_2 L_2 + 3\theta_3 L_3 + \ldots)$$

But this is evidently of two dimensions with respect to planetary masses; and the partial derivatives, mentioned above, have all to be multiplied by factors of one dimension with respect to the same. The last statement is subject to an exception, viz.: when, at the end, the mean motions of the l are derived through the differentiation of F, the last factor is of the dimension zero. But, as we expect to derive the motions of the mean longitudes of the planets from observation, in a practical sense this exception need not

be considered. Thus, the L, G, H will always have the same expressions in terms of any other linear variables we may choose, as a, a', e, e', etc. This property is well illustrated in Delaunay's Lunar Theory. If the values of $\frac{\partial a}{\partial L}$, etc. (given Tom. I, p. 259), are compared with those given at the end of each "Operation" the differences will be found to be divisible by $\frac{n'^4}{n^4}$ until we come to Operation 41, when, on account of l not being present in the argument of the term to be removed by the transformation, the differences are divisible only by $\frac{n'^2}{n^2}$.

These three theorems make evident what is necessary to be done in the proposed method of attacking the problem in hand. In the first place, we assume, since we must set some degree of approximation to be aimed at, that terms of three dimensions may be neglected in the formation of the differential equations. This is the same as to say that, after integration, terms of two dimensions may be passed by, since the effect of this process is to lower the terms by one dimension. Then we develop F into a periodic series, pushing the approximation in the secular portion so as to include terms of two dimensions, but contenting ourselves with terms of one dimension in the periodic portion. Next, by "Operations" of Delaunay we remove its periodic terms from F, term by term. These transformations will be made in F; the consequence will be that the secular part of F will receive accessions of new terms which we preserve, and the periodic portion also new terms all of two dimensions, which we throw aside as unnecessary for our purposes. As many of these "Operations" will be performed as we judge have a significant effect on the secular portion of F. After this is accomplished we lop off from F any periodic terms it may still contain. After combining together the terms which admit addition, the result will be a function F composed exclusively of secular terms. To get the differential equations determining the secular values of the elements of the system, we must subject this F to the same partial differentiations and multiplications by the same factors as in the case where all consideration of terms of two dimensions is neglected. After this is done, the linear elements appearing in the equations have the same signification in both cases: or, which may be more easily comprehended—desiring to include the effect of second-order terms, we do it simply by modifying the form of F and modifying nothing else. Thus is seen the great simplicity of the Delaunay method of proceeding.

THE TERMS OF F TO BE RETAINED IN ITS PRELIMINARY DEVELOPMENT.

In the preliminary development of F in periodic series it is desirable to retain only very exceptionally terms of two dimensions with respect to planetary masses. In the first place, in the interaction of Jupiter and Saturn, these terms are needed because the masses of these planets are large, and because the periods are nearly as 2 to 5. In the second place, in the interaction of Uranus and Neptune, they are needed because the periods are nearly as 1 to 2.

Consider in F the series of terms

$$\sum \frac{m_i m_j}{\Delta_{i,j}}.$$

As there are 8 planets in the system, there will be 28 terms of this type, in 26 of which we can reduce the coordinates of the actual planets to those of their hypothetical planets. Hence, here there will be no difficulty in forming the periodic developments of the reciprocals of the distances, especially as we do not need the periodic portions. But, in the cases of Jupiter-Saturn and Uranus-Neptune, it will be advisable to include terms multiplied by some of the quantities $\varkappa$. Let us suppose that the coordinates of the interior hypothetical planet are x, y, z, while those of the exterior are x', y', z', and adopt the notation

$$xx' + yy' + zz' = rr' \cos H.$$

The reciprocal of the distance, in either of the two cases, will have the expression:

$$\frac{1}{\Delta} = \frac{1}{r'} \left[1 - 2(1-\varkappa)\frac{r}{r'} \cos H + (1-\varkappa)^2 \frac{r^2}{r'^2} \right]^{-\frac{1}{2}}$$

It is plain from the form of the right member of this that its periodic development can be obtained from the ordinary expression for the perturbative function if, in the computation of the quantities $b_s^{(i)}$ of Laplace, we employ the argument $\alpha = (1-\varkappa)\frac{a}{a'}$ instead of $\alpha = \frac{a}{a'}$.

Consider next the middle term of F. It is well known that, for secular perturbations, this term can give rise only to quantities of two dimensions with respect to disturbing forces; hence, according to our plan, this term need be taken into account only in the cases of the interaction of Jupiter and Saturn and again in that of Uranus and Neptune. But we propose neglecting terms of this kind in the latter case because they are not augmented by the small divisor $2n' - n$. Hence, the discussion may be limited to the case of the interaction of Jupiter and Saturn. Here

$$\frac{1}{\Delta_{0,1}} - \frac{1}{r'} = -\varkappa \frac{r}{r'^2} \cos H + \tfrac{1}{2} \varkappa^2 \frac{r^2}{r'^3} (3 \cos^2 H - 1).$$

The first term of the right member is given immediately by the ordinary development of the perturbative function. As the second term is of two dimensions we need consider only its secular portion. In this connection it is proposed to neglect inclinations when we are dealing with terms of two dimensions with respect to disturbing forces. With this limitation it is well known that the secular part of the term under consideration is:

$$\tfrac{1}{4}\,\varkappa^2\,\frac{a^2}{a'^3}\,\frac{1+\tfrac{3}{2}e^2}{(1-e'^2)^{\frac{3}{2}}}$$

and thus is free from the angular elements g, g', h and h'. But as it may be desired to be free from this restriction we will give the rigorous expression. If we denote by A the expression just written and adopt

$$B=\tfrac{15}{16}\,\varkappa^2\,\frac{a^2}{a'^3}\,\frac{e^2}{(1-e'^2)^{\frac{3}{2}}}$$

the secular portion of the term in question is:

$$\begin{aligned}
&A\,(1-\tfrac{3}{2}\sin^2\phi)\,(1-\tfrac{3}{2}\sin^2\phi')\\
&+\tfrac{3}{4}A\sin 2\phi\sin 2\phi'\cos(h-h')\\
&+\tfrac{3}{4}A\sin^2\phi\sin^2\phi'\cos(2h-2h')\\
&+B\sin^2\phi\,(1-\tfrac{3}{2}\sin^2\phi')\cos 2g\\
&-B\sin\phi\cos^2\tfrac{1}{2}\phi\sin 2\phi'\cos(2g+h-h')\\
&+B\sin\phi\sin^2\tfrac{1}{2}\phi\sin 2\phi'\cos(2g-h+h')\\
&+B\cos^4\tfrac{1}{2}\phi\sin^2\phi'\cos(2g+2h-2h')\\
&+B\sin^4\tfrac{1}{2}\phi\sin^2\phi'\cos(2g-2h+2h')
\end{aligned}$$

It will be perceived that the angular element g' is absent from this expression.

On Making the Delaunay Substitutions.

Having now the preliminary development of F it is possible for us to remove the periodic terms of that function by a series of Delaunay's "Operations." If the limitation just stated is adopted, we need retain in the periodic portion of F no term involving inclinations. Let the exposition be limited to the interaction of two planets, and let an accent be attached to the symbols belonging to the outer planet, while those of the inner are without that mark. Then the differential equations satisfied by the elements are:

$$\frac{dL}{dt}=\frac{\partial F}{\partial l}\qquad \frac{dG}{dt}=\frac{\partial F}{\partial g}\qquad \frac{dL'}{dt}=\frac{\partial F}{\partial l'}\qquad \frac{dG'}{dt}=\frac{\partial F}{\partial g'}$$

$$\frac{dl}{dt}=-\frac{\partial F}{\partial L}\qquad \frac{dg}{dt}=-\frac{\partial F}{\partial G}\qquad \frac{dl'}{dt}=-\frac{\partial F}{\partial L'}\qquad \frac{dg'}{dt}=-\frac{\partial F}{\partial G'}$$

The form of F is

$$F=-\Sigma A\cos(il+i'l'-jg+jg')$$

where i, i' and j are integers positive or negative. Suppose that, for the purpose of making a Delaunay "Operation," this function is now limited to two terms, viz., that for which $i = i' = j = 0$, and another for which these integers may be any whatever, except that i and i' must not be both 0. After Delaunay, write this limited F thus:

$$F = -B - A \cos(il + i'l' - jg + jg').$$

For brevity, the argument of the periodic term will be called θ. Then the corresponding differential equations will be:

$$\begin{array}{ll}
\dfrac{dL}{dt} = \quad i\,A \sin\theta & \dfrac{dl}{dt} = \dfrac{\partial B}{\partial L} + \dfrac{\partial A}{\partial L}\cos\theta \\
\dfrac{dL'}{dt} = \quad i'\,A \sin\theta & \dfrac{dl'}{dt} = \dfrac{\partial B}{\partial L'} + \dfrac{\partial A}{\partial L'}\cos\theta \\
\dfrac{dG}{dt} = -j\,A \sin\theta & \dfrac{dg}{dt} = \dfrac{\partial B}{\partial G} + \dfrac{\partial A}{\partial G}\cos\theta \\
\dfrac{dG'}{dt} = \quad j\,A \sin\theta & \dfrac{dg'}{dt} = \dfrac{\partial B}{\partial G'} + \dfrac{\partial A}{\partial G'}\cos\theta
\end{array}$$

In the integration of these equations we have not to go beyond the first power of the disturbing force, and we call to mind that while $\dfrac{\partial B}{\partial L}$ and $\dfrac{\partial B}{\partial L'}$ are of the zero order in this respect, A, $\dfrac{\partial B}{\partial G}$ and $\dfrac{\partial B}{\partial G'}$ are of the first order. Denoting the mean motion of θ by ν, we see that an approximative value will be

$$\nu = i\frac{\partial B}{\partial L} + i'\frac{\partial B}{\partial L'} - j\frac{\partial B}{\partial G} + j\frac{\partial B}{\partial G'}$$

where the symbols involved take their mean values. But, in practice, it will be well to include in ν the corrections of the order of A^2, since they are generally known beforehand. A little consideration will show that we are justified in writing the formulas of transformation thus:

$$\begin{array}{ll}
\text{Replace } L \text{ by } L - i\,\dfrac{A}{\nu}\cos\theta & \text{Replace } l \text{ by } l + \dfrac{1}{\nu}\left[\dfrac{\partial A}{\partial L} - i\,\dfrac{\partial^2 B}{\partial L^2}\dfrac{A}{\nu}\right]\sin\theta \\
\text{“}\quad L' \text{ by } L' - i'\,\dfrac{A}{\nu}\cos\theta & \text{“}\quad l' \text{ by } l' + \dfrac{1}{\nu}\left[\dfrac{\partial A}{\partial L'} - i'\,\dfrac{\partial^2 B}{\partial L'^2}\dfrac{A}{\nu}\right]\sin\theta \\
\text{“}\quad G \text{ by } G + j\,\dfrac{A}{\nu}\cos\theta & \text{“}\quad g \text{ by } g + \dfrac{1}{\nu}\dfrac{\partial A}{\partial G}\sin\theta \\
\text{“}\quad G' \text{ by } G' + j'\,\dfrac{A}{\nu}\cos\theta & \text{“}\quad g' \text{ by } g' + \dfrac{1}{\nu}\dfrac{\partial A}{\partial F'}\sin\theta
\end{array}$$

New secular terms in F can arise only when these substitutions are made in a periodic term of F having the form:

$$-A'\cos[il + i'l' + (j+k)(g' - g)]$$

where k is an integer which may be 0, in which case, $A' = A$. Then, putting γ for $g' - g$, the new secular terms arising in F from making the substitutions in this special term are:

$$\frac{1}{2\nu}\left[i\frac{\partial(AA')}{\partial L} + i'\frac{\partial(AA')}{\partial L'} - j\frac{\partial(AA')}{\partial G} + j\frac{\partial(AA')}{\partial G'} + kA'\left(\frac{\partial A}{\partial G'} - \frac{\partial A}{\partial G}\right)\right.$$
$$\left.-\left(\frac{i^2}{\nu}\frac{\partial^2 B}{\partial L^2} + \frac{i'^2}{\nu}\frac{\partial^2 B}{\partial L'^2}\right)AA'\right]\cos(k\gamma).$$

In the case where $k = 0$, that is, when the substitution is made in the term which gives rise to it, the preceding expression reduces to

$$\frac{1}{2\nu}\left[i\frac{\partial . A^2}{\partial L} + i'\frac{\partial . A^2}{\partial L'} - j\frac{\partial . A^2}{\partial G} + j\frac{\partial . A^2}{\partial G'} - \left(\frac{i^2}{\nu}\frac{\partial^2 B}{\partial L^2} + \frac{i'^2}{\nu}\frac{\partial^2 B}{\partial L'^2}\right)A^2\right]$$

This is to be added to the non-periodic term of F which Delaunay denotes by $-B$.

Partial differentiation with respect to L, L', G, G' is not convenient in practice, therefore we substitute for these the four variables a, a', η, η', of which the third and fourth are defined by the relations

$$1 - \sqrt{1-e^2} = \tfrac{1}{2}\eta^2 \qquad 1 - \sqrt{1-e'^2} = \tfrac{1}{2}\eta'^2.$$

The latter assumptions make the factors multiplying the partial derivatives rational in η and η'. We also make $\mu = a^3n^2$, $\eta' = a'^3n'^2$. Then

$$\frac{\partial A}{\partial L} = \frac{an}{m_0 m}\left[2a\frac{\partial A}{\partial a} + \frac{1-\frac{1}{2}\eta^2}{\eta}\frac{\partial A}{\partial \eta}\right], \qquad \frac{\partial A}{\partial G} = -\frac{an}{m_0 m}\frac{1}{\eta}\frac{\partial A}{\partial \eta}$$
$$\frac{\partial A}{\partial L'} = \frac{a'n'}{m_0 m'}\left[2a'\frac{\partial A}{\partial a'} + \frac{1-\frac{1}{2}\eta'^2}{\eta'}\frac{\partial A}{\partial \eta'}\right], \qquad \frac{\partial A}{\partial G'} = -\frac{a'n'}{m_0 m'}\frac{1}{\eta'}\frac{\partial A}{\partial \eta'}$$

With sufficient approximation

$$\frac{\partial^2 B}{\partial L^2} = -3\frac{an^2}{m_0 m} \qquad \frac{\partial^2 B}{\partial L'^2} = -3\frac{a'n'^2}{m_0 m'}$$

Since AA' is a homogeneous function of a and a' of dimensions -2, we have

$$a\frac{\partial(AA')}{\partial a} + a'\frac{\partial(AA')}{\partial a'} = -2AA'$$

by means of which partial derivatives with respect to a' may be eliminated. For the sake of making the foregoing expression for the augmentation of F more ready in use we adopt the following modification of notation ; instead of A and A' we write $\frac{mm'}{a'}A$ and $\frac{mm'}{a'}A'$, then A and A' become independent of the adopted linear and mass units. As usual, we put α for $\frac{a}{a'}$ and also make

$$f = \tfrac{3}{2}\left(i^2\frac{n^2}{\nu^2}\frac{m'}{m_0}\alpha + i'^2\frac{n'^2}{\nu^2}\frac{m}{m_0}\right) - 2i\frac{n'}{\nu}\frac{m}{m_0}, \qquad h = i\frac{n}{\nu}\frac{m'}{m_0}\alpha - i'\frac{n'}{\nu}\frac{m}{m_0}$$

Then the augmentation of F is given by the formula

$$\delta F = \frac{m m'}{a'} \left\{ f A A' + h a \frac{\partial (A A')}{\partial a} \right.$$
$$+ \frac{1}{2} \frac{n}{\nu} \frac{m' a}{m_0 \eta} \left[(i + j - \tfrac{1}{2} i \eta^2) A \frac{\partial A'}{\partial \eta} + (i + j + k - \tfrac{1}{2} i \eta^2) A' \frac{\partial A}{\partial \eta} \right]$$
$$\left. + \frac{1}{2} \frac{n'}{\nu} \frac{m}{m_0} \frac{1}{\eta'} \left[(i' - j - \tfrac{1}{2} i' \eta'^2) A \frac{\partial A'}{\partial \eta'} + (i' - j - k - \tfrac{1}{2} i' \eta'^2) A' \frac{\partial A}{\partial \eta'} \right] \right\} \cos (k\gamma)$$

It must be borne in mind that, after the substitution is completed, the term having the argument $il + i'l' + j\gamma$ disappears from F, consequently the following substitutions are not to be made in it. Hence, if ν is the number of terms in the group obtained by allowing i and i' to remain constant, but varying j, the number of term substitutions is $\nu \frac{(\nu + 1)}{2}$; and the last substitution of the group can be made only in the term itself.

After all the periodic terms of F, whose removal by "Operations" of Delaunay can sensibly modify the secular portion of this function, have been made to disappear, it is evident the latter will have the form

$$F = \frac{m m'}{a'} \left\{ A_0 + A_1 \eta \eta' \cos \gamma + A_2 \eta^2 \eta'^2 \cos 2\gamma + A_3 \eta^3 \eta'^3 \cos 3\gamma + \dots \right\}$$

where the A are capable of being expressed as power series in η^2 and η'^2. As F no longer contains the angular elements l and l', it is evident that a and a' are to be treated as constants, and, by assigning to the latter, together with the masses, their adopted numerical values, all coefficients of powers of η and η' become expressible in numbers, thus rendering the computation manageable. The variables η, η', g, g' are then determined by the following equations:

$$\frac{d\eta}{dt} = \frac{an}{m_0 m} \frac{1}{\eta} \frac{\partial F}{\partial \gamma}, \quad \frac{d\eta'}{dt} = - \frac{a'n'}{m_0 m'} \frac{1}{\eta'} \frac{\partial F}{\partial \gamma}, \quad \frac{dg}{dt} = \frac{an}{m_0 m} \frac{1}{\eta} \frac{\partial F}{\partial \eta}, \quad \frac{dg'}{dt} = \frac{a'n'}{m_0 m'} \frac{1}{\eta'} \frac{\partial F}{\partial \eta'}.$$

After the integration of these, the mean longitudes result by quadratures from

$$\frac{d(l + g)}{dt} = - \frac{an}{m_0 m} \left[2a \frac{\partial F}{\partial a} - \tfrac{1}{2} \eta \frac{\partial F}{\partial \eta} \right], \quad \frac{d(l' + g')}{dt} = - \frac{a'n'}{m_0 m'} \left[2a' \frac{\partial F}{\partial a'} - \tfrac{1}{2} \eta' \frac{\partial F}{\partial \eta'} \right]$$

It may be interesting to see how much work the proposed method demands. In the case of the interaction of Jupiter and Saturn, some information as to the special groups of terms it is advisable to retain may be got from the New Theory of Jupiter and Saturn (Astr. Papers of the American Ephemeris, Vol. IV, p. 250). Let it be proposed to neglect those groups which give less than 1000 units in the four columns of the table on that

page, and deem it sufficient to carry the approximation in the coefficients of F to terms of the fourth order inclusive with respect to eccentricities, except that, in the great inequality, the fifth-order terms are added. Then we should have the following table of 21 groups corresponding to the indicated values of i and i':

i' i	No. Op.	Term-Sub.	i' i	No. Op.	Term-Sub.	i' i	No. Op.	Term-Sub.
0—1	4	10	3—1	5	15	5—2	6	21
1 0	4	10	3—2	4	10	5—3	5	15
1—1	5	15	3—3	5	15	5—4	4	10
1—2	4	10	3—4	4	10	5—5	5	15
2—1	4	10	4—2	5	15	6—3	4	10
2—2	5	15	4—3	4	10	6—4	5	15
2—3	4	10	4—4	5	15	6—5	4	10

Thus we should have 95 operations of Delaunay, and should have to compute the formula we have given for δF 266 times. The work in the interaction of Uranus and Neptune might be limited to the three groups indicated by the figures 2–1, 4–2, 6–3, and there would be 13 operations of Delaunay and 35 term-substitutions.

MEMOIR No. 74.

Examples of Periplegmatic Orbits.

(Astronomical Journal, Vol. XXIV, pp. 9–14, 1904.)

In the motion of material points it is well known that the determination of the orbits may be considered quite apart from the question what positions upon the orbits the points have at a given time. When the first portion of the problem has been completely investigated, the second is reduced in general to a mere matter of quadratures. Gyldén's later investigations in this line have rendered this division of procedure familiar. Our illustration will be confined to the motion of two points in the same plane.

In this plane, having adopted a pole, let v denote the longitude and r and r' the radii of two orbits in the plane. The line of departure, from which v is measured, may be chosen arbitrarily, but, as r and r' are not in general periodic functions of v, it is not allowable to subtract an integral number of circumferences from the latter, which must be permitted to extend from $-\infty$ to $+\infty$. Then if p and p' are two constants, and we put

$$\frac{p}{r} - 1 = \rho, \qquad \frac{p'}{r'} - 1 = \rho'$$

the differential equations

$$\frac{d^2\rho}{dv^2} + \rho = 0, \qquad \frac{d^2\rho'}{dv^2} + \rho' = 0$$

are, as is well known, those of two conics having a focus at the pole. If, more generally, the differential equations are such that they can be written in the form

$$\frac{d^2\rho}{dv^2} = \frac{\partial V}{\partial \rho}, \qquad \frac{d^2\rho'}{dv^2} = \frac{\partial V}{\partial \rho'}$$

V may be called the orbital potential. The present discussion will be limited to the case where V does not explicitly involve v. In the foregoing simple case we have

$$V = -\tfrac{1}{2}(\rho^2 + \rho'^2)$$

A more general form for this function would be

$$V = f(\rho) + f'(\rho')$$

and then the orbits may be said to be independent of each other, and their determination is evidently a mere matter of quadratures. But, if the differential equations have not this form, nor can be given it through a transformation of variables, the orbits may be said to be *entangled*, it being impossible to determine one of them without the virtual, at least, determination of the other. It is the latter case which demands the employment of Lindstedt's series.

In the simple case adduced V was rational, integral and of two dimensions in ρ and ρ'. In order to construct a very simple case for the application of these series, suppose that V still remains rational and integral, but now involves terms of three dimensions in ρ and ρ'. Were these terms proportional to ρ^3 and ρ'^3, the resulting orbits would be independent, and there would be no occasion for the employment of Lindstedt's series. But let the new terms be proportional to $\rho^2\rho'$ and $\rho\rho'^2$, and the occasion for their use may arise.

Let us suppose that, μ being a constant,

$$2V = -\rho^2 - \rho'^2 - \mu\rho\rho'(\rho + \rho')$$

Then the differential equations will be

$$\frac{d^2\rho}{dv^2} + \rho + \mu(\rho\rho' + \tfrac{1}{2}\rho'^2) = 0$$

$$\frac{d^2\rho'}{dv^2} + \rho' + \mu(\rho\rho' + \tfrac{1}{2}\rho^2) = 0$$

It is desirable to limit as far as possible the number of constant parameters appearing in the equations, and that whether they were there originally or have been introduced by integration. In this connection it will be seen that μ is an unnecessary parameter, for it can be got rid of by multiplying both equations by it, and then replacing $\mu\rho$ and $\mu\rho'$ by ρ and ρ'. Thus, representing the radii by the equations

$$r = \frac{\mu p}{\mu + \rho}, \qquad r' = \frac{\mu p'}{\mu + \rho'}$$

ρ and ρ' will be determined by the equations

$$\frac{d^2\rho}{dv^2} + \rho + \rho\rho' + \tfrac{1}{2}\rho'^2 = 0, \qquad \frac{d^2\rho'}{dv^2} + \rho' + \rho\rho' + \tfrac{1}{2}\rho^2 = 0$$

which differ from the former only in that μ is replaced by unity.

These equations have the integral

$$\frac{d\rho^2}{dv^2} + \frac{d\rho'^2}{dv^2} + \rho^2 + \rho'^2 + \rho\rho'(\rho + \rho') = C^2$$

we write C^2 instead of C in order to avoid a radical sign in some of the following relations. When ρ and ρ' are interchanged, the equations remain

the same; thus the relation $\rho = \rho'$ constitutes a particular integral of the system of differential equations.

Adopt for exhibiting graphically the simultaneous values of ρ and ρ' (simultaneous with reference to the independent variable v) a system of rectangular coordinates, x exhibiting the value of ρ, and y the value of ρ'. Then the representative point P must lie on the negative side of the curve whose equation is

$$x^2 + y^2 + xy(x + y) - C^2 = 0$$

in order that $\frac{d\rho}{dv}$ and $\frac{d\rho'}{dv}$ may be real. This cubic will have a closed branch surrounding the origin if C^2 falls below a certain limit. It crosses the axes of x and y on both sides of the origin at distances therefrom, equal in all four cases to C. Its intersections with the right line whose equation is $x + y = 0$, and which bisects two of the angles made by the axes, are also at a distance C from the origin. On the other hand, its intersections with the line bisecting the remaining angles, whose equation is $x - y = 0$, are given by the roots of the equation

$$2x^2 + 2x^3 - C^2 - 0$$

But this cubic cannot have more than one real root unless C^2 does not exceed $\frac{8}{27}$. This is the condition necessary and sufficient that the original cubic should have a closed branch including the origin. As we wish to confine our attention to the case where the radii are restricted to finite limits, we suppose that C fulfils the mentioned condition, and that the representative point P is always within the closed branch.

When x is at a maximum or minimum in the original cubic, the equation

$$2(1 + x)y + x^2 = 0$$

must be satisfied. Multiply this by $\frac{1}{2}y$ and subtract the product from the cubic; the result is

$$x^2 + \tfrac{1}{2}x^2y - C^2 = 0$$

But the previous equation yields

$$y = -\tfrac{1}{2}\frac{x^2}{1 + x}$$

Hence the quartic

$$x^2 - \tfrac{1}{4}\frac{x^4}{1 + x} = C^2$$

by its roots, which immediately embrace 0 between them, furnishes the limits of both the variables ρ and ρ'. However, we are not under the necessity of solving the quartic for the purpose of obtaining these limits;

evidently, for C we may substitute a function of another constant rendering the solution easy.

The quartic, in a developed form, is

$$x^4 - 4x^3 - 4x^2 + 4C^2x + 4C^2 = 0$$

To remove the second term from this put $x = z + 1$, and we have

$$z^4 - 10z^2 - 4(4 - C^2)z - 7 + 8C^2 = 0$$

We can adopt indeterminates q, q', R, such that the roots of this quartic are

$$\begin{aligned} z_1 &= \sqrt{R} + \sqrt{q + q'\sqrt{R}} \\ z_2 &= -\sqrt{R} + \sqrt{q - q'\sqrt{R}} \\ z_3 &= \sqrt{R} - \sqrt{q + q'\sqrt{R}} \\ z_4 &= -\sqrt{R} - \sqrt{q - q'\sqrt{R}} \end{aligned}$$

Then q, q', R are determined by the equations

$$q + R = 5, \qquad q'R = 4 - C^2, \qquad R^3 - 5R^2 + 2(4 - C^2)R - \left(\frac{4 - C^2}{2}\right)^2 = 0$$

Put, for simplicity, $4 - C^2 = m$, then

$$\begin{aligned} z_1 &= \sqrt{R} + \sqrt{5 - R + mR^{-\frac{1}{2}}} \\ z_2 &= -\sqrt{R} + \sqrt{5 - R - mR^{-\frac{1}{2}}} \\ z_3 &= \sqrt{R} - \sqrt{5 - R + mR^{-\frac{1}{2}}} \\ z_4 &= -\sqrt{R} - \sqrt{5 - R - mR^{-\frac{1}{2}}} \end{aligned}$$

$$R^3 - 5R^2 + 2mR - \tfrac{1}{4}m^2 = 0$$

The solution of the last equation, regarding m as the unknown, is

$$m = 4R \pm 2R\sqrt{R - 1}$$

whence it follows that

$$C^2 = 4(1 - R) \mp 2R\sqrt{R - 1}$$

In order that C may be real R should exceed unity, and the cubic in R has always at least one root greater than 1; for, if we make $R = 1$, the left member becomes $-\frac{1}{4}C^4$, while, for $R = +\infty$, the result is $+\infty$.

If we make $\sqrt{R - 1} = c$, we have

$$C^2 = 2c(1 - c)^2$$

If we adopt the right member of this as a substitute for C^2, it is plain that the roots of the quartic will be expressible without the intervention of cubic radicals. While C^2 goes from 0 to $\frac{8}{27}$, c goes from 0 to $\frac{1}{3}$. In terms of c we have

$$\begin{aligned} x_1 &= 1 + \sqrt{1 + c^2} + \sqrt{4 - c^2 + (4 - 2c)\sqrt{1 + c^2}} \\ x_2 &= 1 - \sqrt{1 + c^2} + \sqrt{4 - c^2 - (4 - 2c)\sqrt{1 + c^2}} \\ x_3 &= 1 + \sqrt{1 + c^2} + \sqrt{4 - c^2 + (4 - 2c)\sqrt{1 + c^2}} \\ x_4 &= 1 - \sqrt{1 + c^2} + \sqrt{4 - c^2 - (4 - 2c)\sqrt{1 + c^2}} \end{aligned}$$

Then x_4 is evidently the lower limit of the values of ρ and ρ' and x_2 the

upper limit of the same. The values of these limits are tabulated below for every 0.01 in c.

LIMITING VALUES OF ρ AND ρ' AS FUNCTIONS OF c.

c	Lower	Upper	c	Lower	Upper
0.00	0.0000	0.0000	0.18	—0.5348	+0.5027
0.01	—0.1404	+0.1403	0.19	0.5462	0.5104
0.02	0.1972	0.1968	0.20	0.5571	0.5175
0.03	0.2399	0.2390	0.21	0.5675	0.5239
0.04	0.2751	0.2735	0.22	0.5775	0.5297
0.05	0.3056	0.3031	0.23	0.5871	0.5348
0.06	0.3326	0.3290	0.24	0.5962	0.5394
0.07	0.3569	0.3520	0.25	0.6050	0.5435
0.08	0.3791	0.3727	0.26	0.6135	0.5470
0.09	0.3996	0.3915	0.27	0.6216	0.5500
0.10	0.4186	0.4086	0.28	0.6295	0.5526
0.11	0.4363	0.4242	0.29	0.6370	0.5546
0.12	0.4529	0.4385	0.30	0.6443	0.5562
0.13	0.4684	0.4516	0.31	0.6513	0.5574
0.14	0.4832	0.4637	0.32	0.6580	0.5581
0.15	0.4971	0.4747	0.33	0.6645	0.5585
0.16	0.5103	0.4849			
0.17	—0.5229	+0.4942	$\frac{1}{3}$	$-\frac{2}{3}$	$+\frac{2}{3}(4-\sqrt{10})$

To illustrate the matter let us take a particular case, the radii being represented by the formulas

$$r = \frac{\mu p}{\mu + \rho}, \qquad r' = \frac{\mu p'}{\mu + \rho'}$$

suppose that the values of the four constants involved are

$$p = 1, \quad p' = 2, \quad \mu = 2, \quad c = 0.2$$

The limiting values of r are

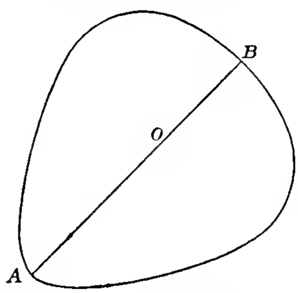
$$r = \frac{2}{2 + 0.5175} = 0.794, \qquad r = \frac{2}{2 - 0.5571} = 1.386$$

and those of r' double these

$$r' = 1.589, \qquad r' = 2.772$$

Here the upper limit of r is less than the lower limit of r'; hence the orbits have no point in common, and do not interfere with each other. We shall call this the quality of noninterference. It will be seen at once that the values of p, p', μ, c can be varied through a considerable range without the failure of this quality. But here is evidently an opportunity to apply Lindstedt's series in integrating the differential equations determining ρ and ρ'. Thus the applicability of these series does not imply dynamical instability in the motions which can take place upon the two orbits.

The form of the cubic circumscribing the values of ρ and ρ' for the special case noted above, where $C^2 = 0.256$, is shown in the adjacent figure



(the scale is $\frac{4}{3}$ inches to the unit). O is the origin, and the right line AOB, passing through that point and bisecting the angle between the axes of coordinates, is the path of the representative point P for the case where $\rho' = \rho$, and the solution of the differential equations is a periodic one. It may be noted that this point in general never attains the closed branch of the cubic curve, as this cannot happen unless the values $\frac{d\rho}{dv} = 0$, $\frac{d\rho'}{dv} = 0$ are simultaneous.*

It is interesting to know whether the orbits are periplegmatic in the sense of Gyldén. With his notation we should have

$$\frac{d^2 \frac{\rho}{\mu}}{dv^2} + \frac{\rho}{\mu} = -\frac{1}{\mu}(\rho\rho' + \tfrac{1}{2}\rho'^2) = P$$

$$\frac{d^2 \frac{\rho'}{\mu}}{dv^2} + \frac{\rho'}{\mu} = -\frac{1}{\mu}(\rho\rho' + \tfrac{1}{2}\rho^2) = P'$$

For the quality in question P and P' must not fall below -1. As the greatest value of $\rho\rho' + \frac{1}{2}\rho'^2$ or $\rho\rho' + \frac{1}{2}\rho^2$ is $\frac{2}{3}$, if μ exceeds this, the orbits will be periplegmatic.

The treatment of the differential equations is, in general, easier if we make the linear transformation:

$$u = \tfrac{1}{2}(\rho + \rho'), \qquad s = \tfrac{1}{2}(\rho - \rho')$$

They then take the form

$$\frac{d^2u}{dz^2} + u + \tfrac{3}{2}u^2 - \tfrac{1}{2}s^2 = 0$$

$$\frac{d^2s}{dv^2} + s - us \qquad\qquad = 0$$

The radii of the orbits are represented by the equations

$$r = \frac{\mu p}{\mu + u + s}, \qquad r' = \frac{\mu p'}{\mu + u - s}$$

The integral, in terms of the new variables, is

$$\frac{du^2}{dv^2} + \frac{ds^2}{dv^2} + u^2 + s^2 + u^3 - us^2 = \tfrac{1}{2}C^2$$

---

* The infinite branch is not given in the diagram, as it is useless for our purposes. The curve is species 67, and is shown in Fig. 71 of Newton's *Enumeratio linearum tertii ordinis*, printed at the end of Dr. Samuel Clarke's Latin translation of Newton's Optics.

The adoption of the solution $s = 0$, satisfying the equations, leads directly to a periodic solution of them. In this case we have the single differential equation

$$\frac{du^2}{dv^2} = \tfrac{1}{2} C^2 - u^2 - u^3$$

to be integrated. Make the substitution

$$u = g + g' \cos 2\psi$$

g and g' being constants; then

$$4 g'^2 (1 - \cos^2 2\psi) \frac{d\psi^2}{dv^2} = \tfrac{1}{2} C^2 - (g + g' \cos 2\psi)^2 - (g + g' \cos 2\psi)^3$$

Let g and g' be so chosen that the right member of this, equated to zero, may have the two roots $\cos 2\psi = \pm 1$. Then g and g' are determined by the equations

$$\tfrac{1}{2} C^2 - (g + g')^2 - (g + g')^3 = 0$$
$$\tfrac{1}{2} C^2 - (g - g')^2 - (g - g')^3 = 0$$

or by

$$\tfrac{1}{2} C^2 - g^2 - g'^2 - g^3 - 3 g g'^2 = 0$$
$$2g + 3g^2 + g'^2 = 0$$

If we divide both members of the last differential equation by $1 - \cos^2 2\psi$ the result is

$$4 g'^2 \frac{d\psi^2}{dv^2} = \tfrac{1}{2} C^2 - g^2 - g^3 + g'^3 \cos 2\psi$$

But, eliminating C^2, this becomes

$$4 \frac{d\psi^2}{dv^2} = 1 + 3g + g' \cos 2\psi = 1 + 3g + g' - 2g' \sin^2 \psi$$

If we put $\sqrt{3} g' = \sin \theta$, then will $3g = \cos\theta - 1$, and

$$\frac{d\psi^2}{dv^2} = \frac{1}{2\sqrt{3}} [\sin(\theta + 60°) - \sin\theta \sin^2\psi]$$

If next

$$k^2 = \frac{\sin\theta}{\sin(\theta + 60°)}$$

we have

$$\frac{d\psi^2}{dv^2} = \tfrac{1}{4} \frac{1}{\sqrt{1 - k^2 + k^4}} (1 - k^2 \sin^2 \psi)$$

and to u may be given the form

$$u = \tfrac{1}{3} \left(\frac{1 + k^2}{\sqrt{1 - k^2 + k^4}} - 1 \right) - \frac{k^2}{\sqrt{1 - k^2 + k^4}} \sin^2 \psi$$

It will be seen that k takes the place of the arbitrary constant C^2 which is

attached to the integral. In the Gudermannian notation for elliptic functions, putting m for $\dfrac{1}{2\sqrt[4]{1-k^2+k^4}}$, and c being an arbitrary constant,

$$\sin\psi = sn\,(mv + c) = sn\,x$$

and

$$u = \frac{1}{\sqrt{1-k^2+k^4}}\left(\frac{1+k^2}{3} - \tfrac{1}{3}\sqrt{1-k^2+k^4} - k^2\,sn^2 x\right)$$

The value of C is of interest; we have

$$\begin{aligned}
\tfrac{1}{2}C^2 &= g^2 + g^3 + g'^2\,(1 + 3g) \\
&= \tfrac{1}{27}(3 - 6\cos\theta + 3\cos^2\theta - 1 + 3\cos\theta - 3\cos^2\theta + \cos^3\theta) + \tfrac{1}{3}(1 - \cos^2\theta)\cos\theta \\
&= \tfrac{2}{27}(1 + 3\cos\theta - 4\cos^3\theta) = \tfrac{2}{27}(1 - \cos 3\theta) \\
C^2 &= \tfrac{8}{27}\sin^2\tfrac{3}{2}\theta
\end{aligned}$$

If C^2 is wanted in terms of k we have

$$C^2 = \tfrac{4}{27}\left(1 - \frac{1 - \frac{3}{2}k^2 - \frac{3}{2}k^4 + k^6}{(1-k^2+k^4)^{\frac{3}{2}}}\right)$$

The argument on which u depends is

$$\frac{\pi}{2K}\,\frac{1}{\sqrt[4]{1-k^2+k^4}}\,v + c$$

where K, as usual, denotes the period of the elliptic integral; or, it is

$$\begin{aligned}
&\frac{1}{\sqrt[4]{1-k^2+k^4}}\,\frac{1}{1 + (\frac{1}{2})^2k^2 + (\frac{1}{2}\cdot\frac{3}{4})^2k^4 + (\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6})^2k^6 + \ldots}\,v + c \\
&= (1 - \tfrac{15}{64}k^4 - \tfrac{15}{64}k^6 - \tfrac{5621}{16384}k^8 + \tfrac{393}{8192}k^{10} + \ldots)\,v + c
\end{aligned}$$

It is to be noted that the square of k is absent from the latter expression, hence this parameter must become quite a large fraction before a marked difference results in the period.

An expression in terms of the nome q may be preferred. The period has the equivalent

$$\frac{2K\sqrt{k'}}{\pi}\sqrt[4]{1 + \frac{k^4}{k'^2}}$$

where $k' = \sqrt{1-k^2}$. But the first factor has the value

$$\frac{2K\sqrt{k'}}{\pi} = 1 + 4\left[-\frac{q^2}{1+q^2} + \frac{q^6}{1+q^4} - \frac{q^{12}}{1+q^6} + \frac{q^{20}}{1+q^8} - \ldots\right]$$

and the second can be derived from

$$\frac{k}{\sqrt{k'}} = 4\sqrt{q}\,\frac{[1 + q^2 + q^6 + q^{12} + \ldots]^2}{[1 + 2q^4 + 2q^{16} + \ldots]^2 - [2q + 2q^9 + 2q^{25} + \ldots]^2}$$

The series for the period or its reciprocal in powers of q is tardily convergent, and it seems better to retain the foregoing expressions where the law of progression is obvious.

If we put $k = \sin\eta$, q may be derived by tentation from the equation

$$\frac{\sin^2 \frac{1}{2}\eta}{(1 + \sqrt{\cos\eta})^2} = \frac{q + q^9 + q^{25} + \ldots}{1 + 2q^4 + 2q^{16} + \ldots}$$

When $k = 1$, the numerator and denominator of the second member become divergent series, but the proper value of q, in this case, is unity. K may be derived from

$$\sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \ldots$$

To have u expressed as a periodic function of its argument, substitute for the transcendental function $cn^2 x$, its equivalent

$$\frac{2\pi^2}{k^2 K^2}\left[\frac{q}{1-q^2} - \frac{2q^2}{1-q^4} + \frac{3q^3}{1-q^6} - \ldots + \frac{q}{1-q^2}\cos\left(\frac{\pi}{K}x\right) + \frac{2q^2}{1-q^4}\cos\left(2\frac{\pi}{K}x\right) + \frac{3q^3}{1-q^6}\cos\left(3\frac{\pi}{K}x\right) + \ldots\right]^*$$

There is still another linear transformation of the differential equations worthy of notice. In order to remove from the potential the terms of three dimensions which are products, let us put

$$\rho = u + hu', \qquad \rho' = u' + hu$$

where h is either of the complex cube roots of unity, or such that

$$h^2 + h + 1 = 0$$

Then

$$\frac{1}{2}\frac{d\rho^2 + d\rho'^2}{dv^2} = -\frac{1}{2}h\frac{du^2 - 4\,du\,du' + du'^2}{dv^2}, \quad V = \frac{1}{2}h(u^2 - 4uu' + u'^2) + \frac{1}{3}(u^3 + u'^3)$$

Hence it is seen that the differential equations take the form

$$\left[\frac{d^2}{dv^2} + 1\right](2u' - u) = \frac{3}{2}h^2u^2, \qquad \left[\frac{d^2}{dv^2} + 1\right](2u - u') = \frac{3}{2}h^2u'^2$$

or, if, as a symbol of operation, we put

$$\mathrm{D} = \frac{2}{3}h\left[\frac{d^2}{dv^2} + 1\right]$$

the simple form

$$\mathrm{D}[2u' - u] = u^2, \qquad \mathrm{D}[2u - u'] = u'^2$$

Thus, if from the double of one of the dependent variables we subtract the other, and on the remainder operate with D, the result is the same as if we squared the latter variable. Simple as are these equations, no completely satisfactory general expressions of the unknowns for an infinite range in longitude have been found.

---

* For these formulas in elliptic functions consult Broch, *Traité Elémentaire des Fonctions Elliptiques*, p. 207, Eq. (124); pp. 210-211, Eqs. (5) and (6); p. 210, Eq. (3); p. 172, Eq. (17).

In applying Lindstedt's series to the integration of these equations we should assume

$$u = \Sigma_{i,i'} A_{i,i'} \varepsilon^{(ik + i'k')v}, \qquad u' = \Sigma_{i,i'} A'_{i,i'} \varepsilon^{(ik + i'k')v}$$

where the A and A' are constants as well as k and k', and i and i' are integers reaching from $-\infty$ to $+\infty$. The substitution of these values in the equations shows that A, A', k, k' must satisfy, for each combination i, i' the conditions

$$[(ik + i'k')^2 + 1](2A'_{i,i'} - A_{i,i'}) = \tfrac{3}{2} h^2 \Sigma_{j,j'} A_{i-j,i'-j'} A_{j,j'}$$
$$[(ik + i'k')^2 + 1](2A_{i,i'} - A'_{i,i'}) = \tfrac{3}{2} h^2 \Sigma_{j,j'} A'_{i-j,i'-j'} A_{j,j'}$$

These equations should suffice for determining the A and A' as well as k and k' in terms of the four arbitrary constants introduced by the integration. But two of these constants are involved in the expressions only through addition to the two elementary arguments kv and $k'v$; thus the mentioned quantities involve only two arbitrary parameters. Since u and u' as periodic functions of v involve only cosines, we have the conditions

$$A_{-i,-i'} = A_{i,i'}, \qquad A'_{-i,-i'} = A'_{i,i'}$$

If, besides k and k', either of the two groups of coefficients A and A' is known, the other is deducible.

The differential equations may be reduced to a system in which all are of the first order; employing for this purpose those in terms of the variables u and s, the closed curve enveloping the area in which the differential coefficients are real has the equation

$$s^2 = \frac{\tfrac{1}{2} C^2 - u^2 - u^3}{1 - u}$$

The maximum value of $|u|$ is then $\frac{2}{3}$, and the maximum of $|s|$ corresponds to the value of u given by the smaller positive root of

$$u^3 - u^2 - u + \tfrac{1}{2} C^2 = 0$$

Thus, if we put

$$\tfrac{1}{2} C^2 = c'(1 + c' - c'^2)$$

it will be found that

$$|s| = \sqrt{2c' + 3c'^2}$$

And if $C^2 = \frac{8}{27}$ we shall have approximately $|s| = 0.392$

In place of the two variables u and s we employ the four u, y, e', l' such that

$$\frac{du}{dv} = -y, \qquad s = e' \cos l', \qquad \frac{ds}{dv} = -e' \sin l'$$

The integral equation will then be expressed in the form

$$y^2 + u^2 + u^3 + e'^2(1 - u\cos^2 l') = \tfrac{1}{2} C^2$$

which gives

$$\tfrac{1}{2} e'^2 = \tfrac{1}{2} \frac{\tfrac{1}{2} C^2 - y^2 - u^2 - u^3}{1 - u\cos^2 l'} = W$$

And the differential equations are

$$\frac{du}{dv} = -y \qquad \frac{dy}{dv} = u + \tfrac{3}{2}u^2 - \tfrac{1}{2}e'^2 \cos^2 l'$$

$$\frac{d\,(e' \cos l')}{dv} = -e' \sin l' \qquad \frac{d\,(e' \sin l')}{dv} = (1-u)\,e' \cos l'$$

The third and fourth are equivalent to

$$\frac{d \,.\log e'}{dv} = -\tfrac{1}{2} u \sin 2\,l', \qquad \frac{dl'}{dv} = 1 - u\cos^2 l'$$

From the latter it is plain that l' and v advance together; thus l' will serve equally well as v for the independent variable. By division and elimination of e'^2, the first and second equations become

$$\frac{du}{dl'} = -\frac{y}{1-u\cos^2 l'} \qquad \frac{dy}{dl'} = \frac{u+\frac{3}{2}u^2}{1-u\cos^2 l'} - \tfrac{1}{2}\frac{\frac{1}{2}C^2 - u^2 - u^3}{(1-u\cos^2 l')^2}\cos^2 l'$$

or, as they may be written

$$\frac{du}{dl'} = \frac{\partial W}{\partial y}, \qquad \frac{dy}{dl'} = -\frac{\partial W}{\partial u}$$

These equations may be still further varied by putting

$$u = e\cos l, \qquad y = e\sin l$$

Then if

$$W = \tfrac{1}{2}\frac{\frac{1}{2}C^2 - e^2 - e^3\cos^3 l}{1 - e\cos l\cos^2 l'}$$

we have

$$\frac{d\,.\frac{1}{2}e^2}{dl'} = \frac{\partial W}{\partial l}, \qquad \frac{dl}{dl'} = -\frac{\partial W}{\partial\,.\frac{1}{2}e^2}$$

After u and y or e and l have been determined in terms of l' through the integration of these equations, v can be found by a quadrature from

$$\frac{dv}{dl'} = \frac{1}{1-u\cos^2 l'}$$

and thence, by inversion, l' in terms of v, and thus the problem completely solved.

W can be developed in an infinite series of the form

$$\Sigma_{i,i'}\, A_{i,i'} \cos\,[il + 2\,i'l']$$

For putting

$$\beta = \frac{2 - u - 2\sqrt{1-u}}{u}$$

we have

$$W = \frac{\frac{1}{2}C^2 - y^2 - u^2 - u^3}{\sqrt{1-u}}\,[\tfrac{1}{2} + \beta\cos 2\,l' + \beta^2\cos 4\,l' + \beta^3\cos 6\,l' + \ldots]$$

From this u and y may be eliminated by substituting their values in terms of e and l.

The integrals of the two differential equations may then be approximated to by a series of Delaunay transformations, as the function W is quite similar to Delaunay's R in the lunar theory. The only noteworthy differences being that here there are only two unknowns in place of Delaunay's six, and only one constant parameter C instead of Delaunay's three n', e', a'.

We may give here Delaunay's rule for making a transformation. If we have integrated the differential equations (L is put for $\frac{1}{2}e^2$)

$$\frac{dL}{dl'} = \frac{\partial W}{\partial l}, \quad \frac{dl}{dl'} = -\frac{\partial W}{\partial L}$$

when W is limited to the terms involving one argument $il + i'l'$ (the constant term is included) and have found in this manner (θ designating the argument)

$$\theta = \theta_0 (l' + c) + \theta_1 \sin \theta_0 (l' + c) + \theta_2 \sin 2\theta_0 (l' + c) + \theta_3 \sin 3\theta_0 (l' + c) + \ldots$$
$$L = L_0 + L_1 \cos \theta_0 (l' + c) + L_2 \cos 2\theta_0 (l' + c) + L_3 \cos 3\theta_0 (l' + c) + \ldots$$

c being a constant, and θ_0, θ_1, θ_2, ... L_0, L_1, L_2, ... being known functions of another constant (e_0 for instance), we can replace

$$L \text{ by } L_0 + L_1 \cos (il + i'l') + L_2 \cos 2(il + i'l') + \ldots$$
$$l \text{ by } l + \frac{\theta_1}{i} \sin (il + i'l') + \frac{\theta_2}{i} \sin 2(il + i'l') + \ldots$$

and we shall have, for determining the new variables e_0, l, precisely the same equations

$$\frac{dL}{dl'} = \frac{\partial W}{\partial l}, \quad \frac{dl}{dl'} = -\frac{\partial W}{\partial L}$$

provided, first, that we put for W the function obtained by making the preceding substitutions in the old function W (complete) augmented by the quantity

$$-\frac{i'}{i}(L - L_0) + \frac{i'}{2i}(\theta_1 L_1 + 2\theta_2 L_2 + 3\theta_3 L_3 + \ldots)$$

second, that we regard the new variable L as connected with e_0 by the relation

$$L = L_0 + \tfrac{1}{2}(\theta_1 L_1 + 2\theta_2 L_2 + 3\theta_3 L_3 + \ldots)$$

MEMOIR No. 75.

The Theorems of Lagrange and Poisson on the Invariability of the Greater Axes in an Ordinary Planetary System.

(Astronomical Journal, Vol. XXIV, pp. 27–29, 1904.)

The remarks of this article follow a line very similar to that of the article in No. 527 of this Journal, and, to avoid the tedium of restating the explanation of the fundamental notation employed, I take the liberty of referring the reader to that place. However, we cannot here use the three theorems there stated, as it is proposed to take account of terms of three dimensions with respect to planetary masses. But we employ the notion there expounded of hypothetical planets, arranging the planets in such an order that the one, for which $i=1$, is that whose greater axis we especially wish to consider; the first actual planet and its hypothetical are then identical. We adopt the distinction there defined between terms periodic and terms secular.

Let there be ν planets in the system. The canonical elements L_i, G_i, H_i, l_i, g_i, h_i and the function F have the significations of the mentioned article, and the differential equations to be satisfied are the same. Let P denote the general periodic argument, and S the general secular argument. Let K denote the general (that is, without any hint of individuality) coefficient usually multiplying the cosine or sine of any argument. We write above each K the number indicating its order of magnitude with respect to planetary masses; thus $\overset{2}{K}$ signifies a coefficient factored in the lowest dimension by squares and products of planetary masses. K, in general, is a function of all the linear elements. Σ will be used to indicate a sum of terms in number either finite or infinite.

The first term of

$$F \text{ or } m_0 \sum_{i=1}^{i=\nu} \frac{m_i}{2a_i}$$

is of the dimension 1 with respect to planetary masses; we denote it by $\overset{1}{F}$. Developing F in a periodic series, we can write it as the sum of three terms, as follows:

$$F = \overset{1}{F} + \Sigma . \overset{2}{K} \cos S + \Sigma . \overset{2}{K} \cos P$$

Here it must be understood that S can receive the value 0, but P not. Also $\overset{2}{K}$ does not mean that the order of the coefficient is 2 in every case, but only that it is never less than 2.

Let us now assume a pure function of the variables $L_1, L_2, \ldots, L_\nu$, which we may write

$$f(L_1, L_2, \ \ldots \ L_\nu)$$

or of the variables $a_1, a_2, \ldots, a_\nu$, to be written

$$f(a_1, a_2, \ \ldots \ a_\nu)$$

f is to be finite, continuous, and of the zero order with respect to planetary masses.

We now propose to apply the principle of the Delaunay transformation to the establishment of the theorems with which we are engaged. Delaunay makes his transformations in the three polar coordinates of the moon; we have to make ours in the one function $f(L_1, L_2, \ldots, L_\nu)$. From the infinite number of periodic arguments P we select one θ, in which the positive or negative integers multiplying the angular variables are prime to each other, and take the part of F, which may be regarded as dependent on the sole argument θ. We call this $[F]$, and write, similarly to Delaunay,

$$[F] = -B - A_1 \cos\theta - A_2 \cos 2\theta - A_3 \cos 3\theta - \ldots$$

$-B$ is the absolute term of F, so that if $\overset{2}{K}_0$ denotes the coefficient of the second part of F when $S = 0$, we have

$$-B = \overset{1}{F} + \overset{2}{K}_0$$

Let us now make the Delaunay transformation necessary to remove from F the terms factored by $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$, ... The formulas for this purpose are, if L denotes any linear variable,

$$\text{Replace } \overset{1}{L} \text{ by } \overset{1}{L} + \overset{3}{M}_0 + \overset{2}{M}_1 \cos\theta + \overset{2}{M}_2 \cos 2\theta + \overset{2}{M}_3 \cos 3\theta + \ldots$$

and, if l denotes any angular variable,

$$\text{Replace } l \text{ by } l + \overset{1}{N}_1 \sin\theta + \overset{1}{N}_2 \sin 2\theta + \overset{1}{N}_3 \sin 3\theta + \ldots$$

where the M and N are functions of the new set of linear variables. By the addition of $\overset{3}{M}_0$ to the first formula we secure the advantage that the new linear variables are the conjugates of the new angular variables. $\overset{3}{M}_0$ is necessarily two orders higher with respect to planetary masses than the term which precedes it. The fact of the rest of the M and N being of the orders indicated above them is due to the circumstance that the motion of the argument θ is of the zero order.

We have now to inquire what changes, if any, are produced in the qualities of the coefficients of the three terms of F by this transformation. It is evident that $[F]$ is reduced by it to a function of the linear variables only. Hence, when the substitution is made in the term $\overset{1}{F}$, the new terms, which arise and are of the second order, precisely cancel the old terms $-A_1 \cos\theta - A_2 \cos 2\theta - \ldots\ldots$, and the remainder are of the form

$$\Sigma\,.\,\overset{3}{K}\cos S + \Sigma\,.\,\overset{3}{K}\cos P$$

Moreover, when the substitution is made in the second and third terms of F, the new terms arising are also of the same form. Hence the new form of F can be written

$$F = \overset{1}{F} + \Sigma\,.\,\overset{2}{K}\cos S + \Sigma\,.\,\overset{2}{K}\cos P$$

where $\overset{1}{F}$ has precisely the same expression as before. Hence the quality of F is unchanged by the execution of the transformation. We need only bear in mind that the coefficients of $\cos\theta$, $\cos 2\theta$, $\ldots\ldots$, are now not of the form $\overset{2}{K}$, but of the form $\overset{3}{K}$.

We may suppose a second Delaunay transformation to be made with the object of removing the terms of F having as periodic arguments the multiples of another θ of the same quality as before. The result will be that after the transformation F will have the same quality as before. Thus it is possible to conceive that an infinity of Delaunay transformations may be made in such a way that the third term of F wholly disappears, and F takes the form

$$F = \overset{1}{F} + \Sigma\,.\,\overset{2}{K}\cos S$$

Let us next inquire what happens when these transformations are made in the expression.

$$f(L_1,\ L_2,\ \ldots\ldots,\ L_\nu)$$

It is quite plain that, after the first transformation depending on the periodic argument θ, we shall have the equation

$$f(L_1,\ L_2,\ \ldots\ldots,\ L_\nu) = f(L_1,\ L_2,\ \ldots\ldots,\ L_\nu) + \Sigma\,.\,\overset{1}{K}\cos S + \Sigma\,.\,\overset{1}{K}\cos P$$

where it must be understood that the L appearing under the functional sign f in the left member have their original signification, but in the right member, their signification as modified by the transformation. That the periodic portion should have coefficients of the form $\overset{1}{K}$ is so

obvious that it needs no formal demonstration; but that the secular portion has coefficients of the form $\overset{2}{K}$ results from the fact that its terms can arise only from the multiplication of two periodic terms $\overset{1}{K}\cos P$ and $\overset{1}{K}\cos P'$, where $P + P'$ or $P - P'$ is an S.

Now make the second Delaunay transformation; it is obvious that we have still the same equation

$$f\,(L_1,\ L_2,\ \ldots.,\ L_\nu) = f\,(L_1,\ L_2,\ \ldots.,\ L_\nu) + \Sigma\,.\overset{2}{K}\cos S + \Sigma\,.\overset{1}{K}\cos P$$

where it is necessary to note only that the L appearing in the second member have the signification as twice modified. Next, suppose that the infinite number of Delaunay transformations conceived to be made for the purpose of removing all periodic terms from the periodic development of F, have also been made here. The result will still be the equation

$$f\,(L_1,\ L_2,\ \ldots.,\ L_\nu) = f\,(L_1,\ L_2,\ \ldots.,\ L_\nu) + \Sigma\,.\overset{2}{K}\cos S + \Sigma\,.\overset{1}{K}\cos P$$

where the L in the second member have the signification as last modified.

Consider now the variability of this last group of the L. Since the angular variables $l_1,\ l_2,\ \ldots.,\ l_\nu$ conjugate to them have altogether disappeared from the last modified expression for F, we have generally

$$\frac{dL_i}{dt} = \frac{\partial F}{\partial l_i} = 0$$

Consequently the last group of the modified L forms a series of constants. Thus, if we please, we may write the foregoing equation

$$f\,(L_1,\ L_2,\ \ldots.,\ L_\nu) = \text{a constant} + \Sigma\,.\overset{2}{K}\cos S + \Sigma\,.\overset{1}{K}\cos P$$

But, for our purpose, it is necessary that the right member of this should appear as an explicit function of the time. Hence a new class of Delaunay transformations must be made, having for object the removal from F of all the terms (the absolute excepted) constituting the second portion $\Sigma\,.\,\overset{2}{K}\cos S$. These transformations would then turn upon the secular arguments of the group S, and, after the requisite infinity of them had been performed, F would be reduced to the absolute term which is a function of the linear variables; thus

$$F = \overset{1}{F} + \overset{2}{K}_0$$

All the linear variables are now constant, for, in addition to what has been stated in reference to the L, we have generally

$$\frac{dG_i}{dt} = \frac{\partial F}{\partial g_i} = 0, \quad \frac{dH_i}{dt} = \frac{\partial F}{\partial h_i} = 0$$

Suppose that all these transformations are made in the term $\Sigma . \overset{1}{K} \cos P$ of $f(L_1, L_2, \ldots, L_\nu)$. It is evident, from the general principles underlying the representation of the integrals of our differential equations by Lindstedt's series, that this term will undergo a change expressed by the apparently tautological equation

$$\Sigma . \overset{1}{K} \cos P = \Sigma . \overset{1}{K} \cos P$$

in the second member of which, however, $\overset{1}{K}$ is absolutely constant, and P is a linear function of the time expressed by $\theta_0(t + c)$, where θ_0 and c are constants, the first being of the zero order with respect to planetary masses. Thus, throughout this second class of transformations, no terms cross over from the portion $\Sigma . \overset{1}{K} \cos P$ to the portion $\Sigma . \overset{2}{K} \cos S$. Then, if we are concerned only about the secular inequalities of $f(L_1, L_2, \ldots, L_\nu)$, we can omit its last term and write

$$f(L_1, L_2, \ldots, L_\nu) = \text{a constant} + \Sigma . \overset{2}{K} \cos S$$

in which it is understood that $\Sigma . \overset{2}{K} \cos S$ has not undergone any of the latter class of transformations.

At this stage we give up the process of approximating to the integrals of our differential equations through Delaunay transformations, and adopt the process of elaborating them in positive integral powers of the time, making use of the generalized theorem of Maclaurin. The equations in terms of the variables last used in the earlier class of transformations have expressions of which the type is

$$\frac{dG_i}{dt} = \frac{\partial F}{\partial g_i}, \qquad \frac{dH_i}{dt} = \frac{\partial F}{\partial h_i}$$
$$\frac{dg_i}{dt} = -\frac{\partial F}{\partial G_i}, \qquad \frac{dh_i}{dt} = -\frac{\partial F}{\partial H_i}$$

where F has the expression

$$F = \overset{1}{F} + \Sigma . \overset{2}{K} \cos S$$

The integrals in series of powers of the time are

$$G_i = \overset{1}{K} + \overset{2}{K}t + \overset{3}{K}t^2 + \ldots, \qquad H_i = \overset{1}{K} + \overset{2}{K}t + \overset{2}{K}t^2 + \ldots$$
$$g = \overset{0}{K} + \overset{1}{K}t + \overset{2}{K}t^2 + \ldots, \qquad h_i = \overset{0}{K} + \overset{1}{K}t + \overset{2}{K}t^2 + \ldots$$

the K being all constant and of the order with respect to planetary masses indicated above them. If we substitute these values in the portion $\Sigma . \overset{2}{K} \cos S$

of the function $f(L_1, L_2, \ldots, L_\nu)$ there arises a constant term equivalent to the value of $\Sigma \,.\, \overset{2}{K} \cos S$ at the origin of time, which coalesces with the preceding constant. This is followed by terms involving t, t^2, t^3, etc. Then it is almost immediately apparent that

$$f(L_1, L_2, \ldots, L_\nu) = \overset{0}{K} + \overset{3}{K}t + \overset{4}{K}t^2 + \overset{5}{K}t^3 + \ldots.$$

It is evident that this relation may also be written

$$f(a_1, a_2, \ldots, a_\nu) = \overset{0}{K} + \overset{3}{K}t + \overset{4}{K}t^2 + \overset{5}{K}t^3 + \ldots.$$

We may therefore state the theorem of Poisson in a more general form than has been customary, as follows:

The secular variation of any finite and continuous function of the zero order with respect to planetary masses of the instantaneous greater axes of the orbits described by the hypothetical planets of a planetary system, when developed in powers of the time, is, at least, of three dimensions in reference to the same masses.

The qualification "at least" is necessary, for the function f may be such that all terms of the third order in the coefficient of t identically vanish, and then $\overset{3}{K}$ must be written $\overset{4}{K}$. In the second place, since the function f probably has an infinite number of maxima and minima, if it so happen that the origin whence t is counted coincides with one of these, the coefficient of t must vanish, not identically, but on account of the special values received by the parameters at that epoch.

It will be perceived that the method here followed for the demonstration of the theorem has marked advantages over those employed heretofore. The truth of the theorem is now so obvious that a formal proof seems scarcely necessary. The Delaunay transformation is to be credited for this advance. An objection may be raised against the method that it is valid only in the case where F never *potentially* becomes infinite.* But, in the opposite case, there seems no valid distinction between secular and periodic inequalities, and Lindstedt's series ought to be rejected as being a possible mode of expressing the coordinates.

* See Transactions of the American Mathematical Society, Vol. I, p. 210. Memoir No. 66.

MEMOIR No. 76.

Comparison of the New Tables of Jupiter and Saturn with the Greenwich Observations of 1889-1900.

(Astronomical Journal, Vol. XXIV, pp. 60, 61, 1904.)

The New Tables of Jupiter and Saturn are founded upon a discussion of observations which ended with the year 1888. They could not, however, be generally used for the calculation of the Ephemerides till 1901. This leaves a gap of twelve years of observations uncompared with the new theories. It seemed that a useful service would be done for astronomy by supplying the lacking comparisons. In this work I have confined myself to the Greenwich observations as the published positions made at other places are desultory in character. Desiring to form normals as near the time of opposition as possible, I have not included observations when the time of culmination was earlier than 10^h. During some portion of the summer of 1891 the instrument appears to have been dismounted; thus only a weak normal for Jupiter could be formed for this year.

It seemed desirable to reduce the normal positions to the standard of Professor Newcomb's *Catalogue of Fundamental Stars* (*Astronomical Papers*, Vol. VIII, Part II). With regard to the right-ascensions, in this memoir (p. 228), $+ 0^s.049$ is given as the correction for the Greenwich Catalogue of 1890. But, being apprehensive that this quantity was not applicable to the whole period 1889–1900, I have determined it at nine different epochs, getting results which range from $+ 0^s.041$ to $+ 0^s.059$. The correction seems to have augmented from the beginning to the end of the period. The systematic correction for the declinations is taken from the table on p. 236 of the same paper, given as applicable to the Greenwich Catalogue of 1890. A comparison of the declinations in the Greenwich volume for 1900 with those of the *Fundamental Catalogue* showed that these systematic corrections have persisted without marked change till the later date.

The columns in the following exhibit scarcely need explanation. In the second column the dates of the first and last observations are given in order that the observations used may be readily identified.

JUPITER.

Date Greenw. M.N.	Interval of Observation	No. of Obsns.	Mean Diff. Syst. from N.A. Corr.		Mean Diff. Syst. from N.A. Corr.		Positions Resulting from Obsn. α	δ
			s	s	″	″	h m s	° ′ ″
1889 June 27	May 21-Aug. 3	20-20	−.032	+.041	+0.01	+.75	18 13 15.289	−23 15 20.84
1890 Aug. 8	July 3-Sept. 3	25-26	+.095	+.046	−0.19	+.69	20 34 41.141	−19 31 17.50
1891 Aug. 22	July 24-Oct. 17	7-6	+.233	+.046	+0.28	+.48	23 6 37.239	− 7 17 11.14
1892 Oct. 13	Sept 8-Nov. 18	24-24	+.274	+.046	+1.21	+.35	1 15 31.330	+ 6 15 5.86
1893 Nov. 19	Oct. 26-Dec. 23	26-27	+.327	+.056	+0.71	+.28	3 36 45.083	+18 14 33.19
1894 Dec. 31	Nov. 15-Jan. 30	29-29	+.216	+.056	+0.58	+.25	6 0 34.102	+23 14 49.63
1896 Feb. 15	Jan. 14-Mar. 4	15-16	+.116	+.050	−0.06	+.27	8 15 5.466	+20 33 23.81
1897 Mar. 10	Jan. 22-Apr. 8	21-21	+.020	+.050	+1.27	+.33	10 23 8.330	+11 31 28.00
1898 Apr. 8	Feb. 26-May 7	27-30	+.040	+.059	+1.10	+.42	12 16 2.969	− 0 0 13.18
1899 May 9	Mar. 29-June 2	31-34	+.089	+.057	+0.93	+.53	14 7 31.066	−11 23 24.94
1900 June 1	Apr. 17-July 6	26-26	+.170	+.054	+0.01	+.70	16 14 36.364	−20 20 49.79

SATURN.

			s	s	″	″	h m s	° ′ ″
1889 Feb. 16	Jan. 8-Mar. 15	19-19	−.138	+.046	+0.21	+.29	9 15 23.788	+17 8 56.20
1890 Mar. 3	Jan. 23-Apr. 5	22-23	−.108	+.046	+0.59	+.32	10 8 59.738	+13 15 37.31
1891 Mar. 17	Feb. 12-Apr. 18	20-20	−.063	+.046	+0.55	+.35	10 59 53.383	+ 8 46 31.80
1892 Apr. 6	Mar. 8-Apr. 30	26-27	−.112	+.046	+1.08	+.39	11 46 24.704	+ 4 11 51.87
1893 Apr. 11	Mar. 4-May 13	41-38	−.230	+.056	+1.97	+.42	12 35 27.096	− 0 52 56.81
1894 Apr. 20	Mar. 15-May 24	24-26	−.281	+.056	+2.18	+.46	13 22 23.815	− 5 41 19.36
1895 Apr. 30	Mar. 13-June 5	27-27	−.314	+.048	+2.37	+.50	14 8 43.764	−10 7 23.03
1896 May 8	Mar. 28-June 15	37-37	−.385	+.050	+2.36	+.58	14 55 36.495	−14 4 53.06
1897 May 16	Apr. 22-June 21	27-27	−.460	+.050	+2.19	+.65	15 43 28.440	−17 24 52.96
1898 June 12	May 12-July 8	16-16	−.418	+.059	+1.32	+.70	16 26 21.771	−19 47 6.38
1899 June 23	May 15-July 20	30-30	−.370	+.057	+1.38	+.72	17 14 55.847	−21 34 32.40
1900 July 8	May 26-Aug. 4	28-28	−.304	+.054	+0.51	+.74	18 2 56.220	−22 28 34.65

As the next step the heliocentric positions of the planets for the dates of their normals were obtained from the New Tables, and the positions of the Sun for the same dates from Professor Newcomb's Tables, neglecting, however, the small terms of nutation. Thus were obtained the apparent geocentric positions, which, with the residuals of the observations are noted below. The latter, although of a systematic character in the case of Saturn, are small enough to render probable the conclusion that the New Tables will represent well the future motion of the planets.

JUPITER.

	α	δ	Obs.−Cal. Δα	Δδ
	h m s	° ′ ″	s	″
1889 June 27	18 13 15.331	—23 15 21.44	—0.042	+0.60
1890 Aug. 8	20 34 41.165	—19 31 17.32	—0.024	—0.18
1891 Aug. 22	23 6 37.219	— 7 17 10.52	+0.020	—0.62
1892 Oct. 13	1 15 31.333	+ 6 15 6.35	—0.003	—0.49
1893 Nov. 19	3 36 45.028	+18 14 33.32	+0.055	—0.13
1894 Dec. 31	6 0 34.087	+23 14 49.49	+0.015	+0.14
1896 Feb. 15	8 15 5.447	+20 33 24.07	+0.019	—0.26
1897 Mar. 10	10 23 8.369	+11 31 27.14	—0.039	+0.86
1898 Apr. 8	12 16 2.941	— 0 0 13.84	+0.028	+0.66
1899 May 9	14 7 31.041	—11 23 25.77	+0.025	+0.83
1900 June 1	16 14 36.344	—20 20 50.69	+0.020	+0.90

• Saturn.

	α			δ			Os.—Cal. Δα	Δδ
	h	m	s	°	′	″	s	″
1889 Feb. 16	9	15	23.797	+17	8	56.55	—0.009	—0.35
1890 Mar. 3	10	8	59.687	+13	15	37.77	+0.051	—0.46
1891 Mar. 17	10	59	53.369	+ 8	46	31.74	+0.014	+0.06
1892 Apr. 6	11	46	24.667	+ 4	11	51.59	+0.037	+0.28
1893 Apr. 11	12	35	27.108	— 0	52	57.24	—0.012	+0.43
1894 Apr. 20	13	22	23.830	— 5	41	19.46	—0.015	+0.10
1895 Apr. 30	14	8	43.789	—10	7	23.19	—0.025	+0.16
1896 May 8	14	55	36.558	—14	4	53.81	—0.063	+0.75
1897 May 16	15	43	28.515	—17	24	53.65	—0.075	+0.69
1898 June 12	16	26	21.838	—19	47	6.86	—0.067	+0.48
1889 June 23	17	14	55.918	—21	34	33.59	—0.071	+1.19
1900 July 8	18	2	56.227	—22	38	35.48	—0.007	+0.83

MEMOIR No. 77.

Development of Functions in Power Series from Special Values.

(Astronomical Journal, Vol. XXIV, pp. 123–128, 1904.)

It is often desirable to develop complicated functions in powers and products of small parameters, and frequently the readiest method for accomplishing this is the derivation of the coefficients from values of the function corresponding to definite values of the parameters. In case we mean to retain all the terms of the development corresponding to definite powers of each parameter, regardless of the order of smallness of the resulting terms, the course to be pursued is plain ; but when we wish to retain only terms of certain degrees of smallness, the process to be followed is not so evident. So far as I am aware there does not exist any treatment of the subject with the mentioned limitation. A general exposition of the principles involved in the method would doubtless demand a complicated notation. But the details of the process in a special case will readily suggest what ought to be done in any other. An example from astronomy will illustrate the matter.

Let W be a function of the radii of two planets. In Gyldèn's method of treating planetary motion it is well known that the latter are to be replaced by the substitutions

$$r = a\frac{1-\eta^2}{1+\eta\cos F}, \qquad r' = a'\frac{1-\eta'^2}{1+\eta'\cos F'}$$

where a and a' are constants having known or assumed values, but the other symbols denote variables, η and η' being always of the first order of smallness. For our purposes it will be more convenient to write these substitutions thus:

$$r = a\frac{1-y}{1+x}, \qquad r' = a'\frac{1-y'}{1+x'}$$

where x and x' are of the first order, and y and y' of the second. It is required to develop W in a series of powers and products of the four parameters x, x' y, y', it being granted that all terms of an order beyond the eighth may be neglected. The questions are: How many special values of W is it necessary to compute, and for what combination of values for the

parameters, and how shall the elimination be conducted if unnecessary labor is to be avoided?

First, we note the number of terms in the development. They are

1	$=$	1	term of the zero order
2	$=$	2	terms of the first "
$3 + 2.1$	$=$	5	terms of the second "
$4 + 2.2$	$=$	8	terms of the third "
$5 + 2.3 + 3.1$	$=$	14	terms of the fourth "
$6 + 2.4 + 3.2$	$=$	20	terms of the fifth "
$7 + 2.5 + 3.3 + 4.1$	$=$	30	terms of the sixth "
$8 + 2.6 + 3.4 + 4.2$	$=$	40	terms of the seventh "
$9 + 2.7 + 3.5 + 4.3 + 5.1$	$=$	55	terms of the eighth "
	Sum $=$	175	

It will be perceived that the number of terms in each order is not a continuous function of the degree of the order; this is because the parameters are not all of the same order. It appears then, that 175 special values of W, provided they correspond to combinations of values of the four parameters having a determinate quality, will enable us to discover the coefficients we are in quest of. The elimination necessary in this method is much facilitated when the several values of the same parameter form an arithmetical progression of which one term is zero; we accordingly adopt this restriction. Let us suppose that the common difference of the values of the parameters x and x' is d, and that of y and y' is d'. Then it is evident that our choice of combinations may be limited to those indicated by the following scheme, where to this is added an exemplification for $d = 0.02$ and $d' = 0.0025$.

$\frac{x}{d}$	$\frac{x'}{d}$	$\frac{y}{d'}$	$\frac{y'}{d'}$	x	x'	y	y'
—4	—4			—0.08	—0.08		
—3	—3			—0.06	—0.06		
—2	—2	—2	—2	—0.04	—0.04	—0.0050	—0.0050
—1	—1	—1	—1	—0.02	—0.02	—0.0025	—0.0025
0	0	0	0	0.00	0.00	0.0000	0.0000
1	1	1	1	0.02	0.02	0.0025	0.0025
2	2	2	2	0.04	0.04	0.0050	0.0050
3	3			0.06	0.06		
4	4			0.08	0.08		

Every one of these specified values for the parameters must be included in our combinations; but we need not employ every combination arising from the preceding scheme. For the latter are in number $= 9 \times 9 \times 5 \times 5 = 2025$,

between 11 and 12 times the number 175, which we have seen to be necessary. Also, by the limitation that no terms beyond the 8th order are to be considered, it results that certain selections of combinations do not afford independent relations between the coefficients. We must avoid such selections. This constitutes the delicate step of the problem.

A system of 175 linear equations with as many unknowns would be unmanageable; we therefore propose to break the coefficients, to be determined, into groups which can be treated separately. It is evident that, as soon as any group of coefficients is determined, it is possible to estimate the values of the terms of W which involve them, correspondent to any values we please of the four parameters. We therefore suppose that, in the treatment of any group, from the special values of W are always subtracted the correspondent special values of the terms involving the coefficients of all the preceding groups; so that the remainders constitute the special values of the function equivalent to the terms still to be determined.

We may write (using the symbol A for the general coefficient),

$$W = \Sigma A \left(\frac{x}{d}\right)^i \left(\frac{x'}{d}\right)^{i'} \left(\frac{y}{d'}\right)^j \left(\frac{y'}{d'}\right)^{j'}$$

where the exponents are integers not negative. It is preferred to determine the coefficients under this form because then the factors of the unknowns in the equations are always integers. After the A are got in this way, the coefficients, corresponding to the form

$$W = \Sigma A\, x^i x'^{i'} y^j y'^{j'}$$

are obtained simply by dividing each A by the factor $d^{i+i'} d'^{j+j'}$ which belongs to it.

The separation of the groups just mentioned is defined by the vanishing or non-vanishing of the exponents i, i', j, j'. Sub-groups may be formed upon the parity or imparity of the same exponent; by taking half the sum and difference of two equations in which one of the parameters receives in one case a positive value, and in the other the correspondent negative value, the values of the others remaining the same, it is evident the equations will be broken into two involving separate classes of coefficients. This operation repeated again with two more equations, and in reference to another parameter, will be called the disintegrating operation.

Each special combination of values for the parameters for each computed value of W will be indicated by writing the correspondent values of

$$\frac{x}{d},\quad \frac{x'}{d},\quad \frac{y}{d'},\quad \frac{y'}{d'}$$

in sequence; and we shall number these combinations in the order in which

they come into use from 1 to 175. Any four combinations to be subjected to the disintegrating operation will be bracketed. When we have occasion to note a group of equations, only the multipliers of the unknowns which are integral will be set down; the sign will be indicated only when it is negative; the absolute term and the sign of equality will be omitted.

Group I is established by the condition $i = i' = j = j' = 0$, and the correspondent coefficient of W is the value for the argument

No. of Comb.	$\frac{x}{d}$	$\frac{x'}{d}$	$\frac{y}{d'}$	$\frac{y'}{d'}$
1	0	0	0	0

Group II is defined by the condition that three out of the four exponents vanish. Four sub-groups are formed by considering which one of the four has the finite value. The combinations to be used are (with the sub-groups indicated),

No. Comb.	Argument	No. Comb.	Argument	No. Comb.	Argument	No. Comb.	Argument
2	—4 0 0 0	10	0 —4 0 0	18	0 0 —2 0	22	0 0 0 —2
3	—3 0 0 0	11	0 —3 0 0	19	0 0 —1 0	23	0 0 0 —1
4	—2 0 0 0	12	0 —2 0 0	1	0 0 0 0	1	0 0 0 0
5	—1 0 0 0	13	0 —1 0 0	20	0 0 1 0	24	0 0 0 1
1	0 0 0 0	1	0 0 0 0	21	0 0 2 0	25	0 0 0 2
6	1 0 0 0	14	0 1 0 0				
7	2 0 0 0	15	0 2 0 0				
8	3 0 0 0	16	0 3 0 0				
9	4 0 0 0	17	0 4 0 0				

The 24 coefficients of the pure powers of the parameters are obtained by differencing the special values of W for the arguments in each group. In the case of the differences of odd orders we suppose half the sum of the adjacent differences to belong to the argument of the function on the same line. Let D^n denote the coefficient belonging to the n^{th} power of the parameter considered. We write in the formulas only the differences necessary in our special case; and thus:

$$1\,!\,D = \Delta - \tfrac{1}{3}\frac{\Delta^3}{2} + \tfrac{1}{3}\cdot\tfrac{2}{5}\frac{\Delta^5}{2^2} - \tfrac{1}{3}\cdot\tfrac{2}{5}\cdot\tfrac{3}{7}\frac{\Delta^7}{2^3}$$

$$2\,!\,D^2 = \Delta^2 - \frac{1}{3\cdot 2}\frac{\Delta^4}{2} + \frac{1\cdot 2}{3\cdot 5\cdot 3}\frac{\Delta^6}{2^2} - \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7\cdot 4}\frac{\Delta^8}{2^3}$$

$$3\,!\,D^3 = \Delta^3 - \tfrac{1}{4}\,\Delta^5 + \tfrac{7}{120}\,\Delta^7$$

$$4\,!\,D^4 = \Delta^4 - \tfrac{1}{6}\,\Delta^6 + \tfrac{7}{240}\,\Delta^8$$

$$5\,!\,D^5 = \Delta^5 - \tfrac{1}{3}\,\Delta^7$$

$$6\,!\,D^6 = \Delta^6 - \tfrac{1}{4}\,\Delta^8$$

$$7\,!\,D^7 = \Delta^7$$

$$8\,!\,D^8 = \Delta^8$$

Group III contains the remaining terms of W for which $j = j' = 0$. Their number is $\frac{7.8}{2} = 28$, and, involving only x and x', they are, in every case, divisible by the product xx'. After this division they will have the form

$$\begin{aligned}
&A_0 \\
&+ A_1 x + A_2 x' \\
&+ A_3 x^2 + A_4 xx' + A_5 x'^2 \\
&+ A_6 x^3 + A_7 x^2x' + A_8 xx'^2 + A_9 x'^3 \\
&+ A_{10}x^4 + A_{11}x^3x' + A_{12}x^2x'^2 + A_{13}xx'^3 + A_{14}x'^4 \\
&+ A_{15}x^5 + A_{16}x^4x' + A_{17}x^3x'^2 + A_{18}x^2x'^3 + A_{19}xx'^4 + A_{20}x'^5 \\
&+ A_{21}x^6 + A_{22}x^5x' + A_{23}x^4x'^2 + A_{24}x^3x'^3 + A_{25}x^2x'^4 + A_{26}xx'^5 + A_{27}x'^6
\end{aligned}$$

In deriving the values of the A we cannot suppose that either $x = 0$ or $x' = 0$, as then the division by xx' becomes nugatory. But by taking the six groups of four combinations each,

No. 26	—1	—1	0	0	No. 30	—2	—1	0	0	No. 34	—1	—2	0	0
27	1	1	0	0	31	2	1	0	0	35	1	2	0	0
28	—1	1	0	0	32	—2	1	0	0	36	—1	2	0	0
29	1	—1	0	0	33	2	—1	0	0	37	1	—2	0	0
No. 38	—2	—2	0	0	No. 42	—3	—1	0	0	No. 46	—1	—3	0	0
39	2	2	0	0	43	3	1	0	0	47	1	3	0	0
40	—2	2	0	0	44	—3	1	0	0	48	—1	3	0	0
41	2	—2	0	0	45	3	—1	0	0	49	1	—3	0	0

and applying to each of these the disintegrating operation, we have, in the first place, the following equations determining A_1, A_6, A_8, A_{15}, A_{17}, A_{19}:

1	1	1	1	1	1
1	4	1	16	4	1
1	1	4	1	4	16
1	4	4	16	16	16
1	9	1	81	9	1
1	1	9	1	9	81

Then by elimination in order, commencing at the left, we obtain the following groups:

1	0	5	1	0
0	1	0	1	5
1	1	5	5	5
1	0	10	1	0
0	1	0	1	10

1	0	1	5
1	0	4	5
0	5	0	0
1	0	1	10

0	3	0
5	0	0
0	0	5

By returning on this elimination we evidently can get the values of the six mentioned coefficients. Next, it is plain that the six coefficients A_2, A_7, A_9, A_{16}, A_{18}, A_{20}, and again the group of six, A_4, A_{11}, A_{13}, A_{22}, A_{24}, A_{26} are

determined by equations having the same integral coefficients as in the group just treated, the absolute terms being generally different. There still remains to be determined the group of ten coefficients, A_0, A_3, A_5, A_{10}, A_{13}, A_{14}, A_{21}, A_{23}, A_{25}, A_{27}. The six groups of four combinations afford each one equation for this purpose. Then four additional equations are necessary. We select the following four combinations for giving these equations:

No. 50	3	2	0	0
51	2	3	0	0
52	4	1	0	0
53	1	4	0	0

As we know the values of 18 coefficients of the group we can subtract from the special values of W the special values of the corresponding terms. We thus obtain 10 equations involving only the last group of 10 coefficients. They are as follows:

1	1	1	1	1	1	1	1	1	1
1	4	1	16	4	1	64	16	4	1
1	1	4	1	4	16	1	4	16	64
1	4	4	16	16	16	64	64	64	64
1	9	1	81	9	1	729	81	9	1
1	1	9	1	9	81	1	9	81	729
1	9	4	81	36	16	729	324	144	64
1	4	9	16	36	81	64	144	324	729
1	16	1	256	16	1	4096	256	16	1
1	1	16	1	16	256	1	16	256	4096

The elimination being conducted in the mentioned order, we have the groups

1	0	5	1	0	21	5	1	0
0	1	0	1	5	0	1	5	21
1	1	5	5	5	21	21	21	21
1	0	10	1	0	91	10	1	0
0	1	0	1	10	0	1	10	91
8	3	80	35	15	728	323	143	63
3	8	15	35	80	63	143	323	728
1	0	17	1	0	273	17	1	0
0	1	0	1	17	0	1	17	273

1	0	1	5	0	1	5	21
1	0	4	5	0	16	20	21
0	1	0	0	14	1	0	0
1	0	1	10	0	1	10	91
3	40	27	15	560	283	135	63
1	0	4	10	0	16	40	91
0	1	0	0	21	1	0	0
1	0	1	17	0	1	17	273

0	1	0	0	5	5	0
1	0	0	14	1	0	0
0	0	1	0	0	1	14
5	3	0	70	35	15	0
0	3	5	0	15	35	70
1	0	0	21	1	0	0
0	0	1	0	0	1	21

1	0	0	5	5	0		1	0	0	1	14		0	1	0	0
0	1	0	0	1	14		0	0	1	0	0		0	0	1	0
3	0	0	30	15	0		1	0	0	4	14		1	0	0	0
3	5	0	15	35	70		0	1	0	0	0		0	0	0	1
0	0	1	0	0	0		1	0	0	1	21					
0	1	0	0	1	21											

By returning on the elimination the values of the 10 coefficients are obtained.

Group IV contains those of the remaining terms of W for which $i = i' = 0$. They are divisible by yy'; after which they take the form

$$A_0 + A_1 y + A_2 y' + A_3 y^2 + A_4 yy' + A_5 y'^2$$

Here it is not allowed to suppose that either y or y' vanishes. We may take the group of six combinations

$$\left.\begin{array}{rrrrr} \text{No. } 54 & 0 & 0 & -1 & -1 \\ 55 & 0 & 0 & 1 & 1 \\ 56 & 0 & 0 & -1 & 1 \\ 57 & 0 & 0 & 1 & -1 \end{array}\right\} \qquad \begin{array}{rrrrr} \text{No. } 58 & 0 & 0 & 2 & 1 \\ 59 & 0 & 0 & 1 & 2 \end{array}$$

The disintegrating operation, performed on the bracketed four, gives the values of A_1, A_2, A_4 besides one equation between A_0, A_3, A_5. In the two last combinations, by subtracting the terms corresponding to the former coefficients, we have the three equations,

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{array}$$

which evidently suffice for determining A_0, A_3, A_5.

Group V contains those of the remaining terms of W for which $i' = j' = 0$. These terms are all divisible by xy, after which they take the form

$$\begin{array}{l} A_0 \\ + A_1 x \\ + A_2 x^2 + A_3 y \\ + A_4 x^3 + A_5 xy \\ + A_6 x^4 + A_7 x^2 y + A_8 y^2 \\ + A_9 x^5 + A_{10} x^3 y + A_{11} xy^2 \end{array}$$

To determine these 12 coefficients we compute the values of W for the eight combinations

$$\left.\begin{array}{rrrrr} \text{No. } 60 & -1 & 0 & -1 & 0 \\ 61 & 1 & 0 & 1 & 0 \\ 62 & -1 & 0 & 1 & 0 \\ 63 & 1 & 0 & -1 & 0 \end{array}\right\} \qquad \left.\begin{array}{rrrrr} \text{No. } 64 & -2 & 0 & -1 & 0 \\ 65 & 2 & 0 & 1 & 0 \\ 66 & -2 & 0 & 1 & 0 \\ 67 & 2 & 0 & -1 & 0 \end{array}\right\}$$

The application of the disintegrating operation furnishes for determining A_3, A_7, and again for A_5, A_{10}, two equations each, which groups are identically the same as far as the numerical multipliers of the unknowns are concerned ; they are

$$\begin{matrix} 1+1 \\ 1+4 \end{matrix}$$

The disintegrating operation furnishes besides two equations for determining A_0, A_2, A_6, A_8 and again two for A_1, A_4, A_9, A_{11} which groups are identical as far as the coefficients of the unknowns are concerned. We therefore need four additional values of W, and choose the combinations

No. 68	1	0	2	0
69	−1	0	2	0
70	3	0	1	0
71	−3	0	1	0

From the special values of W there can be subtracted in addition the values of the terms which correspond to the eight previously determined coefficients of this group. Thus our equations for determining both groups of four coefficients have the form, and, with those derived by elimination, are

1	1	1	1		3	15	0		0	3
1	4	16	1		0	0	3		40	0
1	1	1	4		8	80	0			
1	9	81	1							

Groups VI, VII and VIII are defined severally by the conditions $i = j' = 0$, and $i' = j = 0$, and $i = j = 0$; also the equations belonging to them, in respect to the integral factors multiplying the unknowns, are the same. Hence we need only set down the combinations:

GROUP VI.

No. 72	0	−1	−1	0	No. 76	0	−2	−1	0	No. 80	0	1	2	0
73	0	1	1	0	77	0	2	1	0	81	0	−1	2	0
74	0	−1	1	0	78	0	−2	1	0	82	0	3	1	0
75	0	1	−1	0	79	0	2	−1	0	83	0	−3	1	0

GROUP VII.

No. 84	−1	0	0	−1	No. 88	−2	0	0	−1	No. 92	1	0	0	2
85	1	0	0	1	89	2	0	0	1	93	−1	0	0	2
86	−1	0	0	1	90	−2	0	0	1	94	3	0	0	1
87	1	0	0	−1	91	2	0	0	−1	95	−3	0	0	1

GROUP VIII.

No. 96	0	−1	0	−1	No. 100	0	−2	0	−1	No. 104	0	1	0	2
97	0	1	0	1	101	0	2	0	1	105	0	−1	0	2
98	0	−1	0	1	102	0	−2	0	1	106	0	3	0	1
99	0	1	0	−1	103	0	2	0	−1	107	0	−3	0	1

We come now to the terms in which only one exponent is zero. Group IX is characterized by the condition $j' = 0$; the terms are all divisible by $xx'y$. After this is made the expression under consideration is

$$
\begin{aligned}
&A_0 \\
&+ A_1\, x + A_2 x' \\
&+ A_3\, x^2 + A_4\, xx' + A_5\, x'^2 \\
&+ A_6\, x^3 + A_7\, x^2x' + A_8\, xx'^2 + A_9\, x'^3 \\
&+ A_{10}x^4 + A_{11}x^3x' + A_{12}x^2x'^2 + A_{13}xx'^3 + A_{14}x'^4 \\
&+ A_{15}y \\
&+ A_{16}xy + A_{17}x'y \\
&+ A_{18}x^2y + A_{19}xx'y + A_{20}x'^2y + A_{21}y^2
\end{aligned}
$$

For determining these coefficients we need values of W for 22 combinations. Of the latter we choose first the four groups of four each:

No. 108	1	1	1	0	No. 116	1	2	1	0
109	—1	1	1	0	117	—1	2	1	0
110	—1	—1	1	0	118	—1	—2	1	0
111	1	—1	1	0	119	1	—2	1	0
112	2	1	1	0	120	1	1	—1	0
113	—2	1	1	0	121	—1	1	—1	0
114	—2	—1	1	0	122	—1	—1	—1	0
115	2	—1	1	0	123	1	—1	—1	0

By applying the disintegrating operation to each of the four groups we arrive at four equations determining A_1, A_6, A_8, A_{16}, and at four determining A_2, A_7, A_9, A_{17}, and at four determining A_4, A_{11}, A_{13}, A_{19}. These three groups of equations agree in having the same numerical multipliers for the unknowns. Their general statement, and the consequent steps of elimination, are

$$
\begin{array}{rrrr}
1 & 1 & 1 & 1 \\
1 & 4 & 1 & 1 \\
1 & 1 & 4 & 1 \\
1 & 9 & 1 & -1
\end{array}
\qquad
\begin{array}{rrr}
3 & 0 & 0 \\
0 & 3 & 0 \\
8 & 0 & -2
\end{array}
\qquad
\begin{array}{rr}
3 & 0 \\
0 & 6
\end{array}
$$

In addition four equations are afforded for the determination of the 10 remaining coefficients A_0, A_3, A_5, A_{10}, A_{12}, A_{14}, A_{15}, A_{18}, A_{20}, A_{21}. Six more equations are therefore needed to complete the determination of these. We choose the combinations

No. 124	2	2	1	0	No. 126	1	3	1	0	No. 128	2	1	—1	0
125	3	1	1	0	127	1	1	2	0	129	1	2	—1	0

Knowing the values of the 12 coefficients of the group previously determined it is possible to subtract from the absolute terms of the 6 last equations the

special values of the terms involving these. Thus we have the following equations with the consequent steps of elimination :

$$
\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 4 & 1 & 16 & 4 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 4 & 1 & 4 & 16 & 1 & 1 & 1 & 1 \\
1 & 9 & 1 & 81 & 9 & 1 & -1 & -1 & -1 & 1 \\
1 & 4 & 4 & 16 & 16 & 16 & 1 & 4 & 4 & 1 \\
1 & 9 & 1 & 81 & 9 & 1 & 1 & 9 & 1 & 1 \\
1 & 1 & 9 & 1 & 9 & 81 & 1 & 1 & 9 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 4 \\
1 & 4 & 1 & 16 & 4 & 1 & -1 & -4 & -1 & 1 \\
1 & 1 & 4 & 1 & 4 & 16 & -1 & -1 & -4 & 1
\end{array}
\qquad
\begin{array}{ccccccccc}
1 & 0 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 5 & 0 & 0 & 0 & 0 \\
4 & 0 & 40 & 4 & 0 & -1 & -1 & -1 & 0 \\
1 & 1 & 5 & 5 & 5 & 0 & 1 & 0 & 0 \\
1 & 0 & 10 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 10 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \\
3 & 0 & 15 & 3 & 0 & -2 & -5 & -2 & 0 \\
0 & 3 & 0 & 3 & 15 & -2 & -2 & -5 & 0
\end{array}
$$

$$
\begin{array}{cccccccc}
1 & 0 & 1 & 5 & 0 & 0 & 0 & 0 \\
0 & 20 & 0 & 0 & -1 & -1 & -1 & 0 \\
1 & 0 & 4 & 5 & 0 & 1 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 10 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & -2 & -5 & -2 & 0 \\
3 & 0 & 3 & 15 & -2 & -2 & -5 & 0
\end{array}
\qquad
\begin{array}{ccccccc}
20 & 0 & 0 & -1 & -1 & -1 & 0 \\
0 & 3 & 0 & 0 & 1 & 0 & 0 \\
5 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 \\
0 & 0 & 0 & -2 & -5 & -2 & 0 \\
0 & 0 & 0 & -2 & -2 & -5 & 0
\end{array}
\qquad
\begin{array}{cccccc}
3 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & -5 & -1 & 0 \\
0 & 5 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 3 \\
0 & 0 & -2 & -5 & -2 & 0 \\
0 & 0 & -2 & -2 & -5 & 0
\end{array}
$$

$$
\begin{array}{ccccc}
0 & -1 & -5 & -1 & 0 \\
5 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 3 \\
0 & 2 & 5 & 2 & 0 \\
0 & 2 & 2 & 5 & 0
\end{array}
\qquad
\begin{array}{cccc}
1 & 5 & 1 & 0 \\
1 & 1 & 1 & 3 \\
2 & 5 & 2 & 0 \\
2 & 2 & 5 & 0
\end{array}
\qquad
\begin{array}{ccc}
4 & 0 & -3 \\
5 & 0 & 0 \\
8 & -3 & 0
\end{array}
\qquad
\begin{array}{cc}
0 & 15 \\
1 & -2
\end{array}
$$

The 10 equations are therefore independent and suffice for determining the 10 coefficients.

Group X is characterized by the condition $j = 0$; the terms are all divisible by $xx'y'$. The process to be followed is identical with that which just precedes; the numerical coefficients of the equations are the same; we need only set down the combinations to be used:

$$
\left.\begin{array}{lrrrr}
\text{No. } 130 & 1 & 1 & 0 & 1 \\
131 & -1 & 1 & 0 & 1 \\
132 & -1 & -1 & 0 & 1 \\
133 & 1 & -1 & 0 & 1
\end{array}\right\}
\qquad
\left.\begin{array}{lrrrr}
\text{No. } 134 & 2 & 1 & 0 & 1 \\
135 & -2 & 1 & 0 & 1 \\
136 & -2 & -1 & 0 & 1 \\
137 & 2 & -1 & 0 & 1
\end{array}\right\}
\qquad
\left.\begin{array}{lrrrr}
\text{No. } 138 & 1 & 2 & 0 & 1 \\
139 & -1 & 2 & 0 & 1 \\
140 & -1 & -2 & 0 & 1 \\
141 & 1 & -2 & 0 & 1
\end{array}\right\}
$$

$$
\left.\begin{array}{lrrrr}
\text{No. } 142 & 1 & 1 & 0 & -1 \\
143 & -1 & 1 & 0 & -1 \\
144 & -1 & -1 & 0 & -1 \\
145 & 1 & -1 & 0 & -1
\end{array}\right\}
\qquad
\begin{array}{lrrrr}
\text{No. } 146 & 2 & 2 & 0 & 1 \\
147 & 3 & 1 & 0 & 1 \\
148 & 1 & 3 & 0 & 1
\end{array}
\qquad
\begin{array}{lrrrr}
\text{No. } 149 & 1 & 1 & 0 & 2 \\
150 & 2 & 1 & 0 & -1 \\
151 & 1 & 2 & 0 & -1
\end{array}
$$

Group XI is characterized by the condition $i' = 0$; the terms are all

divisible by xyy'. The division performed the expression to be treated takes the form

$$A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 y' + A_5 xy' + A_6 y + A_7 xy$$

It will be found that the 8 coefficients here involved can be obtained from the combinations

No. 152	1	0	1	1	No. 156	1	0	1	—1
153	—1	0	1	1	157	—1	0	1	—1
154	—1	0	—1	1	158	2	0	1	1
155	1	0	—1	1	159	—2	0	1	1

The first four give the values of

$$A_0 + A_2 + A_4\,,\ A_1 + A_3 + A_5\,,\ A_6\,,\ A_7$$

the two following the values of

$$A_0 + A_2 - A_4\,,\ A_1 + A_3 - A_5$$

and the two last the values of

$$A_0 + 4A_2\,,\ A_1 + 4A_3$$

Group XII is characterized by the condition $i = 0$; the terms are all divisible by $x'yy'$. The coefficients are derived by equations of exactly the same form as in the preceding group. It is only necessary to set down the combinations:

No. 160	0	1	1	1	No. 164	0	1	1	—1
161	0	—1	1	1	165	0	—1	1	—1
162	0	—1	—1	1	166	0	2	1	1
163	0	1	—1	1	167	0	—2	1	1

The last group of terms to be considered is that of the 8 in which all the exponents have finite values. After division by $xx'yy'$ they have the form

$$A_0 + A_1 x + A_2 x^2 + A_3 x' + A_4 xx' + A_5 x'^2 + A_6 y + A_7 y'$$

The following combinations will enable us to arrive at the coefficients:

No. 168	1	1	1	1	No. 172	1	1	—1	1
169	—1	1	1	1	173	1	1	1	—1
170	—1	—1	1	1	174	2	1	1	1
171	1	—1	1	1	175	1	2	1	1

The first four give the values of

$$A_0 + A_2 + A_5 + A_6 + A_7\,,\ A_1\,,\ A_3\,,\ A_4$$

The next two being added we have the values of

$$A_0 + A_2 + A_5\,,\ A_6\,,\ A_7$$

The addition of the two last enables us to have the values of A_0, A_2, A_5.

A few words may be added regarding the choice of the elements forming the combined arguments for the special values of W. It is not pretended that that adopted in the preceding is the best. There is a lack of symmetry which, at the moment, I am unable to remove. The latitude in the matter appears to be great; as would be anticipated when 175 things are to be chosen from 2025. It is important, however, to keep the integers multiplying d and d' as small as consists with the condition that the selected combinations afford independent equations. At least, there is no need of going outside of the scheme on page 297.

MEMOIR No. 78.

Deduction of the Power Series Representing a Function from Special Values of the Latter.

(American Journal of Mathematics, Vol. XXVII, pp. 203–216, 1905.)

I have already treated this matter in another place,* but the exposition there is by illustration only and quite incomplete. The subject needs a more general presentation, which will be the endeavor here.

The treatment of the question is much facilitated or, in many cases, even rendered possible, by the application of two principles. The first is the isolation of groups in the assemblage of linear equations through the attribution of zero values to some of the parameters involved. The second is the disintegration of the equations by comparison when corresponding positive and negative values are given to one or more of the parameters.

Here it is expedient to adopt a peculiar notation Let F denote the function to be treated and x the general parameter. The formulæ to be written in what follows will be limited to the case where there are four parameters ; the modifications to be made when there are more or less will be obvious. We use i for the general integral exponent always not negative, and A for the general coefficient. There is here no necessity for the employment of accents or subscripts to distinguish quantities of the same kind. The parameters will be known as the first, second, third and fourth. In designating any one of these all must be written ; thus, the third parameter is $x^0x^0xx^0$. Accordingly, we write the equation

$$F = \Sigma A\, x^i x^i x^i x^i,$$

where the i's are not necessarily the same. Let subscripts attached to F denote the special values of the function correspondent to special values of the parameters; and, as we have to distinguish between significant and zero values for the latter, let us suppose that i always denotes a positive integer; consequently, the value $i = 0$ is excluded from the summations Σ. The function F must undergo a sort of differencing in reference to zero values for the parameter; a differencing which is more general than the ordinary,

*Astronomical Journal, No. 567. Memoir No. 77.

since it often involves more than one variable. This mode of operating is, for the adopted case, depicted in the following system of equations:

$$\Sigma\,.\,Ax^0x^0x^0x^0 = F_{0000} = \overset{0}{F}_{0000},$$

$$\left\{\begin{aligned}
\Sigma\,.\,Ax^ix^0x^0x^0 &= F_{x000} - F_{0000} = \overset{1}{F}_{x000},\\
\Sigma\,.\,Ax^0x^ix^0x^0 &= F_{0x00} - F_{0000} = \overset{1}{F}_{0x00},\\
\Sigma\,.\,Ax^0x^0x^ix^0 &= F_{00x0} - F_{0000} = \overset{1}{F}_{00x0},\\
\Sigma\,.\,Ax^0x^0x^0x^i &= F_{000x} - F_{0000} = \overset{1}{F}_{000x},
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\Sigma\,.\,Ax^ix^ix^0x^0 &= F_{xx00} - \overset{1}{F}_{x000} - \overset{1}{F}_{0x00} - F_{0000} = \overset{2}{F}_{xx00},\\
\Sigma\,.\,Ax^ix^0x^ix^0 &= F_{x0x0} - \overset{1}{F}_{x000} - \overset{1}{F}_{00x0} - F_{0000} = \overset{2}{F}_{x0x0},\\
\Sigma\,.\,Ax^ix^0x^0x^i &= F_{x00x} - \overset{1}{F}_{x000} - \overset{1}{F}_{000x} - F_{0000} = \overset{2}{F}_{x00x},\\
\Sigma\,.\,Ax^0x^ix^ix^0 &= F_{0xx0} - \overset{1}{F}_{0x00} - \overset{1}{F}_{00x0} - F_{0000} = \overset{2}{F}_{0xx0},\\
\Sigma\,.\,Ax^0x^ix^0x^i &= F_{0x0x} - \overset{1}{F}_{0x00} - \overset{1}{F}_{000x} - F_{0000} = \overset{2}{F}_{0x0x},\\
\Sigma\,.\,Ax^0x^0x^ix^i &= F_{00xx} - \overset{1}{F}_{00x0} - \overset{1}{F}_{000x} - F_{0000} = \overset{2}{F}_{00xx},
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\Sigma\,.\,Ax^ix^ix^ix^0 &= F_{xxx0} - \overset{2}{F}_{xx00} - \overset{2}{F}_{x0x0} - \overset{2}{F}_{0xx0} - \overset{1}{F}_{x000} - \overset{1}{F}_{0x00} - \overset{1}{F}_{00x0} - F_{0000} = \overset{3}{F}_{xxx0},\\
\Sigma\,.\,Ax^ix^ix^0x^i &= F_{xx0x} - \overset{2}{F}_{xx00} - \overset{2}{F}_{x00x} - \overset{2}{F}_{0x0x} - \overset{1}{F}_{x000} - \overset{1}{F}_{0x00} - \overset{1}{F}_{000x} - F_{0000} = \overset{3}{F}_{xx0x},\\
\Sigma\,.\,Ax^ix^0x^ix^i &= F_{x0xx} - \overset{2}{F}_{x0x0} - \overset{2}{F}_{x00x} - \overset{2}{F}_{00xx} - \overset{1}{F}_{x000} - \overset{1}{F}_{00x0} - \overset{1}{F}_{000x} - F_{0000} = \overset{3}{F}_{x0xx},\\
\Sigma\,.\,Ax^0x^ix^ix^i &= F_{0xxx} - \overset{2}{F}_{0xx0} - \overset{2}{F}_{0x0x} - \overset{2}{F}_{00xx} - \overset{1}{F}_{0x00} - \overset{1}{F}_{00x0} - \overset{1}{F}_{000x} - F_{0000} = \overset{3}{F}_{0xxx},
\end{aligned}\right.$$

$$\begin{aligned}
\Sigma\,.\,Ax^ix^ix^ix^i = F_{xxxx} &- \overset{3}{F}_{xxx0} - \overset{3}{F}_{xx0x} - \overset{3}{F}_{x0xx} - \overset{3}{F}_{0xxx} - \overset{2}{F}_{xx00} - \overset{2}{F}_{x0x0}\\
&- \overset{2}{F}_{x00x} - \overset{2}{F}_{0xx0} - \overset{2}{F}_{0x0x} - \overset{2}{F}_{00xx} - \overset{1}{F}_{x000}\\
&- \overset{1}{F}_{0x00} - \overset{1}{F}_{00x0} - \overset{1}{F}_{000x} - F_{0000} \qquad = \overset{4}{F}_{xxxx}.
\end{aligned}$$

The number of these equations is $16 = 2^4$, and, generally, if there are k parameters, the number is 2^k. It will be readily perceived that $\overset{0}{F}$ is the term of the series independent of the parameters; that the $\overset{1}{F}$ are functions of the single significant parameter appearing in their subscripts, without a term independent of that parameter; that the $\overset{2}{F}$ are functions of the two significant parameters appearing in their subscripts, without any terms independent of one or both parameters; that the $\overset{3}{F}$ are functions of the three significant parameters appearing in their subscripts, without any terms independent of one, two or all of these parameters; and, finally, that $\overset{4}{F}$ is a function of all four parameters, but without any terms independent of one, two, three or all of these parameters. Thus each F of a definite

superscript involves no terms included in the F of smaller superscripts. By this device we have broken the system of linear equations for the determination of the coefficients into 16 groups, each of which can be treated independently of the others.

It is not necessary that the computations should be made by the equations just written. The last involves no less than 16 terms, and labor will be saved by eliminating some of the F. The 5 equations at the beginning remaining unmodified, it will be preceived the following system is equivalent to the former:

$$\left\{\begin{aligned}
\overset{2}{F}_{xx00} &= F_{xx00} - \overset{1}{F}_{x000} - F_{0x00}, \\
\overset{2}{F}_{x0x0} &= F_{x0x0} - \overset{1}{F}_{00x0} - F_{x000}, \\
\overset{2}{F}_{x00x} &= F_{x00x} - \overset{1}{F}_{000x} - F_{x000}, \\
\overset{2}{F}_{0xx0} &= F_{0xx0} - \overset{1}{F}_{00x0} - F_{0x00}, \\
\overset{2}{F}_{0x0x} &= F_{0x0x} - \overset{1}{F}_{000x} - F_{0x00}, \\
\overset{2}{F}_{00xx} &= F_{00xx} - \overset{1}{F}_{00x0} - F_{000x},
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\overset{3}{F}_{xxx0} &= F_{xxx} - \overset{2}{F}_{xx00} - F_{x0x0} - F_{0xx0} + F_{00x0}, \\
\overset{3}{F}_{xx0x} &= F_{xx0x} - \overset{2}{F}_{xx00} - F_{0x0x} - F_{x00x} + F_{000x}, \\
\overset{3}{F}_{x0xx} &= F_{x0xx} - \overset{2}{F}_{00xx} - F_{x00x} - F_{x0x0} + F_{x000}, \\
\overset{3}{F}_{0xxx} &= F_{0xxx} - \overset{2}{F}_{00xx} - F_{0x0x} - F_{0xx0} + F_{0x00},
\end{aligned}\right.$$

$$\overset{4}{F}_{xxxx} = F_{xxxx} - \overset{3}{F}_{xxx0} - \overset{3}{F}_{xx0x} - \overset{2}{F}_{xx00} - F_{x0xx} - F_{0xxx} + F_{00xx}.$$

These formulæ are not the unique ones of their type, but the $\overset{2}{F}$ admit two different forms, the $\overset{3}{F}$ three and $\overset{4}{F}$ six. All are obtained by making certain transpositions between the x and 0 of the subscripts. They need not be given here, as their employment has no advantage over those just written.

Each of the $\overset{i}{F}$ is evidently divisible by the product of the significant parameters in its subscript. The functions thus obtained may be considered as one step nearer the result of elimination. We may use G to denote them. Thus:

$$G_{0000} = \frac{1}{x^0x^0x^0x^0} F_{0000}, \qquad G_{x000} = \frac{1}{xx^0x^0x^0} \overset{1}{F}_{x000}, \qquad G_{0x00} = \frac{1}{x^0xx^0x^0} \overset{1}{F}_{0x00}, \text{ etc.,}$$

$$G_{xx00} = \frac{1}{xxx^0x^0} \overset{2}{F}_{xx00}, \text{ etc.,} \qquad G_{xxx0} = \frac{1}{xxxx^0} \overset{3}{F}_{xxx0}, \text{ etc.,} \qquad G_{xxxx} = \frac{1}{xxxx} \overset{4}{F}_{xxxx}.$$

We come now to the application of the second principle. In the first place consider F when involving only a single significant parameter as

F_{x000}, and let F_{+000} and F_{-000} denote the values of F_{x000} for corresponding positive and negative values of x; then it is plain we shall have

$$\Sigma A\,(x^2)^i x^0 x^0 x^0 = \tfrac{1}{2}[F_{+000} + F_{-000}],$$
$$xx^0x^0x^0\;\Sigma A\,(x^2)^i x^0x^0x^0 = \tfrac{1}{2}[F_{+000} - F_{-000}],$$

where the A of the first equation are distinct from the A of the second, and where it is now necessary to allow i to assume the value 0.

Next, supposing F involves two significant parameters as F_{xx00}, then we shall have the four equations

$$\Sigma A\;(x^2)^i xx^0x^0 = \tfrac{1}{2}[F_{+x00} + F_{-x00}],$$
$$xx^0x^0x^0\;\Sigma A\;(x^2)^i xx^0x^0 = \tfrac{1}{2}[F_{+x00} - F_{-x00}],$$
$$\Sigma Ax\,(x^2)^i x^0x^0 = \tfrac{1}{2}[F_{x+00} + F_{x-00}],$$
$$x^0xx^0x^0\;\Sigma Ax\,(x^2)^i x^0x^0 = \tfrac{1}{2}[F_{x+00} - F_{x-00}].$$

By taking half the sum and half the difference of certain of these, we obtain the four equations

$$\Sigma Ax^{2i}x^{2i}x^0x^0 = \tfrac{1}{4}[F_{++00} + F_{+-00} + F_{-+00} + F_{--00}],$$
$$\Sigma Ax^{2i+1}x^{2i}x^0x^0 = \tfrac{1}{4}[F_{++00} + F_{+-00} - F_{-+00} - F_{--00}],$$
$$\Sigma Ax^{2i}x^{2i+1}x^0x^0 = \tfrac{1}{4}[F_{++00} - F_{+-00} + F_{-+00} - F_{--00}],$$
$$\Sigma Ax^{2i+1}x^{2i+1}x^0x^0 = \tfrac{1}{4}[F_{++00} - F_{+-00} - F_{-+00} + F_{--00}],$$

where the coefficients A are distinct for each. The second, third and fourth are divisible severally by $xx^0x^0x^0$, $x^0xx^0x^0$ and xxx^0x^0. By making these divisions we shall be a step nearer the result of elimination. The rule of the signs connecting the form F in the group of four equations may seem a little obscure, but a consideration of the successive operations of taking half the sum and difference shows that the sign of each F is given by raising the signs in the subscripts to the same powers as the corresponding parameters have in the left members of the equations. As here, the even integer $2i$ may be dropped out of the exponents, we perceive that the signs in question are given by the expressions

$$(+)^0(+)^0,\quad (+)^0(-)^0,\quad (-)^0(+)^0,\quad (-)^0(-)^0,$$
$$(+)^1(+)^0,\quad (+)^1(-)^0,\quad (-)^1(+)^0,\quad (-)^1(-)^0,$$
$$(+)^0(+)^1,\quad (+)^0(-)^1,\quad (-)^0(+)^1,\quad (-)^0(-)^1,$$
$$(+)^1(+)^1,\quad (+)^1(-)^1,\quad (-)^1(+)^1,\quad (-)^1(-)^1,$$

In case F involves three significant parameters, as F_{xxx0}, we have entirely analogous equations which, for brevity, we write as follows:

$$\Sigma Ax^{2i}x^{2i}x^{2i}x^0 = \tfrac{1}{2^3}\,S(\pm)^0(\pm)^0(\pm)^0\,F_{\pm\pm\pm0},$$
$$\Sigma Ax^{2i+1}x^{2i}x^{2i}x^0 = \tfrac{1}{2^3}\,S(\pm)^1(\pm)^0(\pm)^0\,F_{\pm\pm\pm0},$$
$$\Sigma Ax^{2i}x^{2i+1}x^{2i}x^0 = \tfrac{1}{2^3}\,S(\pm)^0(\pm)^1(\pm)^0\,F_{\pm\pm\pm0},$$
$$\Sigma Ax^{2i}x^{2i}x^{2i+1}x^0 = \tfrac{1}{2^3}\,S(\pm)^0(\pm)^0(\pm)^1\,F_{\pm\pm\pm0},$$
$$\Sigma Ax^{2i+1}x^{2i+1}x^{2i}x^0 = \tfrac{1}{2^3}\,S(\pm)^1(\pm)^1(\pm)^0\,F_{\pm\pm\pm0},$$
$$\Sigma Ax^{2i+1}x^{2i}x^{2i+1}x^0 = \tfrac{1}{2^3}\,S(\pm)^1(\pm)^0(\pm)^1\,F_{\pm\pm\pm0},$$
$$\Sigma Ax^{2i}x^{2i+1}x^{2i+1}x^0 = \tfrac{1}{2^3}\,S(\pm)^0(\pm)^1(\pm)^1\,F_{\pm\pm\pm0},$$
$$\Sigma Ax^{2i+1}x^{2i+1}x^{2i+1}x^0 = \tfrac{1}{2^3}\,S(\pm)^1(\pm)^1(\pm)^1\,F_{\pm\pm\pm0}.$$

The connection of the ambiguous signs in these equations will be readily understood.

In case F has all its parameters significant, there are 16 equations analogous to the preceding; we need write but one as a type of all:

$$\Sigma A x^{2i} x^{2i} x^{2i} x^{2i} = \frac{1}{2^4} S (\pm)^0 (\pm)^0 (\pm)^0 (\pm)^0 F_{\pm\pm\pm\pm}.$$

Thus, by the application of the two principles of zero values and of pairs of values of opposite signs, we succeed in breaking the system of linear equations to be solved into several subordinate systems entirely independent of each other. When the number of parameters is 2, it is evident that the number of these subordinate systems is

$$1.1 + 2.2 + 4.1 = 9 = 3^2;$$

where there are 3 parameters this number is

$$1.1 + 2.3 + 4.3 + 8.1 = 27 = 3^3;$$

and when the number of parameters is 4 (the case we have been treating) the number is

$$1.1 + 2.4 + 4.6 + 8.4 + 16.1 = 81 = 3^4;$$

hence, in the general case, where there are k parameters, the number of independent subordinate systems is 3^k.

After having shown the applicability of zero and parity values of the parameters for breaking the system of linear equations into detached portions, it remains to show what principles should guide us in selecting the values of the parameters for which the special values of the function are to be computed. As in the former memoir we suppose that the values of each parameter are taken from an arithmetical progression of which one term is zero. Let d denote the common difference in this progression which, although it may be different for each parameter, we designate by the same letter, just as before we employed x and i. Then, in the first instance, the power series will be derived in the form

$$F = \Sigma A \left(\frac{x}{d}\right)^i \left(\frac{x}{d}\right)^i \left(\frac{x}{d}\right)^i \left(\frac{x}{d}\right)^i.$$

As it is necessary to cut off the power series at some limit, it is desirable to choose the d in such a way that the neglected terms should vitiate as little as possible the derived values of the A. The smaller are the d the smaller is this vitiation; but practical considerations set a limit to this diminution. Suppose we are going to quantities of the 10th order of smallness in the x, and decide to halve the d; then, as $2^{10} = 1024$, it will be necessary to add 3 more decimals in our computations; and if the d are diminished to a tenth, 10 decimals must be added, which procedure could

not generally be entertained. Thus nice judgment is required in deciding on the magnitude of the d. As good a rule for the choice as can be given is to divide the range over which the parameter is supposed to play by the number of significant exponents it is to receive in the power series. Then the selection of the values of the parameters should be such that, in a graphical exhibition, they would be arranged as nearly as possible in a symmetrical manner about the origin; and, in a space of k dimensions if k is the number of parameters, they should be contained within the ellipsoid whose axes are the several ranges.

As the computation of special values of the function constitutes much the larger part of the labor incident to the method, it is desirable to insist on the limitation that no more special values are to be computed than terms are to be retained. But some restrictions must be put on the employment of the two principles given for the purpose of disintegrating the system of linear equations, and on the selection of values of the parameters for which the special values of the function are to be computed.

If F is computed for $x = id$, $x = id$, $x = id$, $x = id$, we shall call $iiii$ the argument of the value of F. Here i is integral, but may be zero or negative. Then, in each group of linear equations obtained by the application of the first principle, it is plain that the zeros of the arguments used must fall in the same place as the zero exponents of the parameters; thus, when we are treating the group whose type is $[i00i]$, the arguments of the special values used must be of the type $i00i$, where, however, i can be negative. The first principle can always be used, but it is desirable to limit the selection of arguments in the following manner:—Dividing the terms into Division I, where all the exponents are zeros, Division II, when all but one are zeros, and III where all but two are zeros, and so on; if we have used an argument such as $iii0$ in Division IV, it is necessary to use the arguments $ii00$, $i0i0$, $0ii0$ in the preceding or here Division III, understanding that the i in the second case are identical with those standing in the same place in the first. Hence the proper method of selecting the arguments to be used seems to be to commence at Division I, for which, in the case we exhibit, the argument is 0000, and get the arguments for Division II by substituting for one of the zeros an integer positive or negative. Then the arguments for Division III are got from these by substituting for one of the remaining zeros positive or negative integers, and so on to the end. These integers should constitute in each case an arithmetical progression having zero near the middle of it.

With regard to the application of the second principle, that of parity

values, it often cannot be employed without introducing non-independent equations. The remedy for this state of things is to cut down the operation to a half stage or even to a quarter stage, and, in some cases, not to employ it at all.

These matters cannot be well set forth without the help of an example. We adopt that of the preceding memoir. It is characterized by saying that it involves four parameters, two of which are regarded as of the first, and two of the second order of smallness; and all terms above the eighth order are to be neglected. This demands the presence of 175 terms in the power series. How they are disintegrated into 81 subgroups by the application of our two principles in shown in the following table. As each term is sufficiently characterized by the exponents of the four parameters, nothing else is set down, and the terms of each subgroup are connected by the sign +. In addition, the exponents of the first term are set down as they are, but the following terms of the line are divided by the first term, as the quotients are more useful than the terms themselves.

Division.	Group.	Sub-group.	Divisor	Quotients.
I	1	1	0000	
II	2	2	2000	+2000+4000+6000
		3	1000	“ “ “
	3	4	0200	+0200+0400+0600
		5	0100	“ “ “
	4	6	0020	+0020
		7	0010	“
	5	8	0002	+0002
		9	0001	“
III	6	10	2200	+2000+0200+4000+2200+0400
		11	1200	“ “ “ “ “
		12	2100	“ “ “ “ “
		13	1100	+ “ “ “ “ +6000+4200+2400+0600
				“
	7	14	2020	“
		15	1020	“
		16	2010	“ +0020 “
		17	1010	“ “ “
	8	18	2002	“
		19	1002	“
		20	2001	“ +0002 “
		21	1001	“ “ “

Division.	Group.	Sub-group.	Divisor.	Quotients.					
III	9	22	0220	+0200					
		23	0120	"					
		24	0210	"	+0020	+0400			
		25	0110	"	"	"			
	10	26	0202	"					
		27	0102	"					
		28	0201	"	+0002	"			
		29	0101	"	"	"			
	11	30	0022						
		31	0012						
		32	0021						
		33	0011	+0020	"				
IV	12	34	2220						
		35	1220						
		36	2120						
		37	2210	+2000	+0200				
		38	1120	"	"				
		39	1210	"	"				
		40	2110	"	"				
		41	1110	"	"	+0020	+4000	+2200	+0400
	13	42	2202						
		43	1202						
		44	2102						
		45	2201	"	"				
		46	1102	"	"				
		47	1201	"	"				
		48	2101	"	"				
		49	1101	"	"	+0002	"	"	"
	14	52	2012						
		53	2021						
		54	1012						
		55	1021						
		56	2011	"					
		57	1011	"					
	15	60	0212						
		61	0221						
		62	0112						
		63	0121						
		64	0211	+0200					
		65	0111	"					
V	16	76	2211						
		77	2111						
		78	1211						
		79	1121						
		80	1112						
		81	1111	+2000	"				

The 16 groups in the table are the result of the application of the first principle; the 81 sub-groups result from the further application of the second principle. It will be noticed that 14 out of the 81 sub-groups do not appear in the table; this is because their terms are all above the 8th order. The success of the application of the two principles is well shown by the table. Out of 81 sub-groups there is only one (the 13th) which consists of as many as 10 equations and 10 unknowns; two groups have 7, and three have 6; and 23 groups consist of a single equation giving the value of one coefficient each.

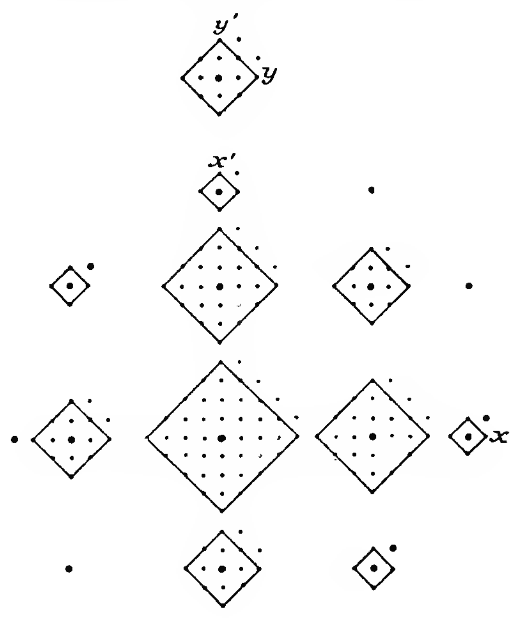
The following table shows a selection of arguments which may be employed in our illustrative example. The ambiguous signs must be taken in every possible combination; thus three in one argument denote eight different arguments.

Division.	Group.	Arguments.									
I	1	0 0 0 0									
II	2	±1 0 0 0	±2 0 0 0	±3 0 0 0	±4 0 0 0						
	3	0±1 0 0	0±2 0 0	0±3 0 0	0±4 0 0						
	4	0 0±1 0	0 0±2 0								
	5	0 0 0±1	0 0 0±2								
III	6	±1±1 0 0	±2±1 0 0	±1±2 0 0	±2±2 0 0	±3±1 0 0	±1±3 0 0	1 4 0 0	2 3 0 0	3 2 0 0	4 1 0 0
	7	±1 0±1 0	±2 0±1 0	±1 0 2 0	±3 0 1 0						
	8	±1 0 0±1	±2 0 0±1	±1 0 0 2	±3 0 0 1						
	9	0±1±1 0	0±2±1 0	0±1 2 0	0±3 1 0						
	10	0±1 0±1	0±2 0±1	0±1 0 2	0±3 0 1						
	11	0 0±1±1	0 0 2 1	0 0 1 2							
IV	12	±1±1±1 0	±2±1 1 0	±1±2 1 0	2 1 −1 0	1 2 −1 0	2 2 1 0	3 1 1 0	1 3 1 0	1 1 2 0	
	13	±1±1 0±1	±2±1 0 1	±1±2 0 1	2 1 0 −1	1 2 0 −1	2 2 0 1	3 1 0 1	1 3 0 1	1 1 0 2	
	14	±1 0±1 1	±1 0 1 −1	±2 0 1 1							
	15	0±1±1 1	0±1 1 −1	0±2 1 1							
V	16	±1±1 1 1	1 1 −1 1	1 1 1 −1	2 1 1 1	1 2 1 1					

It will be seen from this table that the second principle has not in every case been pushed to its limit. Thus in Div. IV, Group 12, if we employ the 8 arguments $\pm 1 \pm 1 \pm 1\ 0$ we get as many independent relations between the sought coefficients; but, if we annex the 8 auguments $\pm 2 \pm 1 \pm 1\ 0$, we do not get 8 additional relations but only 5. This is

explained by the fact (consult the arrangement of terms in Group 12 in the first table) that the first 8 give the values of 3 coefficients, and the second 8 also give them.

We will catalogue all the deviations from a complete parity treatment in the foregoing table. In Group 6, the parity treatment, here involving two steps, has been applied only in six cases, while four arguments are without it; to have applied it to the latter would have introduced superfluous relations. In Groups 7–10 we have two instances of parity treatment to two steps, and two to one step. In Group 11 one instance of this treatment to two steps and two arguments without it. In Groups 12 and 13 one instance to three steps, two to two and six arguments without it. In Groups 14 and 15 one instance to two steps and two to one. In fine, in Group 16 one instance to two steps and four arguments without it.



But it is much easier to comprehend the principles which should be followed in the choice of the arguments through a graphical exhibition. The 175 arguments in our example, since they are to four elements, can be represented in a space of four dimensions. By drawing in this space $3.5 = 15$ planes properly chosen, the points representing the arguments will all lie in these planes. We adopt here for the coordinates the notation of the first memoir, viz., $xx'yy'$. In the adjacent diagram the upper oblique square with its two adjacent points constitutes a table of contents or index

to the graphs of the 15 planes shown below; it bears on the coordinates y and y' or the third and fourth constituents of the argument. These graphs are placed relatively to each other as the points of the index which belong to them. By this device we are enabled to represent on a plane, sufficiently for our purposes, a space of four dimensions. Moreover, the graphs are placed so as not to interfere with each other, the coordinates x and x' being measured from the central point of the oblique squares. The introduction of the latter into the diagram has no other object than to enable the eye to grasp quickly the law of distribution of the points.

It will be perceived that 5 of the graphs reduce to a single point; they may be called oblique squares to side 0. Next 4 graphs consist of oblique squares to side 1 and they all have one point exterior to the square. Again there are 3 graphs to side 2 with 2 exterior points. Next 2 graphs to side 3 with 3 exterior points; and finally, a single graph to side 4 with 4 exterior points. With regard to these exterior points, it must be explained that the positions they may have in the diagram are not unique. Let us suppose that the positions lying nearest the perimeter of an oblique square and exterior to it are called the adjacent points; they are in number four times the number expressing the side of the square, and they can be joined by straight lines so as to form rectangles. Then the exterior points must be distributed in such a way that each rectangle shall receive one and but one point at some one of its angles. It is not necessary that a similar arrangement should be adopted for all or for some of the graphs; it may be varied at will. In the diagram the exterior points are, in all cases, placed to the upper and right side of the square. As to the arrangement of the squares in reference to the magnitude of their sides, it will be perceived that on the one hand the limit is a square of the side 0, and on the other a square of side 1; and, as we pass inwards towards the centre, at every step the side augments by 2 but when we arrive at the middle column, it is only a half-step on the right hand, while it is a whole step on the left. This is for an even number of parameters; for an odd number, the half steps do not exist.

The number of points in each graph is shown by the following scheme:

$$\left.\begin{array}{l} 1.1 \\ 1.1+2.3 \\ 1.1+2.3+3.5 \\ 1.1+2.3+3.5+4.7 \\ 1.1+2.3+3.5+4.7+5.9 \end{array}\right\} = 175.$$

The regularity apparent in the diagram is due to the tabulation of the points under the headings of two of the parameters. However, after the diagram

is formed, it will not be difficult to distribute the arguments under the headings of the groups. It will be noticed that the exterior points are each a half-unit distant from the perimeters of the squares. As we have placed them they may be included in a rectangle having one more column in one direction than in the other.

When we have the arguments of the special values which determine the coefficients of a sub-group, it is easy to write, with the assistance of the first table, the determinant belonging to the solution. Thus, in Sub-group 13 of our example the determinant is

$$\begin{vmatrix}
1^0.1^0 & 1^2.1^0 & 1^0.1^2 & 1^4.1^0 & 1^2.1^2 & 1^0.1^4 & 1^6.1^0 & 1^4.1^2 & 1^2.1^4 & 1^0.1^6 \\
2^0.1^0 & 2^2.1^0 & 2^0.1^2 & 2^4.1^0 & 2^2.1^2 & 2^0.1^4 & 2^6.1^0 & 2^4.1^2 & 2^2.1^4 & 2^0.1^6 \\
1^0.2^0 & 1^2.2^0 & 1^0.2^2 & 1^4.2^0 & 1^2.2^2 & 1^0.2^4 & 1^6.2^0 & 1^4.2^2 & 1^2.2^4 & 1^0.2^6 \\
2^0.2^0 & 2^2.2^0 & 2^0.2^2 & 2^4.2^0 & 2^2.2^2 & 2^0.2^4 & 2^6.2^0 & 2^4.2^2 & 2^2.2^4 & 2^0.2^6 \\
3^0.1^0 & 3^2.1^0 & 3^0.1^2 & 3^4.1^0 & 3^2.1^2 & 3^0.1^4 & 3^6.1^0 & 3^4.1^2 & 3^2.1^4 & 3^0.1^6 \\
1^0.3^0 & 1^2.3^0 & 1^0.3^2 & 1^4.3^0 & 1^2.3^2 & 1^0.3^4 & 1^6.3^0 & 1^4.3^2 & 1^2.3^4 & 1^0.3^6 \\
4^0.1^0 & 4^2.1^0 & 4^0.1^2 & 4^4.1^0 & 4^2.1^2 & 4^0.1^4 & 4^6.1^0 & 4^4.1^2 & 4^2.1^4 & 4^0.1^6 \\
3^0.2^0 & 3^2.2^0 & 3^0.2^2 & 3^4.2^0 & 3^2.2^2 & 3^0.2^4 & 3^6.2^0 & 3^4.2^2 & 3^2.2^4 & 3^0.2^6 \\
2^0.3^0 & 2^2.3^0 & 2^0.3^2 & 2^4.3^0 & 2^2.3^2 & 2^0.3^4 & 2^6.3^0 & 2^4.3^2 & 2^2.3^4 & 2^0.3^6 \\
1^0.4^0 & 1^2.4^0 & 1^0.4^2 & 1^4.4^0 & 1^2.4^2 & 1^0.4^4 & 1^6.4^0 & 1^4.4^2 & 1^2.4^4 & 1^0.4^6
\end{vmatrix}$$

There is no need of proving that these determinants are non-vanishing, as they are all met with in the problem of drawing a parabolic curve through a definite number of distinct points in a space of two or more dimensions.

MEMOIR No. 79.

Integrals of Planetary Motion Suitable for an Indefinite Length of Time.

(Astronomical Journal, Vol. XXV, pp. 1–12, 1905.)

The desirableness of having in our possession a feasible method for procuring expressions for the coordinates of the planets not limited to short intervals of time about an adopted epoch cannot be disputed. The general theory of such expressions is now well known, and the difficulties attending the subject are reduced to those of elaboration. It must be confessed that when all the parameters involved are left indeterminate in the formulas the latter have a degree of complexity that is truly frightful.

Gyldén devoted the latter years of his life to the investigation of this matter. But he left everything in an incomplete state, and it is obvious that he undertook too much in endeavoring to provide for the eight major planets of the solar sytems at once.

Is it not likely, that, leaving supplementary efforts for the future, a more satisfactory result may be attained by advancing with a lighter load? In accordance with such a view I have undertaken to mark out a practicable route in treating the simplest case suggested by the constitution of the solar system. Let us suppose that Jupiter and Saturn are alone considered, and that they are made to move in the same plane. Also let the masses of the three bodies concerned, the two constants called protometers by Gyldén, and the two constants attached severally to the integrals of living force and the conservation of areas, be known numerical quantities. It is proposed to treat the problem thus limited. The deviations in passing from the ideal to the actual case can afterwards be estimated by methods similar to Lagrange's variation of arbitrary constants. It is evident that the coefficients of the various inequalities brought out in the treatment of the simplified problem will be functions of two indeterminate constants, which we may designate by e_0 and e_0', and which are the moduli of the deviations of the orbits from circularity. These functions admit of development in powers and products of these constants, the multipliers always turning out as numbers. It is the latter circumstance which renders the treatment at all practicable. If we were to insist on the four linear elements, as well as the

masses being kept indeterminate in the coefficients, we should find the latter incapable of expression.

The fundamental conceptions of Gyldén are employed in the following treatment, and I desire to express my high sense of their value, nevertheless, in the interests of brevity, many modifications have been made in the ulterior procedures.

The motions of Jupiter and Saturn relative to the Sun are treated simultaneously, that is, as if we were concerned with the motion of a single point in a space of four dimensions.

Let the following scheme show the notation for masses and rectangular coordinates:

	Masses	Coordinates	
Sun	M		
Jupiter	m	ξ	η
Saturn	m'	ξ'	η'

The motion of the planets relative to the Sun is bound up in the expressions of two functions; *first*, T the living force deduced by multiplying half the product of every two masses by the square of the velocity of one relative to the other, adding the three terms thus obtained and dividing by the mass of the system; *second*, Ω the potential function equivalent to the sum of the three terms given by dividing the product of every two masses by their distance. Thus:

$$T = (M + m + m')^{-1}\left[Mm\frac{d\xi^2 + d\eta^2}{2dt^2} + Mm'\frac{d\xi'^2 + d\eta'^2}{2dt^2} + mm'\frac{(d\xi' - d\xi)^2 + (d\eta' - d\eta)^2}{2dt^2}\right]$$

$$\Omega = \frac{Mm}{\sqrt{\xi^2 + \eta^2}} + \frac{Mm'}{\sqrt{\xi'^2 + \eta'^2}} + \frac{mm'}{\sqrt{(\xi' - \xi)^2 + (\eta' - \eta)^2}}\,*$$

By putting

$$\mu_1 = \frac{(M + m')\,m}{M + m + m'}, \qquad \mu_2 = \frac{(M + m)\,m'}{M + m + m'}, \qquad \mu_3 = \frac{mm'}{M + m + m'}$$

we may write

$$T = \mu_1\frac{d\xi^2 + d\eta^2}{2\,dt^2} + \mu_2\frac{d\xi'^2 + d\eta'^2}{2\,dt^2} - \mu_3\frac{d\xi\,d\xi' + d\eta\,d\eta'}{dt^2}$$

But it is convenient to have a form of this function consisting of two terms, each involving two coordinates. We get this at the expense of complicating the form of Ω. Submitting to this, however, we introduce two hypothetical planets for the actual. Let the coordinates of the former be x, y and x', y'

* These equations are identical with those adopted by Lagrange in his *Essai*, except that here the third coordinate is made to vanish. One may consult Laplace, *Mécanique Céleste*, *Première Partie*, Liv. II, Art. 9, especially Equation (7); Tom. I, p. 131, Old Ed.

connected with the coordinates of the actual planets ($\varkappa$ being a constant) by the equations

$$\xi = x + \varkappa x', \qquad \eta = y + \varkappa y', \qquad \xi' = x' + \varkappa x, \qquad \eta' = y' + \varkappa y.$$

Now consider the quadratic form

$$\mu_1 \xi^2 + \mu_2 \xi'^2 - 2\mu_3 \xi\xi'$$

By the substitution this becomes

$$[\mu_1 + \mu_2 \varkappa^2 - 2\mu_3 \varkappa]\, x^2 + [\mu_2 + \mu_1 \varkappa^2 - 2\mu_3 \varkappa]\, x'^2 + 2\,[(\mu_1 + \mu_2)\varkappa - \mu_3(1 + \varkappa^2)]\, xx'$$

If $\varkappa$ is adopted so as to satisfy the equation

$$\frac{\varkappa}{1 + \varkappa^2} = \frac{\mu_3}{\mu_1 + \mu_2}$$

(we can take the smaller root of the quadratic) the term in xx' will disappear. Instead of eliminating $\varkappa$ eliminate μ_3 from the expression, and put

$$\mathrm{m} = \frac{1 - \varkappa^2}{1 + \varkappa^2}(\mu_1 - \mu_2 \varkappa^2), \qquad \mathrm{m}' = \frac{1 - \varkappa^2}{1 + \varkappa^2}(\mu_2 - \mu_1 \varkappa^2)$$

and the quadratic form becomes

$$\mathrm{m}x^2 + \mathrm{m}'x'^2$$

This linear transformation being applied to the living force T, gives rise to the expression

$$T = \mathrm{m}\frac{dx^2 + dy^2}{2dt^2} + \mathrm{m}'\frac{dx'^2 + dy'^2}{2dt^2}$$

or, using r for the radius and v for the longitude, in terms of polar coordinates, to

$$T = \mathrm{m}\frac{dr^2 + r^2dv^2}{2dt^2} + \mathrm{m}'\frac{dr'^2 + r'^2dv'^2}{2dt^2}$$

But since we wish Ω to involve only three variables, we further transform by making

$$v = \tfrac{1}{2}(\psi + \phi), \qquad v' = \tfrac{1}{2}(\psi + \phi)$$

which leads to

$$T = \mathrm{m}\frac{dr^2 + \frac{1}{4}r^2(d\psi + d\phi)^2}{2dt^2} + \mathrm{m}'\frac{dr'^2 + \frac{1}{4}r'^2(d\psi - d\phi)^2}{2dt^2}$$

Denoting the variables severally conjugate to r, r', ϕ, ψ by the symbols s, s', u, w, we shall have

$$s = \mathrm{m}\frac{dr}{dt}, \qquad s' = \mathrm{m}'\frac{dr'}{dt}, \qquad u = \tfrac{1}{4}\mathrm{m}r^2\frac{d\psi + d\phi}{dt} - \tfrac{1}{4}\mathrm{m}'r'^2\frac{d\psi - d\phi}{dt}$$

$$w = \tfrac{1}{4}\mathrm{m}r^2\frac{d\psi + d\phi}{dt} + \tfrac{1}{4}\mathrm{m}'r'^2\frac{d\psi - d\phi}{dt}$$

The derivatives being eliminated from T by means of these equations,

$$T = \frac{1}{2\mathrm{m}}\left[\left(\frac{w + u}{r}\right)^2 + s^2\right] + \frac{1}{2\mathrm{m}'}\left[\left(\frac{w - u}{r'}\right)^2 + s'^2\right]$$

The potential function in terms of the variables last adopted is

$$\Omega = \frac{Mm}{\sqrt{r^2 + 2\varkappa rr'\cos\phi + \varkappa^2 r'^2}} + \frac{Mm'}{\sqrt{r'^2 + 2\varkappa rr'\cos\phi + \varkappa^2 r^2}} + \frac{\dfrac{mm'}{1 - \varkappa}}{\sqrt{r'^2 - 2rr'\cos\phi + r^2}}$$

Putting F for $\Omega - T$, the differential equations of the problem are

$$\frac{ds}{dt} = \frac{\partial F}{\partial r}, \quad \frac{ds'}{dt} = \frac{\partial F}{\partial r'}, \quad \frac{du}{dt} = \frac{\partial F}{\partial \phi}, \quad \frac{dw}{dt} = \frac{\partial F}{\partial \psi}$$

$$\frac{dr}{dt} = -\frac{\partial F}{\partial s}, \quad \frac{dr'}{dt} = -\frac{\partial F}{\partial s'}, \quad \frac{d\phi}{dt} = -\frac{\partial F}{\partial u}, \quad \frac{d\psi}{dt} = -\frac{\partial F}{\partial w}$$

But F does not contain ψ, hence $\frac{dw}{dt} = 0$ and $w = h$ a constant. This value may be substituted in T, and thus

$$T = \frac{1}{2m}\left[\left(\frac{h+u}{r}\right)^2 + s^2\right] + \frac{1}{2m'}\left[\left(\frac{h-u}{r'}\right)^2 + s'^2\right]$$

Employing this in F the equations of the problem are the independent system of six:

$$\frac{ds}{dt} = \frac{\partial F}{\partial r}, \quad \frac{ds'}{dt} = \frac{\partial F}{\partial r'}, \quad \frac{du}{dt} = \frac{\partial F}{\partial \phi}$$

$$\frac{dr}{dt} = -\frac{\partial F}{\partial s}, \quad \frac{dr'}{dt} = -\frac{\partial F}{\partial s'}, \quad \frac{d\phi}{dt} = -\frac{\partial F}{\partial u}$$

with the equation $w = h$, and the equation (to be treated by a quadrature)

$$\frac{d\psi}{dt} = -\frac{\partial F}{\partial h}$$

We have still one more integral of the problem, viz., $F = C$ a constant, and t is not explicitly involved in the equations. Thus we may dispense with t as the independent variable and employ some other in its place. It is well known that Gyldén's aim was to determine the radius of each planet as a function of its longitude, and thus he adopts the latter as the independent variable. But we are almost necessitated to have only one independent variable through the whole treatment of the problem. It will be advantageous to select a variable already contained in the equations. The only one suitable appears to be ϕ or the elongation of the hypothetical planets; this like t can be regarded as passing from $-\infty$ to $+\infty$, and $\frac{d\phi}{dt}$ never vanishes.* It will be seen that Ω involves this variable through the function $\cos\phi$, hence, no elaboration of this factor is needed with the proposed choice.

By division of the differential equations we obtain

$$\frac{ds}{d\phi} = -\frac{\frac{\partial F}{\partial r}}{\frac{\partial F}{\partial u}}, \quad \frac{ds'}{d\phi} = -\frac{\frac{\partial F}{\partial r'}}{\frac{\partial F}{\partial u}}, \quad \frac{du}{d\phi} = -\frac{\frac{\partial F}{\partial \phi}}{\frac{\partial F}{\partial u}}$$

$$\frac{dr}{d\phi} = -\frac{\frac{\partial F}{\partial s}}{\frac{\partial F}{\partial u}}, \quad \frac{dr'}{d\phi} = -\frac{\frac{\partial F}{\partial s'}}{\frac{\partial F}{\partial u}}, \quad \frac{dt}{d\phi} = -\frac{1}{\frac{\partial F}{\partial u}}$$

* The Julian year being the unit of time, a rude computation has given 81211″ and 50122″ as the greatest and least values of $\frac{d\phi}{dt}$.

Also we may write

$$\frac{d\psi}{d\phi} = \frac{\dfrac{\partial F}{\partial h}}{\dfrac{\partial F}{\partial u}}$$

A simpler form may be given to these equations; by solving the equation $F = C$, u being regarded as the unknown, we arrive at

$$u = W \text{ a function of } r, r', s, s', \phi$$

Then we may write

$$\frac{ds}{d\phi} = \frac{\partial W}{\partial r}, \qquad \frac{ds'}{d\phi} = \frac{\partial W}{\partial r'}, \qquad \frac{du}{d\phi} = \frac{\partial W}{\partial \phi}$$

$$\frac{dr}{d\phi} = -\frac{\partial W}{\partial s}, \qquad \frac{dr'}{d\phi} = -\frac{\partial W}{\partial s'}, \qquad \frac{dt}{d\phi} = -\frac{\partial W}{\partial C}$$

to which may be added

$$\frac{d\psi}{d\phi} = -\frac{\partial W}{\partial h}$$

The four differential equations bearing on the variables r, r', s, s' constitute an independent system to be integrated by itself. Thus, r, r', s, s' will be got as functions of ϕ the independent variable. The remainder of the work on the problem consists, first, of a quadrature executed on the equation

$$\frac{dt}{d\phi} = -\frac{\partial W}{\partial C}$$

by means of which t will be obtained as a function of ϕ, and, by inversion, ϕ as a function of t; and, second, of a quadrature executed on the equation

$$\frac{d\psi}{d\phi} = -\frac{\partial W}{\partial h}$$

by which ψ will be had as a function of ϕ, and thence of t.

It will be seen that these operations introduce six additional arbitrary constants, which, with h and C, make up the eight demanded by the problem.

If it is thought undesirable to keep h and C evident in the expression of W, we can have recourse to the equations

$$\frac{d\phi}{dt} = -\frac{\partial F}{\partial u}, \qquad \frac{dv}{dt} = \frac{h + W}{\mathrm{m}r^2}, \qquad \frac{dv'}{dt} = \frac{h - W}{\mathrm{m}'r'^2}$$

We have now to consider the derivation of W. This is obtained from the solution of a quadratic. This quadratic is

$$\frac{(h + W)^2}{\mathrm{m}r^2} + \frac{(h - W)^2}{\mathrm{m}'r'^2} = 2(\Omega - C) - \frac{s^2}{\mathrm{m}} - \frac{s'^2}{\mathrm{m}'}$$

To simplify the solution of this we can put

$$W = hV, \qquad \frac{1}{\mathrm{m}r^2} = \rho^2 \cos^2 \upsilon, \qquad \frac{1}{\mathrm{m}'r'^2} = \rho^2 \sin^2 \upsilon$$

$$\frac{h+W}{\sqrt{\mathrm{m}r}} = \sqrt{2(\Omega - C) - \frac{s^2}{\mathrm{m}} - \frac{s'^2}{\mathrm{m}'}} \cos \nu$$

$$\frac{h-W}{\sqrt{\mathrm{m}'r'}} = \sqrt{2(\Omega - C) - \frac{s^2}{\mathrm{m}} - \frac{s'^2}{\mathrm{m}'}} \sin \nu$$

Whence may be derived

$$\sin(\upsilon + \nu) = \frac{h\rho \sin 2\upsilon}{\sqrt{2(\Omega - C) - \frac{s^2}{\mathrm{m}} - \frac{s'^2}{\mathrm{m}'}}}, \qquad W = h\frac{\sin(\upsilon - \nu)}{\sin(\upsilon + \nu)}$$

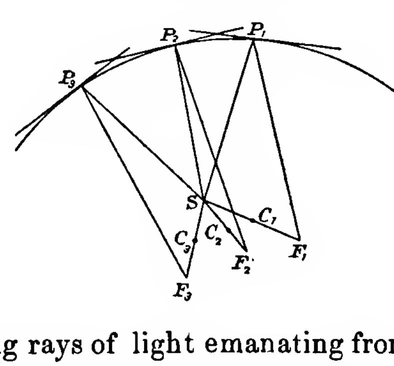
The quadratic in V can be given the form

$$V^2 + 2V\cos 2\upsilon + 1 = \frac{1}{h^2\rho^2}\left[2(\Omega - C) - \frac{s^2}{\mathrm{m}} - \frac{s'^2}{\mathrm{m}'}\right]$$

This equation will be more useful to us in deriving the value of V or W than the equation involving the auxiliary angle ν.

Since W involves no less than four square radicals, it is sufficiently plain that, with the four dependent variables r, r', s, s', nothing can be accomplished in the line of integration. We are therefore led to make a transformation of variables such that the radicals in the expression of W may be got rid of. Gyldén's notions relative to this step in the treatment of the planetary problem are valuable, but, what is singular, he has never given a philosophical presentation of them. We adopt the essential part of them, reserving the privilege of making extensive modifications in the remainder.

We bear in mind that it is always possible to study the form of orbits without regard to the question as to what particular points the planets occupy at stated times. To show what, in fact, is at the bottom of Gyldén's principles, the annexed diagram is drawn. Let the curve $P_1P_2P_3$ be a portion of the relative orbit of a planet about the sun S. Suppose we have it in our power to draw the tangents to the curve at the points P_1, P_2, P_3. These tangents may be regarded as the traces of mirrors perpendicular to the plane of the orbit, and SP_1, SP_2, SP_3 being rays of light emanating from the Sun let the directions of the reflected rays be P_1F_1, P_2F_2, P_3F_3. Next adopt a linear magnitude a, named the protometer by Gyldén, and take the points F so that generally, $SP + PF$ shall equal double this. The fixed point S being called the occupied focus of the curve, the points F_1, F_2, F_3



may be called the empty foci of the curve severally belonging to the points P_1, P_2, P_3 and correspondent to the protometer a. As the point P moves along the curve from P_1 through P_2 to P_3 the general empty focus F will move on a curve starting from F_1, passing through F_2 to F_3. As we have drawn the radii SP, so we may draw the radii SF. Note the general quotient

$$\frac{SF}{SP + PF}$$

of which the denominator is constant; this is called the eccentricity (Gyldén's diastem) of the orbit at the point P correspondent to protometer a. Also the direction of SF is that of the apsides at P correspondent to protometer a; and the difference of the directions SP and FS taken so as to augment with the motion of P is called the true anomaly (Gyldén's diastematic argument) at P correspondent to protometer a.

The properties of the orbit may then be studied in the path of the empty focus F. There is nothing which necessitates a determinate value for a, but practical considerations lead us to adopt a value making F move much more slowly than P. It is easy to see that a value may be adopted such that when P makes a movement of the order of the solar mass, F makes a movement of the order of the disturbing planetary masses. If we have no other information as to a proper value for a, we may use the semiaxis of the instantaneous ellipse which prevails at any moment, or half the sum of the radii at a perihelion and an aphelion passage if the latter are consecutive. The protometer is a superabundant constant; if it is left indeterminate in the integrals of the problem, on their substitution in the original differential equations, the latter will fail to be satisfied unless a condition is established enabling us to reduce the number of introduced constants by a unit. It is not necessary, however, that the eliminated constant should be a protometer; we may elect to remove one of the others.

In the theory of Jupiter and Saturn, by properly adopting a and a', the empty foci of the two orbits may be made to move so slowly that, omitting minor oscillations, they do not accomplish what may be called a relative revolution in their positions in less than 54000 years.

Precisely as we have had a protometer and empty focus for the planet's path, we may have similar things for the path of F. Here the protometer will generally be smaller than the first, and the path of the second order wholly contained within the path of the first F. In stable planetary motion it is to be expected that when the operation of establishing an empty focus is repeated many times, the movement of the last F may be small enough

to be neglected, and we thus shall have an empty focus as fixed as the occupied one S.

Limiting our attention to the empty focus of the first order, we see that when we have adopted a value for the protometer and know the position of the corresponding empty focus with the longitude of the planet, we know the position of the latter in space as well as its velocities, it being, of course, assumed that we know the value of W or V at the moment concerned. For convenience in graphic exhibition we have supposed in the diagram that we had the power of drawing the tangent to the planetary path at any point. It does not, at first sight, seem that the knowledge of the variable

$$s = \mathrm{m}\,\frac{dr}{dt}$$

could be tantamount to this; but when it is noted that

$$\frac{dr}{dt} = \frac{s}{\mathrm{m}}, \qquad \frac{dv}{dt} = h\,\frac{1+V}{\mathrm{m}r^2}$$

we see that

$$\frac{dr}{dv} = \frac{r^2 s}{h\,(1+V)} = \frac{r^2 s}{h+W}$$

Thus, if we know the values of s, h and W, the tangent can be drawn.

It is apparent from all this that the planet may be conceived to move at each moment in an ellipse with a constant major-axis, but with the eccentricity and the line of apsides in constant variation. The principle of the moving empty focus has been invoked chiefly to find a transformation of the four variables r, r', s, s' suitable for the purpose of enabling us to get rid of the square radicals which appear in W.

Let us call the protometer a, the eccentricity e, and the true anomaly f (Gyldén's symbols for the latter are η and F.) Then we propose to replace the four variables r, r', s, s' by the four f, f', e, e'. We immediately have

$$r = a\,\frac{1-e^2}{1+e\cos f}, \qquad r' = a'\,\frac{1-e'^2}{1+e'\cos f'}$$

and it only remains to consider what functions of f, f', e, e' we shall substitute for the variables s, s'. To this end we appeal to some properties of intermediate orbits. Let us put

$$\begin{aligned}
\Delta^2 &= r^2 + 2\kappa r r' \cos\phi + \kappa^2 r'^2 \\
\Delta'^2 &= r'^2 + 2\kappa r r' \cos\phi + \kappa^2 r^2 \\
\Delta''^2 &= r'^2 - 2 r r' \cos\phi + r^2
\end{aligned}$$

Then the potential function will have the expression

$$\Omega = \frac{Mm}{\Delta} + \frac{Mm'}{\Delta'} + \frac{\dfrac{mm'}{1-\varkappa}}{\Delta''}$$

For the intermediate orbits we may suppose that r takes the place of Δ, and r' the places of Δ' and Δ''. By putting

$$Mm = \mu \mathrm{m}\,, \qquad Mm' + \frac{mm'}{1-\varkappa} = \mu'\mathrm{m}'$$

we can write

$$\Omega = \frac{\mu \mathrm{m}}{r} + \frac{\mu'\mathrm{m}'}{r'} + R$$

where

$$R = -\,Mm\varkappa r'\,\frac{2r\cos\phi + \varkappa r'}{r\Delta\,(\Delta + r)} - Mm'\varkappa r\,\frac{2r'\cos\phi + \varkappa r}{r'\Delta'\,(\Delta' + r')} + \frac{mm'}{1-\varkappa}\,r\,\frac{2r'\cos\phi - r}{r'\Delta''\,(\Delta'' + r')}$$

Let the intermediate orbits be founded upon the potential

$$\Omega = \frac{\mu \mathrm{m}}{r} + \frac{\mu'\mathrm{m}'}{r'}$$

Then it is plain that the variables s and s' in these intermediate orbits will have expressions in terms of f, f', e, e', as follow:

$$s = \mathrm{m}\sqrt{\frac{\mu}{a}}\,\frac{e}{\sqrt{1-e^2}}\sin f, \qquad s' = \mathrm{m}'\sqrt{\frac{\mu'}{a'}}\,\frac{e'}{\sqrt{1-e'^2}}\sin f'$$

which are the functions we shall use.

Compute now the two Jacobians

$$\Gamma = \frac{\partial s}{\partial e}\frac{\partial r}{\partial f} - \frac{\partial s}{\partial f}\frac{\partial r}{\partial e}\,, \qquad \Gamma' = \frac{\partial s'}{\partial e'}\frac{\partial r'}{\partial f'} - \frac{\partial s'}{\partial f'}\frac{\partial r'}{\partial e'}$$

Then the differential equations in terms of the new variables will be

$$\Gamma\frac{de}{d\phi} = \frac{\partial W}{\partial f}\,, \qquad \Gamma\frac{df}{d\phi} = -\frac{\partial W}{\partial e}\,, \qquad \Gamma'\frac{de'}{d\phi} = \frac{\partial W}{\partial f'}\,, \qquad \Gamma'\frac{df'}{d\phi} = -\frac{\partial W}{\partial e'}$$

We have

$$\Gamma = \frac{\mathrm{m}\sqrt{\mu a}}{\sqrt{1-e^2}}\,\frac{e\sin^2 f + [2e + (1+e^2)\cos f]\,e\cos f}{(1+e\cos f)^2} = \frac{\mathrm{m}\sqrt{\mu a}\,e}{\sqrt{1-e^2}}$$

consequently

$$\frac{de}{d\phi} = \frac{1}{\mathrm{m}\sqrt{\mu a}}\,\frac{\sqrt{1-e^2}}{e}\,\frac{\partial W}{\partial f}\,, \qquad \frac{de'}{d\phi} = \frac{1}{\mathrm{m}'\sqrt{\mu' a'}}\,\frac{\sqrt{1-e'^2}}{e'}\,\frac{\partial W}{\partial f'}$$

$$\frac{df}{d\phi} = -\frac{1}{\mathrm{m}\sqrt{\mu a}}\,\frac{\sqrt{1-e^2}}{e}\,\frac{\partial W}{\partial e}\,, \qquad \frac{df'}{d\phi} = -\frac{1}{\mathrm{m}'\sqrt{\mu' a'}}\,\frac{\sqrt{1-e'^2}}{e'}\,\frac{\partial W}{\partial e'}$$

These equations have not the canonical form, but it is easy to reduce them to it. For brevity put

$$\frac{1}{\mathrm{m}\sqrt{\mu a}} = k\,, \qquad \frac{1}{\mathrm{m}'\sqrt{\mu' a'}} = k'$$

Then we adopt the variables η and η' of the order of the squares of the eccentricities to replace e and e' and such that

$$\eta = \frac{1}{k}\left(1 - \sqrt{1-e^2}\right), \qquad \eta' = \frac{1}{k'}\left(1 - \sqrt{1-e'^2}\right)$$

whence it follows that

$$e = \sqrt{2k\eta - k^2\eta^2}, \qquad e' = \sqrt{2k'\eta' - k'^2\eta'^2}$$

With this choice of variables we have to submit to the slight inconvenience of having half powers of η and η' in the expression of W. The differential equations become

$$\frac{d\eta}{d\phi} = \frac{\partial W}{\partial f}, \qquad \frac{d\eta'}{d\phi} = \frac{\partial W}{\partial f'}$$
$$\frac{df}{d\phi} = -\frac{\partial W}{\partial \eta}, \qquad \frac{df'}{d\phi} = -\frac{\partial W}{\partial \eta'}$$

The two protometers a and a' will be superabundant constants, but we can bring it about that there shall be only one such constant by supposing that a and a' are adopted not in an arbitrary manner, but so as to fulfil the equation

$$\frac{\mu \mathrm{m}}{a} + \frac{\mu' \mathrm{m}'}{a'} = 2C$$

Thus we eliminate C which is replaced by the two constants a and a'. This restriction does not impair the suitableness of these constants for our purpose, and it brings about a marked reduction in the complexity of the quadratic equation which determines W. Here, with profit, we may introduce the variable semi-parameters

$$p = a\,(1 - e^2) \quad , \quad p' = a'(1 - e'^2)$$

By putting the values

$$\frac{\mu \mathrm{m}}{r} = \frac{\mu \mathrm{m}}{p}(1 + e\cos f), \qquad \frac{s^2}{\mathrm{m}} = \frac{\mu \mathrm{m}}{p} e^2 \sin^2 f$$

into the expression

$$\frac{\mu \mathrm{m}}{r} - \frac{\mu \mathrm{m}}{2a} - \frac{s^2}{2\mathrm{m}}$$

it becomes

$$\frac{\mu \mathrm{m}}{p}[1 + e\cos f - \tfrac{1}{2}(1 - e^2) - \tfrac{1}{2} e^2 \sin^2 f] = \frac{\mu \mathrm{m} p}{2r^2}$$

Thus the equation for W takes the form

$$\frac{(h + W)^2}{\mathrm{m} r^2} + \frac{(h - W)^2}{\mathrm{m}' r'^2} = \frac{\mu \mathrm{m} p}{r^2} + \frac{\mu' \mathrm{m}' p'}{r'^2} + 2R$$

where the variables s and s' have been eliminated and replaced by p and p', or, what is the same thing, by e and e'. At first sight, it might appear possible to get rid of the arbitrary constant h by putting it equal to a function of the protometers, and thus escape having any superabundant constants. But h essentially depends on the moduli of the departure of the orbits from circularity, hence there is an incongruity in supposing that h depends on a and a'. After the integration is accomplished, we shall find that the differen-

tial equations are not satisfied unless a condition, which may be put in the form

$$h = \text{funct.}\ (a, a', e_0, e'_0)$$

is fulfilled. But we suppose that the numerical values of both h and C have been derived from observation before the investigation is commenced. It is apparent therefore that any incongruity in the values assigned to a and a' is simply thrown on the values of the constants e_0 and e'_0. As the values of the latter are supposed to be determined after the investigation is completed, we need not pay any attention to the matter.

Employing the preceding auxiliary quantities ρ and v, and putting

$$K = \frac{\mu \mathrm{m}^2 a}{h^2}(1 - e^2), \qquad K' = \frac{\mu' \mathrm{m}'^2 a'}{h^2}(1 - e'^2), \qquad X = \frac{2R}{h^2 \rho^2}$$

the quadratic for V takes the form

$$\begin{aligned} V^2 + 2V \cos 2v + 1 &= K \cos^2 v + K' \sin^2 v + X \\ &= \tfrac{1}{2}(K + K') + \tfrac{1}{2}(K - K') \cos 2v + X \end{aligned}$$

Let us make

$$A = \sqrt{\tfrac{1}{2}(K + K') - 1}, \qquad M = 2\left[\frac{K - K'}{4} - A\right]$$

Then

$$[V + \cos 2v]^2 = A^2 + (2A + M) \cos 2v + \cos^2 2v + X$$

If we put

$$N = A + \cos 2v \quad , \quad N^2 Q = M \cos 2v + X$$

we have

$$[V + \cos 2v]^2 = N^2 (1 + Q)$$

whence it follows that

$$V = A + \frac{NQ}{1 + \sqrt{1 + Q}}$$

The radical in this expression must have the positive sign.

The preceding formulas have been given such a shape that the greatest degree of accuracy may be attained by the use of logarithms of a definite number of decimals. We note that X is of the order of the planetary masses; and, in the case of Jupiter and Saturn, the numerical value of cos $2v$ is always less than $\frac{1}{11}$, and M of the order of the squares of the eccentricities, hence $M \cos 2v$ may be considered as of the same order as X. Thus Q, always within the limits ± 0.004, is of the same order. A is then quite an approximate value of V. The computation of the latter is facilitated by having a table of $\log \dfrac{2}{1 + \sqrt{1 + Q}}$ for small values of Q. We have preferred to derive V instead of W, because it is independent of the assumed linear and temporal units; to have W multiply V by the constant h.

It is proposed to develop V in series suitable for use in the further prosecution of the subject by the employment of special values. It will be found convenient to have the development in two forms. First, as a power series of four rectangular coordinates, so to speak, and second as a series of periodic terms depending on arguments whose constituents are ϕ, f and f'. It is comparatively easy to pass from one to the other of these forms. We prefer to attack the development by way of the first form. In the elaboration of this matter is seems a trifle easier to employ parameters somewhat different from those previously suggested. We adopt the four following:

$$e \cos f = x \quad , \quad e' \cos f' = x' \quad , \quad e^2 = y \quad , \quad e'^2 = y'$$

We shall then have

$$V = \Sigma \, . \, A x^i x'^{i'} y^j y'^{j'}$$

where the A are periodic functions of the independent variable ϕ, such that

$$A = C_0 + C_1 \cos \phi + C_2 \cos 2\phi + C_3 \cos 3\phi + \; . \; .$$

the C being constants. The object of the procedure is to discover the values of the C from the special values of V corresponding to chosen values of the five parameters x, x', y, y', ϕ.

The second or polar form for V may be given in terms of η and η' instead of e and e'; it is

$$V = \Sigma [A_{00} + A_{10}\eta + A_{01}\eta' + A_{20}\eta^2 + A_{11}\eta\eta' + A_{02}\eta'^2 + \, . \, .] \, \eta^{\left|\frac{i}{2}\right|} \eta'^{\left|\frac{i'}{2}\right|} \cos (j\phi + if + i'f')$$

where j may not receive negative integral values, while i and i' do. The A in this expression are constants whose numerical values result from the proposed method.

With the chosen parameters the radii have the expressions

$$r = a \frac{1 - y}{1 + x}, \qquad r' = a' \frac{1 - y'}{1 + x'}$$

Let us suppose that, in the considered development of V, all terms of an order greater than the eighth with respect to eccentricities may be neglected; and that the quantities A are to be pushed so as to stop with the term C_{15} cos 15ϕ; then it is evident that the number of constant coefficients C is

$$16 \; [1.5.9 + 2.4.7 + 3.3.5 + 4.2.3 + 5.1.1] = 2800$$

We shall thus be obliged to compute 2800 special values of V. But, not to be too greatly appalled at this, we see that very large portions of the computations involved are identical throughout certain groups in the 2800 values. In order to save labor we must arrange our work in such a manner that there are no virtual repetitions even of arithmetical operations. For instance, having to make our computations for the 16 values of ϕ, viz., 0°, 12°, 24°,

. . . 180°, we notice that the only way ϕ is involved in V is by the factor $\cos\phi$; hence, when ϕ is in the second quadrant, the terms having it as factor are to be got by negativing the corresponding terms when ϕ was in the first quadrant.

It is impossible to give here such a development of V as has just been described, nevertheless I propose to exemplify the process by giving some details for the special value $\phi = 60°$. We must then compute 175 values of V. First, it is necessary to mention the values adopted for the masses, the two protometers and the two constants C and h added severally to the equations of living forces and conservation of areas. Let the Julian year be the unit of time, and the Earth's mean distance from the Sun the linear unit. Let us assume the data:

	Mass in terms of Sun's mass	n	log (n in terms of the radian)
Earth	$\frac{1}{328000}$	1295977″.4238	0.798 1723 029
Jupiter	$\frac{1}{1047.355}$	109256″.61518	9.724 0226 085
Saturn	$\frac{1}{3501.6}$	43996″.08754	9.328 9889 243

Whence follow the values of the logarithms of the masses:

$$\log M = 1.596\,3432\,817 \quad , \quad \log m = 8.576\,2493\,713$$
$$\log m' = 8.052\,0767\,483$$

Thence are derived the values of the constants employed in the preceding:

$$\log \mu_1 = 8.575\,8350\,290 \quad , \quad \log \mu_2 = 8.051\,9528\,568$$
$$\log \mu^3 = 5.031\,4444\,859 \quad , \quad \log \kappa \ = 6.341\,8974\,798$$
$$\log \mathrm{m} = 8.575\,8349\,808 \quad , \quad \log \mathrm{m}' = 8.051\,9527\,448$$
$$\log \mu \ = 1.596\,7576\,722 \quad , \quad \log \mu' \ = 1.596\,8818\,368$$

To get the values of a and a', we have the equation already agreed upon,

$$\frac{\mu \mathrm{m}}{a} + \frac{\mu' \mathrm{m}'}{a'} = 2C$$

A discussion of ephemerides, derived from the New Tables of Jupiter and Saturn, gives

$$2C = 0.33268\,25845$$

It may be arbitrarily assumed that the ratio may be obtained from the equation

$$\alpha^3 = \frac{a^3}{a'^3} = \frac{\mu}{\mu'} \frac{n'^2}{n^2}$$

Thence $\log \alpha = 9.7366028\,224$; and the two equations combined give

$$\log a = 0.716\,2344\,631 \quad , \quad \log a' = 0.979\,6316\,407$$

From the same discussion of ephemerides we get

$$h = 0.37893\,10781 \quad , \quad \log h = 9.578\,5602\,254$$

Having now the values of the necessary constants, the formulas for the

special value $\phi = 60°$ may be set down. The special values of x and x' are selected from the arithmetical progression $-0.08, -0.06, \ldots, 0.06, 0.08$; those of y and y' from the progression $-0.0050, -0.0025, 0.0000, 0.0025, 0.0050$. Modifying the significations of r and Δ, the following formulas which involve constants are given (the numbers within brackets are common logarithms):

$$r = \frac{1-y}{1+x}, \qquad r' = \frac{1-y'}{1+x'}$$

$$\Delta^2 = 1 + 0.00040\,29904 \frac{r'}{r} + 0.00000\,01624 \frac{r'^2}{r^2}$$

$$\Delta'^2 = 1 + 0.00011\,98120 \frac{r}{r'} + 0.00000\,00144 \frac{r^2}{r'^2}$$

$$\Delta''^2 = 1 - 0.54525\,89745 \frac{r}{r'} + 0.29730\,73492 \frac{r^2}{r'^2}$$

$$\rho^2 = 0.98106\,11839 \frac{1}{r^2} + 0.97450\,47801 \frac{1}{r'^2}$$

$$\rho^2 \cos 2\upsilon = 0.98106\,11839 \frac{1}{r^2} - 0.97450\,47801 \frac{1}{r'^2}$$

$$X = \frac{-[7.205\,5623\,922]}{\rho^2} \frac{r'}{r^2} \frac{1+0.00040\,29904 \frac{r'}{r}}{\Delta(\Delta+1)}$$

$$\frac{-[5.891\,1982\,364]}{\rho^2} \frac{r}{r'^2} \frac{1+0.00011\,98120 \frac{r}{r'}}{\Delta'(\Delta'+1)}$$

$$\frac{+[6.529\,3022\,860]}{\rho^2} \frac{r}{r'^2} \frac{1-0.54525\,89745 \frac{r}{r'}}{\Delta''(\Delta''+1)}$$

The following table gives the values of A and $\log M$ for the only combinations of the values of y and y' that are used; d' represents 0.0025.

Arguments $\frac{y}{d}$	$\frac{y'}{d'}$	A	$\log M$
0	0	.42653 77301 599	7.680 9425 816n
—2	0	.43244 64917 631	8.062 1565 242n
—1	0	.42950 22721 059	7.913 1787 711n
1	0	.42355 24392 043	7.134 7777 062n
2	0	.42054 59573 717	7.324 5524 816
0	—2	.42751 44143 044	7.879 9102 207n
0	—1	.42702 63514 632	7.791 7616 788n
0	1	.42604 85484 730	7.531 6407 641n
0	2	.42555 88044 701	7.302 0514 923n
—1	—1	.42998 75246 308	7.981 1656 474n
1	1	.42305 98056 562	5.584 9170 982
2	1	.42004 97978 427	7.546 6291 400
1	2	.42256 65977 901	7.158 9463 351
—1	1	.42901 64707 212	7.832 4670 973n
1	—1	.42404 45004 359	7.441 7072 155n

The 175 values of V and of the derived function G which serve better for the determination of the coefficients follow; all are to 13 places of decimals; the horizontal lines delimit the 16 groups; the first G is omitted as it is identical with the corresponding V.*

No.	Argument i	i	i	i	V	G
1	0	0	0	0	.42607 46965 557	. . . . . .
2	—4	0	0	0	.42658 69065 999	— 12 80525 110$\frac{1}{2}$
3	—3	0	0	0	.42643 33658 278	— 11 95564 240$\frac{1}{3}$
4	—2	0	0	0	.42629 88798 411	— 11 20916 427
5	—1	0	0	0	.42618 01784 162	— 10 54818 605
6	1	0	0	0	.42598 03943 278	— 9 43022 279
7	2	0	0	0	.42589 56291 947	— 8 95336 805
8	3	0	0	0	.42581 90638 313	— 8 52109 081$\frac{1}{3}$
9	4	0	0	0	.42574 95984 151	— 8 12745 351$\frac{1}{2}$
10	0	—4	0	0	.42569 15912 989	+ 9 57763 142
11	0	—3	0	0	.42577 87633 688	+ 9 86443 956$\frac{1}{3}$
12	0	—2	0	0	.42587 12863 433	+ 10 17051 062
13	0	—1	0	0	.42596 97198 701	+ 10 49766 856
14	0	1	0	0	.42618 69350 754	+ 11 22385 197
15	0	2	0	0	.42630 72560 656	+ 11 62797 549$\frac{1}{2}$
16	0	3	0	0	.42643 66015 374	+ 12 06349 939
17	0	4	0	0	.42657 60587 540	+ 12 53405 495$\frac{3}{4}$
18	0	0	—2	0	.43202 73981 346	—297 63507 894$\frac{1}{2}$
19	0	0	—1	0	.42905 14744 700	—297 67779 143
20	0	0	1	0	.42309 71356 838	—297 75608 719
21	0	0	2	0	.42011 88635 244	—297 79165 156$\frac{1}{2}$
22	0	0	0	—2	.42699 90167 611	— 46 21601 027
23	0	0	0	—1	.42654 09550 516	— 46 62584 959
24	0	0	0	1	.42560 00405 323	— 47 46560 234
25	0	0	0	2	.42511 67793 591	— 47 89585 983
26	—1	—1	0	0	.42606 56238 373	— 95778 933
27	1	1	0	0	.42608 34132 997	— 92195 478
28	—1	1	0	0	.42630 31492 094	— 1 07322 735
29	1	—1	0	0	.42588 36915 790	— 82739 368
30	—2	—1	0	0	.42617 31408 338	— 1 03811 608$\frac{1}{2}$
31	2	1	0	0	.42599 06615 124	— 86031 010
32	—2	1	0	0	.42643 44613 541	— 1 16714 966$\frac{1}{2}$
33	2	—1	0	0	.42580 61300 834	— 77387 871$\frac{1}{2}$
34	—1	—2	0	0	.42595 86133 903	— 90774 067$\frac{1}{2}$
35	1	2	0	0	.42619 34273 162	— 97632 607$\frac{1}{2}$

*The reader is referred to *A. J.*, No. 567, and *Amer. Jour. Math.*, Vol. XXVII, for further explanation. Memoirs Nos. 77, 78.

No.	Argument i	i	i	i	V		G
36	−1	2	0	0	.42643 55401 477	−	1 14011 108
37	1	−2	0	0	.42579 27061 368	−	78610 107
38	−2	−2	0	0	.42605 61728 808	−	98241 869¾
39	2	2	0	0	.42609 17946 109	−	90985 234¼
40	−2	2	0	0	.42658 11299 314	−	1 24226 451
41	2	−2	0	0	.42572 16599 647	−	73602 456
42	−3	−1	0	0	.42629 44381 336	−	1 13170 028⅔
43	3	1	0	0	.42590 71252 144	−	80590 455⅓
44	−3	1	0	0	.42658 39221 886	−	1 27726 137
45	3	−1	0	0	.42573 58809 032	−	72645 858⅓
46	−1	−3	0	0	.42585 83853 549	−	86199 581⅓
47	1	3	0	0	.42631 12095 383	−	1 03632 570⅔
48	−1	3	0	0	.42657 85136 233	−	1 21434 084⅔
49	1	−3	0	0	.42570 69070 294	−	74819 628⅓
50	3	2	0	0	.42600 05443 053	−	85131 726½
51	2	3	0	0	.42619 96703 738	−	96439 671
52	4	1	0	0	.42583 15338 857	−	75757 622¾
53	1	4	0	0	.42643 76450 170	−	1 10278 772¾
54	0	0	−1	−1	.42950 85717 572	−	91612 087
55	0	0	1	1	.42261 31404 584	−	93392 020
56	0	0	−1	1	.42858 61411 708	−	93227 242
57	0	0	1	−1	.42357 25708 394	−	97766 597
58	0	0	2	1	.41962 55137 645	−	93468 682½
59	0	0	1	2	.42212 03727 050	−	94228 911
60	−1	0	−1	0	.42924 09891 117	+	8 40327 812
61	1	0	1	0	.42307 81172 646	+	7 52838 087
62	−1	0	1	0	.42311 83237 274	+	8 42938 169
63	1	0	−1	0	.42888 20629 923	+	7 51092 498
64	−2	0	−1	0	.42945 42659 496	+	8 93040 971
65	2	0	1	0	.42306 09573 591	+	7 14445 181½
66	−2	0	1	0	.42314 20669 271	+	8 96260 210½
67	2	0	−1	0	.42872 98052 077	+	7 13009 506½
68	1	0	2	0	.42017 52952 802	+	7 53669 918½
69	−1	0	2	0	.42005 55058 656	+	8 44197 596½
70	3	0	1	0	.42304 54020 198	+	6 79663 534⅔
71	−3	0	1	0	.42316 88492 105	+	9 56519 151⅓
72	0	−1	−1	0	.42886 99701 766	−	7 65276 078
73	0	1	1	0	.42312 68659 512	−	8 25082 523
74	0	−1	1	0	.42306 88639 735	−	7 67049 753
75	0	1	−1	0	.42924 59664 401	−	8 22534 504
76	0	−2	−1	0	.42870 02053 051	−	7 39294 762½
77	0	2	1	0	.42315 82494 227	−	8 57228 855
78	0	−2	1	0	.42304 18786 785	−	7 40766 035½

No.	Argument i	i	i	i	V		G
79	0	2	—1	0	.42945 48709 669	—	8 54184 935
80	0	1	2	0	.42006 58396 977	—	8 26311 732
81	0	—1	2	0	.42016 74658 131	—	7 67894 871½
82	0	3	1	0	.42319 15078 648	—	8 91776 002⅓
83	0	—3	1	0	.42301 60263 708	—	7 16079 579⅔
84	—1	0	0	—1	.42667 87841 359	+	3 23472 238
85	1	0	0	1	.42553 54568 960	+	2 97185 916
86	—1	0	0	1	.42567 22868 258	+	3 32355 670
87	1	0	0	—1	.42641 76925 208	+	2 89603 029
88	—2	0	0	—1	.42683 38236 934	+	3 43426 782
89	2	0	0	1	.42547 74072 869	+	2 82170 578
90	—2	0	0	1	.42575 35990 131	+	3 53124 023
91	2	0	0	—1	.42630 68649 045	+	2 75113 930½
92	1	0	0	2	.42508 26989 054	+	3 01108 871
93	—1	0	0	2	.42515 48692 123	+	3 36960 036½
94	3	0	0	1	.42542 49737 198	+	2 68553 039⅔
95	—3	0	0	1	.42584 57440 965	+	3 76552 359⅔
96	0	—1	0	—1	.42640 64709 066	—	2 95074 594
97	0	1	0	1	.42567 97523 215	—	3 25267 305
98	0	—1	0	1	.42552 53433 701	—	3 02795 234
99	0	1	0	—1	.42668 48515 369	—	3 16579 656
100	0	—2	0	—1	.42628 04966 680	—	2 85290 856
101	0	2	0	1	.42576 50630 001	—	3 37685 210½
102	0	—2	0	1	.42545 51492 114	—	2 92594 457½
103	0	2	0	—1	.42683 92013 805	—	3 28434 095
104	0	1	0	2	.42516 30638 533	—	3 29770 127½
105	0	—1	0	2	.42507 31605 566	—	3 06789 415½
106	0	3	0	1	.42585 66437 708	—	3 51005 810⅔
107	0	—3	0	1	.42538 90076 639	—	2 83001 061
108	1	1	1	0	.42310 58853 156	+	72573 314
109	—1	1	1	0	.42315 03254 566	+	84608 117
110	—1	—1	1	0	.42308 80287 354	+	75546 116
111	1	—1	1	0	.42305 16036 942	+	65157 969
112	2	1	1	0	.42308 70161 097	+	67673 426
113	—2	1	1	0	.42317 67229 171	+	92086 353½
114	—2	—1	1	0	.42310 94231 961	+	81951 505
115	2	—1	1	0	.42303 59835 544	+	60898 343½
116	1	2	1	0	.42313 50711 015	+	76833 097½
117	—1	2	1	0	.42318 42690 634	+	89853 122½
118	—1	—2	1	0	.42305 92346 103	+	71613 508½
119	1	—2	1	0	.42302 61988 561	+	61917 123
120	1	1	—1	0	.42906 01583 824	+	71770 322

No.	Argument i	i	i	i	V		G
121	—1	1	—1	0	.42945 45641 449	+	83507 896
122	—1	—1	—1	0	.42904 24359 261	+	74710 089
123	1	—1	—1	0	.42871 52871 682	+	64545 325
124	2	2	1	0	.42311 42982 545	+	71553 125½
125	3	1	1	0	.42306 99606 090	+	63351 528
126	1	3	1	0	.42316 58611 180	+	81538 145⅓
127	1	1	2	0	.42012 76460 949	+	72970 945½
128	2	1	—1	0	.42889 36957 902	+	66975 928
129	1	2	—1	0	.42925 07517 229	+	75906 224
130	1	1	0	1	.42560 87731 296	+	28239 922
131	—1	1	0	1	.42575 94463 515	+	32845 370
132	—1	—1	0	1	.42559 09517 703	+	29400 000
133	1	—1	0	1	.42546 64928 980	+	25407 726
134	2	1	0	1	.42554 51840 905	+	26356 082
135	—2	1	0	1	.42584 95142 874	+	35697 541
136	—2	—1	0	1	.42566 45095 958	+	31800 333
137	2	—1	0	1	.42541 34343 104	+	23766 943
138	1	2	0	1	.42568 69244 747	+	29858 160
139	—1	2	0	1	.42585 31449 857	+	34832 647½
140	—1	—2	0	1	.42551 48200 089	+	27896 587½
141	1	—2	0	1	.42540 14542 605	+	24166 680
142	1	1	0	—1	.42654 96567 531	+	27127 051
143	—1	1	0	—1	.42683 65607 230	+	31478 283
144	—1	—1	0	—1	.42653 18979 640	+	28241 336
145	1	—1	0	—1	.42629 39283 270	+	24460 144
146	2	2	0	1	.42561 71694 733	+	27834 530¾
147	3	1	0	1	.42548 79163 701	+	24693 325⅔
148	1	3	0	1	.42577 04609 992	+	31635 453
149	1	1	0	2	.42512 55279 918	+	28820 700
150	2	1	0	—1	.42642 84864 186	+	25343 846
151	1	2	0	—1	.42669 06826 634	+	28648 324
152	1	0	1	1	.42262 47874 148	+	9467 885
153	—1	0	1	1	.42259 99658 292	+	11271 013
154	—1	0	—1	1	.42874 35361 705	+	11159 250
155	1	0	—1	1	.42844 55092 763	+	9390 084
156	1	0	1	—1	.42352 36690 008	+	9230 923
157	—1	0	1	—1	.42362 72033 168	+	10972 342
158	2	0	1	1	.42263 51464 200	+	8750 876
159	—2	0	1	1	.42258 49628 601	+	12420 162½
160	0	1	1	1	.42260 92414 414	—	11025 497
161	0	—1	1	1	.42261 61124 863	—	9642 193
162	0	—1	—1	1	.42843 39600 940	—	9563 068

No.	Argument. i	i	i	i	V		G
163	0	1	—1	1	.42874 91980 464	—	10916 360
164	0	1	1	—1	.42363 50324 033	—	10733 551
165	0	—1	1	—1	.42351 38515 611	—	9400 844
166	0	2	1	1	.42260 43481 118	—	11845 194½
167	0	—2	1	1	.42261 82134 327	—	9055 463
168	1	1	1	1	.42262 19060 681	+	1558 900
169	—1	1	1	1	.42259 48582 121	+	1955 294
170	—1	—1	1	1	.42260 40168 769	+	1622 880
171	1	—1	1	1	.42262 68459 023	+	1309 122
172	1	1	—1	1	.42859 48403 716	+	1531 925
173	1	1	1	—1	.42358 13051 049	+	1505 625
174	2	1	1	1	.42263 31294 705	+	1411 817
175	1	2	1	1	.42261 81500 213	+	1716 091½

From the preceding data is derived the following development of V for the special case of $\phi = 60°$. It is given in both of the forms,

$$V = \Sigma A \left(\frac{x}{d}\right)^i \left(\frac{x'}{d}\right)^i \left(\frac{y}{d'}\right)^i \left(\frac{y'}{d'}\right)^i, \qquad V = \Sigma' A\, x^i x'^i y^i y'^i$$

as the first more readily than the second, enables us to see how well the development represents the function in the region played over by the special values of the parameters. The coefficients of the first form are in units of the 13th decimal, and the fraction is appended, so that if substituted in the linear equations they should rigorously reproduce the special values of V. In the second form the coefficients are carried to such a number of decimals as the case seems to warrant.

Fact		A		A
			+	0.42607 46965 567
x	—	9 95887 358 $\frac{89}{150}$	—	0.00497 94367 94
x^2	+	55734 500 $\frac{129}{280}$	+	0.01393 36251 1
x^3	—	3024 283 $\frac{43}{45}$	—	0.03780 35494
x^4	+	163 188 $\frac{431}{480}$	+	0.10199 306
x^5	—	8 771 $\frac{289}{720}$	—	0.27410 63
x^6	+	472 $\frac{19}{120}$	+	0.73775
x^7	—	27 $\frac{4009}{5040}$	—	2.1715
x^8	+	1 $\frac{541}{1120}$	+	5.793
x'	+	10 84799 091 $\frac{11}{840}$	+	0.00542 39954 55
x'^2	+	36266 880 $\frac{5233}{10080}$	+	0.00906 67201 3
x'^3	+	1275 486 $\frac{3}{32}$	+	0.01594 3576
x'^4	+	42 241 $\frac{841}{1152}$	+	0.02640 108
x'^5	+	1 447 $\frac{29}{40}$	+	0.04524 14
x'^6	+	48 $\frac{577}{2880}$	+	0.07531

Fact		A^*		A
x'^7	+	1 $\frac{449}{672}$	+	0.1303
x'^8	+	$\frac{2035}{40320}$	+	0.197
y	—	297 71813 066 $\frac{1}{6}$	—	1.19087 25226
y^2	—	3914 945 $\frac{1}{2}$	—	0.06263 9128
y^3	+	119 135 $\frac{1}{6}$	+	0.76246 51
y^4	+	157 $\frac{1}{2}$	+	0.40320
y'	—	47 04232 293 $\frac{2}{3}$	—	0.18816 92917
y'^2	—	41984 770 $\frac{1}{3}$	—	0.67175 6325
y'^3	—	340 302 $\frac{5}{6}$	—	2.17793 81
y'^4	—	2 867 $\frac{1}{6}$	—	7.33994
$x\ x'$	—	93780 675 $\frac{11863}{12600}$	—	0.02344 51690
$x\ x'^2$	—	5193 965 $\frac{61}{80}$	—	0.06492 4563
$x\ x'^3$	—	243 845 $\frac{3037}{7200}$	—	0.15253 026
$x\ x'^4$	—	10 525 $\frac{281}{288}$	—	0.32893 7
$x\ x'^5$	—	433 $\frac{209}{288}$	—	0.6777
$x\ x'^6$	—	18 $\frac{121}{288}$	—	1.439
$x\ x'^7$	—	$\frac{3007}{5040}$	—	2.3
$x^2\ x'$	+	6979 301 $\frac{71}{720}$	+	0.08724 1264
$x^2\ x'^2$	+	516 869 $\frac{121}{720}$	+	0.32304 323
$x^2\ x'^3$	+	30 598 $\frac{7}{288}$	+	0.95618 8
$x^2\ x'^4$	+	1 590 $\frac{265}{288}$	+	2.4858
$x^2\ x'^5$	+	80 $\frac{141}{160}$	+	6.319
$x^2\ x'^6$	+	3 $\frac{421}{480}$	+	15.1
$x^3\ x'$	—	478 942 $\frac{53}{72}$	—	0.29933 92
$x^3\ x'^2$	—	44 994 $\frac{23}{144}$	—	1.40606 7
$x^3\ x'^3$	—	3 190 $\frac{77}{240}$	—	4.9849
$x^3\ x'^4$	—	203 $\frac{67}{144}$	—	15.90
$x^3\ x'^5$	—	10 $\frac{71}{720}$	—	39.4
$x^4\ x'$	+	31 293 $\frac{5}{144}$	+	0.97790 7
$x^4\ x'^2$	+	3 418 $\frac{43}{48}$	+	5.3420
$x^4\ x'^3$	+	300 $\frac{59}{72}$	+	23.50
$x^4\ x'^4$	+	21 $\frac{7}{36}$	+	82.8
$x^5\ x'$	—	2 008 $\frac{173}{900}$	—	3.1378
$x^5\ x'^2$	—	270 $\frac{13}{60}$	—	21.111
$x^5\ x'^3$	—	13 $\frac{47}{50}$	—	54.5
$x^6\ x'$	+	131 $\frac{77}{120}$	+	10.284
$x^6\ x'^2$	+	18 $\frac{17}{18}$	+	74.0
$x^7\ x'$	—	7 $\frac{1327}{2520}$	—	29.4
$y\ y'$	—	92493 335 $\frac{1}{2}$	—	1.47989 337
$y\ y'^2$	—	810 144 $\frac{1}{2}$	—	5.18492 48
$y^2\ y'$	—	79 822	—	0.51086 08
$y^2\ y'^2$	—	2 567	—	6.57152
$y\ y'^3$	—	8 059 $\frac{5}{6}$	—	20.6332
$y^3\ y'$	+	1 908 $\frac{5}{6}$	+	4.8866
$x\ y$	+	7 94380 088 $\frac{23}{40}$	+	1.58876 0177

Fact		A		A
$x\ y^2$	+	1064 072 $\frac{11}{24}$	+	0.85125 80
$x\ y^3$	−	14 452 $\frac{1}{3}$	−	4.62475
$x^2\ y$	−	44703 924 $\frac{443}{720}$	−	4.47039 246
$x^2\ y^2$	−	213 940 $\frac{13}{16}$	−	8.55763
$x^2\ y^3$	+	798	+	12.77
$x^3\ y$	+	2426 062 $\frac{3}{4}$	+	12.13031 37
$x^3\ y^2$		24 914 $\frac{1}{24}$	+	49.8281
$x^4\ y$	−	130 322 $\frac{109}{144}$	−	32.58067
$x^4\ y^2$	−	2 251 $\frac{3}{16}$	−	225.12
$x^5\ y$	+	7 442 $\frac{61}{120}$	+	93.0314
$x^6\ y$	−	399 $\frac{113}{180}$	−	249.77
$x'\ y$	−	7 94043 613 $\frac{37}{60}$	−	1.58807 2272
$x'\ y^2$	−	1064 265 $\frac{1}{4}$	−	0.85141 22
$x'\ y^3$	+	14 419 $\frac{11}{12}$	+	4.61457
x'^2y	−	28791 354 $\frac{29}{180}$	−	2.87913 542
x'^2y^2	−	192 587 $\frac{17}{24}$	−	7.70351
x'^2y^3	+	513 $\frac{7}{12}$	+	8.22
x'^3y	−	955 406 $\frac{5}{8}$	−	4.77703 3
x'^3y^2	−	16 158 $\frac{1}{4}$	−	32.316
x'^4y	−	31 920 $\frac{17}{18}$	−	7.98024
x'^4y^2	−	998 $\frac{7}{24}$	−	99.83
x'^5y	−	1 114 $\frac{7}{40}$	−	13.927
x'^6y	−	37 $\frac{43}{90}$	−	22.17
$x\ y'$	+	3 09681 436 $\frac{13}{24}$	+	0.61936 2873
$x\ y'^2$	+	4092 615 $\frac{5}{8}$	+	3.27409 25
$x\ y'^3$	+	49 027	+	15.68864
$x^2\ y'$	−	17205 607 $\frac{5}{16}$	−	1.72056 07
$x^2\ y'^2$	−	323 490 $\frac{13}{48}$	−	12.93961
$x^2\ y'^3$	−	5 189 $\frac{5}{6}$	−	83.037
$x^3\ y'$	+	920 969 $\frac{5}{24}$	+	4.60484 6
$x^3\ y'^2$	+	23 964 $\frac{1}{8}$	+	47.9282
$x^4\ y'$	−	48 800 $\frac{31}{48}$	−	12.2002
$x^4\ y'^2$	−	1 645 $\frac{47}{48}$	−	16.460
$x^5\ y'$	+	2 780 $\frac{1}{2}$	+	34.756
$x^6\ y'$	−	142 $\frac{23}{24}$	−	89.35
$x'\ y'$	−	3 09524 667 $\frac{19}{30}$	−	0.61904 9335
$x'\ y'^2$	−	4089 869 $\frac{11}{12}$	−	3.27165 59
$x'\ y'^3$	−	48 809 $\frac{11}{12}$	−	15.61917
x'^2y'	−	10978 473 $\frac{151}{240}$	−	1.09784 74
$x'^2y'^2$	−	241 189 $\frac{11}{12}$	−	9.64759
$x'^2y'^3$	−	4 189 $\frac{5}{12}$	−	67.031
x'^3y'	−	355 319 $\frac{5}{6}$	−	1.77659 9
$x'^3y'^2$	−	12 202 $\frac{1}{3}$	−	24.4047
x'^4y'	−	11 611 $\frac{2}{3}$	−	2.90292
$x'^4y'^2$	−	562 $\frac{1}{3}$	−	5.623

Fact		A		A
x'^5y'	—	399 $\frac{13}{15}$	—	4.998
x'^6y'	—	8 $\frac{43}{80}$	—	5.41
$x\ x'\ y$	+	73449 149 $\frac{19}{20}$	+	7.34491 499
$x\ x'^2y$	+	4018 099 $\frac{5}{16}$	+	20.09049 7
$x\ x'^3y$	+	206 947 $\frac{41}{72}$	+	51.7369
$x\ x'^4y$	+	8 370 $\frac{11}{16}$	+	104.634
$x\ x'^5y$	—	1 404 $\frac{17}{60}$	—	877.7
$x^2\ x'\ y$	—	5490 364 $\frac{3}{4}$	—	27.45182 4
$x^2\ x'^2y$	—	398 400 $\frac{11}{16}$	—	99.6002
$x^2\ x'^3y$	—	24 455 $\frac{1}{24}$	—	305.688
$x^2\ x'^4y$	—	1 263 $\frac{31}{48}$	—	789.8
$x^3\ x'\ y$	+	405 964 $\frac{17}{18}$	+	101.49123
$x^3\ x'^2y$	+	36 048 $\frac{3}{4}$	+	450.609
$x^3\ x'^3y$	—	5 648 $\frac{7}{36}$	—	3530.1
$x^4\ x'\ y$	—	25 841 $\frac{5}{24}$	—	323.015
$x^4\ x'^2y$	—	2 768 $\frac{11}{12}$	—	1730.6
$x^5\ x'\ y$	—	1 328 $\frac{3}{20}$	—	830.1
$x\ x'\ y^2$	+	395 042 $\frac{5}{6}$	+	15.80171
$x\ x'^2y^2$	+	56 817 $\frac{3}{4}$	+	113.635
$x\ x'^3y^2$	+	9 569 $\frac{1}{6}$	+	956.92
$x^2\ x'\ y^2$	—	65 076 $\frac{1}{2}$	—	130.153
$x^2\ x'^2y^2$	—	9 230 $\frac{3}{4}$	—	923.07
$x^3\ x'\ y^2$	+	14 373 $\frac{1}{2}$	+	1437.35
$x\ x'\ y^3$	—	1 288 $\frac{1}{6}$	—	20.611
$x\ x'\ y'$	+	28165 660 $\frac{1}{3}$	+	2.81656 603
$x\ x'^2y'$	+	1502 262 $\frac{1}{12}$	+	7.51131 0
$x\ x'^3y'$	+	78 840 $\frac{17}{24}$	+	19.7102
$x\ x'^4y'$	+	3 017	+	37.712
$x\ x'^5y'$	—	798 $\frac{19}{24}$	—	499.2
$x^2\ x'\ y'$	—	2074 903 $\frac{5}{8}$	—	10.37451 8
$x^2\ x'^2y'$	—	143 737 $\frac{1}{24}$	—	35.9343
$x^2\ x'^3y'$	—	8 889 $\frac{5}{12}$	—	111.118
$x^2\ x'^4y'$	—	759 $\frac{2}{3}$	—	474.8
$x^3\ x'\ y'$	+	152 297 $\frac{17}{24}$	+	38.07443
$x^3\ x'^2y'$	+	17 398 $\frac{5}{12}$	+	217.480
$x^3\ x'^3y'$	—	3 092	—	1932.5
$x^4\ x'\ y'$	—	7 475 $\frac{5}{24}$	—	93.440
$x^4\ x'^2y'$	—	3 405 $\frac{1}{24}$	—	2128.2
$x^5\ x'\ y'$	—	1 043 $\frac{1}{8}$	—	652.0
$x\ x'\ y'^2$	+	559 677 $\frac{2}{3}$	+	22.38711
$x\ x'^2y'^2$	+	46 714	+	93.428
$x\ x'^3y'^2$	+	4 773 $\frac{1}{2}$	+	477.3
$x^2\ x'\ y'^2$	—	58 162 $\frac{1}{4}$	—	116.324
$x^2\ x'^2y'^2$	—	5 391 $\frac{3}{4}$	—	539.17
$x^3\ x'\ y'^2$	+	8 824 $\frac{1}{3}$	+	882.43

Fact		A	A
$x\ x'\ y'^3$	$+$	$8\,114\,\frac{1}{6}$	$+\ 129.827$
$x\ \ y\ y'$	$+$	$10116\,126\,\frac{1}{3}$	$+\ 8.09290\,10$
$x^2\ \ y\ y'$	$-$	$872\,393\,\frac{17}{24}$	$-\ 34.89575$
$x^3\ \ y\ y'$	$+$	$72\,023\,\frac{5}{12}$	$+\ 144.047$
$x^4\ \ y\ y'$	$-$	$5\,252\,\frac{13}{24}$	$-\ 525.3$
$x\ \ y\ y'^2$	$+$	$133\,908\,\frac{1}{4}$	$+\ 42.8506$
$x^2\ \ y\ y'^2$	$-$	$15\,427\,\frac{1}{4}$	$-\ 246.84$
$x\ \ y^2y'$	$+$	$47\,391$	$+\ 15.1651$
$x^2\ \ y^2y'$	$-$	$8\,490\,\frac{1}{2}$	$-\ 135.848$
$x'\ \ y\ y'$	$-$	$10114\,627\,\frac{1}{3}$	$-\ 8.09170\,2$
$x'^2\ \ y\ y'$	$-$	$669\,572\,\frac{7}{24}$	$-\ 26.78289$
$x'^3\ \ y\ y'$	$-$	$38\,828\,\frac{5}{12}$	$-\ 77.657$
$x'^4\ \ y\ y'$	$-$	$1\,927\,\frac{11}{24}$	$-\ 192.7$
$x'\ \ y\ y'^2$	$-$	$133\,323$	$-\ 42.6634$
$x'^2\ \ y\ y'^2$	$-$	$12\,648\,\frac{1}{2}$	$-\ 202.38$
$x'\ \ y^2y'$	$-$	$47\,064\,\frac{3}{4}$	$-\ 15.0607$
$x'^2\ \ y^2y'$	$-$	$7\,502\,\frac{1}{4}$	$-\ 120.036$
$x\ x'\ y\ y'$	$+$	$1543\,618\,\frac{1}{2}$	$+\ 61.3447$
$x\ x'^2y\ y'$	$+$	$145\,548$	$+\ 291.096$
$x\ x'^3y\ y'$	$+$	$10\,767\,\frac{1}{2}$	$+1076.7$
$x^2\,x'\ y\ y'$	$-$	$177\,538$	$-\ 355.076$
$x^2\,x'^2y\ y'$	$-$	$20\,659$	-2065.9
$x^3\,x'\ y\ y'$	$+$	$17\,038$	$+1503.8$
$x\ x'\ y^2y'$	$+$	$13\,487\,\frac{1}{2}$	$+\ 215.800$
$x\ x'\ y\ y'^2$	$+$	$26\,637\,\frac{1}{2}$	$+\ 426.20$

In order to have V or W as a function of the four variables f, f', η, η' the preceding expression must be transformed by making the substitutions $x = \sqrt{2k\eta - k^2\eta^2}\cos f'$, $x' = \sqrt{2k'\eta' - k'^2\eta'^2}\cos f$, $y = 2k\eta - k^2\eta^2$, $y' = 2k'\eta' - k'^2\eta'^2$

It is proposed to accomplish the integrations the problem involves by Delaunay transformations. Selecting an argument $j\phi + if + i'f'$, j, i and i' being integers prime to each other, such a transformation of the four variables f, f' η, η' is made as shall make the periodic terms of W depending on this argument disappear. When all the sensible periodic terms have been got rid of by a series of these operations W will be reduced to a function of η and η'. As the differential equations retain their canonical form throughout the whole of this process, in the final stage, η and η' become constant, and if we put

$$-\frac{\partial W}{\partial \eta} = \varkappa, \qquad -\frac{\partial W}{\partial \eta'} = \varkappa'$$

$\varkappa$ and $\varkappa'$ will be constants, and the final expressions for f and f' will be

$$f = \varkappa(\phi + c), \qquad f' = \varkappa'(\phi + c')$$

c and c' being constants.

In accordance with the principles of the Delaunay method, the mentioned transformations must be made not only in the function W, but also in four functions designed to define the positions of the two planets in the common orbital plane. There is considerable latitude for choice here, but the four functions I propose are these:

$$\frac{a}{r} = (1 - k\eta)^{-2} [1 + \sqrt{2k\eta - k^2\eta^2} \cos f],$$

$$\frac{a'}{r'} = (1 - k'\eta')^{-2} [1 + \sqrt{2k'\eta' - k'^2\eta'^2} \cos f']$$

$$\frac{dt}{d\phi} = \left[\frac{h + W}{\mathrm{m}r^2} - \frac{h - W}{\mathrm{m}'r'^2}\right]^{-1} = \frac{\tan(\upsilon + \nu)}{h\rho^2 \sin 2\upsilon}$$

$$\frac{d\psi}{d\phi} = \left[\frac{h + W}{\mathrm{m}r^2} + \frac{h - W}{\mathrm{m}'r'^2}\right]\left[\frac{h + W}{\mathrm{m}r^2} - \frac{h - W}{\mathrm{m}'r'^2}\right]^{-1} = \frac{\cos(\upsilon - \nu)}{\cos(\upsilon + \nu)}$$

When computing the special values of V or W, since we employ the values of r and r', it is very little additional labor to derive the special values of the right members of the third and fourth equations of the just-given group. By applying the same method as for V we have infinite series for $\frac{dt}{d\phi}$ and $\frac{d\psi}{d\phi}$ of the same character as for the former quantity, and the Delaunay transformations can be made in them in precisely the same way.

When the latter are concluded we shall have $\frac{a}{r}$ and $\frac{a'}{r'}$ as functions of the independent variable ϕ; but before we can have t and ψ as similar functions, it will be necessary, in each case, to execute an integration with reference to ϕ, which will be easy, as each term is of the form

$$K \cos(\varkappa\phi + \beta)$$

With this operation I regard the solution of the problem as completed. The assertion may need justification. We are in the habit of using tables by inversion, the general theory of interpolation sufficing for the purpose. Although tables of anti-logarithms have been published, they are seldom used, and no tables have ever been computed for furnishing the arc to a given sine or tangent. Let this notion be applied to tables for Jupiter and Saturn; let them be constructed so as to give, in the first instance, the time at which the hypothetical planets have a definite elongation ϕ. The computation being made for a series of values of ϕ as 720°, 721°, 722° . . ., by interpolation we find the value of ϕ corresponding to a definite time; with this as argument we can enter another division of the tables and get the corresponding values of $\frac{a}{r}$ and $\frac{a'}{r'}$ and ψ; thus the positions of the hypotheti-

cal planets are known; whence it is possible to get those of the actual. In this way an analytical inversion of series is avoided.

In this connection I must state my conviction that Gyldén's device of the reduced time is without sensible advantage.

The application of Delaunay transformations will be treated in another memoir.

MEMOIR No. 80.

Application of the Delaunay Transformation in the Planetary Theories.

(This memoir appears here for the first time.)

The carrying out of the method of treating the theories of Jupiter and Saturn, proposed in the preceding memoir, requires the execution of about 2500 of Delaunay transformations. However, in all excepting about 150, we may limit ourselves, as far as the determining the formulæ of transformation are concerned, to the first power of the disturbing force. But in making the substitutions in the five quantities W, $\frac{a}{r}$, $\frac{a'}{r}$, $\frac{dt}{d\phi}$, $\frac{d\psi}{d\phi}$, with respect to the new terms which arise, having the arguments on which depend the secular and great inequalities, it may be desirable, in some cases, to take account of the second power. But, to be consistent, we adopt the rule that, in reference to η and η', no terms above 4 dimensions are to be admitted into W, and none above $3\frac{1}{2}$ dimensions in the remaining quantities.

In the present use of Delaunay transformations, the latter may be distributed into three classes; the determining argument being $j\phi \pm if \pm i'f'$, we have

Case I.—When i and i' are both even;
Case II.—When, of i and i', one is odd;
Case III.—When both i and i' are odd.

It will be easier to illustrate each case by an example than to write the generalized formulæ. We select then the three arguments $\phi - 2f + 2f'$, $3\phi + 2f - f'$, $\phi - f + f'$ for treatment. The last is the most difficult of all the transformations; on it depend what Gyldén calls the elementary, but which are more generally known as the secular terms. It is recommended that this be the last transformation made.

Case I.

In our example for illustration the truncated W has the form (the constant is omitted as unnecessary for our purposes)

$$\begin{aligned}W = {} & a_1\eta + a_2\eta' + a_3\eta^2 + a_4\eta\eta' + a_5\eta'^2 + a_6\eta^3 + a_7\eta^2\eta' + a_8\eta\eta'^2 + a_9\eta'^3 \\ & + a_{10}\eta^4 + a_{11}\eta^3\eta' + a_{12}\eta^2\eta'^2 + a_{13}\eta\eta'^3 + a_{14}\eta'^4 \\ & + [a_{15} + a_{16}\eta + a_{17}\eta' + a_{18}\eta^2 + a_{19}\eta\eta' + a_{20}\eta'^2]\,\eta\eta'\cos(\varphi - 2f + 2f') \\ & + a_{21}\eta^2\eta'^2\cos 2(\varphi - 2f + 2f').\end{aligned}$$

We note that the a from a_1 to a_{14} are of the zero order with reference to planetary masses, but that from a_{15} to a_{21} they are of the first. In integrating, terms of the third order will be neglected. With regard to eccentricities, no terms of higher order than η^4 will be retained in the expressions for η and η', nor than η^3 in the expressions for f and f'. Wherever convenient θ will be written for $\phi - 2f + 2f'$.

The differential equations being

$$\frac{d\eta}{d\varphi} = \frac{\partial W}{\partial f}, \quad \frac{d\eta'}{d\varphi} = \frac{\partial W}{\partial f'}, \quad \frac{df}{d\varphi} = -\frac{\partial W}{\partial \eta}, \quad \frac{df'}{d\varphi} = -\frac{\partial W}{\partial f'},$$

we evidently have the relation

$$\frac{d\eta}{d\varphi} + \frac{d\eta'}{d\varphi} = 0.$$

The integral is

$$\eta' = \text{const.} - \eta.$$

Instead of using this relation to eliminate η' from the equations, as Delaunay does, we allow η' to stand, keeping in mind, however, that whenever any function as H is to be partially differentiated with respect to η, we must take the operation

$$\frac{\partial H}{\partial \eta} - \frac{\partial H}{\partial \eta'}.$$

As our purpose in making these integrations is solely to discover proper transformations for the variables, we need have no special symbols for the introduced arbitrary constants. As the latter are to become variable we designate them by the symbols of the belonging variables. The sign of equality is to be interpreted by the phrase "to be replaced by". Thus the equation

$$\eta = \eta + \text{function}\ (\eta),$$

the η in the right member is an arbitrary constant, which, to form the transformation, is changed into a variable. By this convention a great deal of writing is saved.

In accordance with the foregoing explanation we get two differential equations *virtually* involving but two variables, viz., η and θ, from the relations

$$\frac{d\eta}{d\varphi} = \frac{\partial W}{\partial f}, \quad \frac{d\theta}{d\varphi} = 1 + 2\frac{\partial W}{\partial \eta}.$$

Written at length they are

$$\frac{d\eta}{d\varphi}=2\left[a_{15}+a_{16}\eta+a_{17}\eta'+a_{18}\eta^2+a_{19}\eta\eta'+a_{20}\eta'^2\right]\eta\eta'\sin\theta+4a_{21}\eta^2\eta'^2\sin 2\theta,$$

$$\begin{aligned}\frac{d\theta}{d\varphi}=1&+2a_1-2a_2+(4a_3-2a_4)\eta+(2a_4-4a_5)\eta'+(6a_6-2a_7)\eta^2+(4a_7-4a_8)\eta\eta'\\&+(2a_8-6a_9)\eta'^2+(8a_{10}-2a_{11})\eta^3+(6a_{11}-4a_{12})\eta^2\eta'+(4a_{12}-6a_{13})\eta\eta'^2\\&+(2a_{13}-8a_{14})\eta'^3+2\left[-a_{15}\eta+a_{15}\eta'-a_{16}\eta^2+(2a_{16}-2a_{17})\eta\eta'\right.\\&\left.+a_{17}\eta'^2-a_{18}\eta^3+(3a_{18}-2a_{19})\eta^2\eta'+(2a_{19}-3a_{20})\eta\eta'^2+a_{20}\eta'^3\right]\cos\theta\\&+4a_{21}\left[-\eta+\eta'\right]\eta\eta'\cos 2\theta.\end{aligned}$$

After η and θ have been derived from the integration of these two equations, η' will also be known, and it suffices for the completion of the problem that f should be found by a quadrature from the equation

$$\begin{aligned}\frac{df}{d\varphi}=&-\left[a_1+2a_3\eta+a_4\eta'+3a_6\eta^2+2a_7\eta\eta'+a_8\eta'^2+4a_{10}\eta^3+3a_{11}\eta^2\eta'+2a_{12}\eta\eta'^2+a_{13}\eta'^3\right]\\&-\left[a_{15}\eta'+2a_{16}\eta\eta'+a_{17}\eta'^2+3a_{18}\eta^2\eta'+2a_{19}\eta\eta'^2+a_{20}\eta'^3\right]\cos\theta-2a_{21}\eta\eta'^2\cos 2\theta.\end{aligned}$$

The differential equations, to be first integrated, may be written

$$\begin{aligned}\frac{d\eta}{d\varphi}&=Q_1\sin\theta+2Q_2\sin 2\theta,\\\frac{d\theta}{d\varphi}&=P+\frac{\partial Q_1}{\partial\eta}\cos\theta+\frac{\partial Q_2}{\partial\eta}\cos 2\theta.\end{aligned}$$

We proceed by successive approximations, and, assuming the form of the series representing the values of η and θ, we make

$$\begin{aligned}\eta&=\eta+\eta_1\cos\theta'+\eta_2\cos 2\theta',\\\theta&=\theta'+\theta_1\sin\theta'+\theta_2\sin 2\theta',\end{aligned}$$

where $\theta'=\theta_0(\phi+c)$ and η (of the right member of the first equation) and c are the arbitrary constants. θ_0 is a constant and a function of the constants η and η'.

On substituting the expressions just given in the differential equations we get the following which suffice for determining the five constant quantities $\eta_1,\ \theta_1,\ \eta_2,\ \theta_2,\ \theta_0$:

$$\begin{aligned}\theta_0\eta_1&=-Q_1,\\\theta_0\eta_2&=-\frac{1}{4}\left(\theta_1Q_1+\eta_1\frac{\partial Q_1}{\partial\eta}\right)-Q_2,\\\theta_0\theta_1&=\eta_1\frac{\partial P}{\partial\eta}+\frac{\partial Q_1}{\partial\eta},\\\theta_0\theta_2&=\frac{1}{4}\left(\eta_1\frac{\partial^2 Q_1}{\partial\eta^2}+\theta_1\frac{\partial Q_1}{\partial\eta}\right)+\frac{1}{2}\frac{\partial Q_2}{\partial\eta},\\\theta_0&=P.\end{aligned}$$

Their solution gives

$$\eta_1 = -\frac{Q_1}{P}, \qquad \eta_2 = \frac{1}{P}\left[\frac{1}{4}\eta_1^2\,\frac{\partial P}{\partial \eta} - Q_2\right],$$

$$\theta_1 = -\frac{\partial \eta_1}{\partial \eta}, \qquad \theta_2 = \frac{1}{P}\left[\frac{1}{4}\left(\eta_1\frac{\partial^2 Q_1}{\partial \eta^2} + \theta_1\frac{\partial Q_1}{\partial \eta}\right) + \frac{1}{2}\frac{\partial Q_2}{\partial \eta}\right].$$

For brevity we now introduce some new constants. Putting $\nu = 1 + 2\,(a_1 - a_2)$, we make

$$\nu b_0 = -2a_{15},\ \nu b_1 = -2a_{16},\ \nu b_2 = -2a_{17},\ \nu b_3 = -2a_{18},\ \nu b_4 = -2a_{19},$$
$$\nu b_5 = -2a_{20},\ \nu b_6 = -2a_{21},$$
$$\nu c_1 = 2a_4 - 4a_3,\ \nu c_2 = 4a_5 - 2a_4,\ \nu c_3 = 2a_7 - 6a_6 + \nu c_1^2,$$
$$\nu c_4 = 4a_8 - 4a_7 + 2\nu c_1 c_2,\ \nu c_5 = 6a_9 - 2a_8 + \nu c_2^2,$$
$$h_0 = b_0,\ h_1 = b_1 + b_0 c_1,\ h_2 = b_2 + b_0 c_2,\ h_3 = b_3 + b_0 c_3 + b_1 c_1,$$
$$h_4 = b_4 + b_0 c_4 + b_1 c_2 + b_2 c_1,\ h_5 = b_5 + b_0 c_5 + b_2 c_2.$$

Then

$$\frac{\nu}{P} = 1 + c_1\eta + c_2\eta' + c_3\eta^2 + c_4\eta\eta' + c_5\eta'^2,$$

$$-\frac{Q_1}{\nu} = [b_0 + b_1\eta + b_2\eta' + b_3\eta^2 + b_4\eta\eta' + b_5\eta'^2]\eta\eta',$$

$$\eta_1 = [h_0 + h_1\eta + h_2\eta' + h_3\eta^2 + h_4\eta\eta' + h_5\eta'^2]\eta\eta',$$

$$\eta_2 = \left[\frac{1}{4}h_0^2(c_2 - c_1) - b_6\right]\eta^2\eta'^2,$$

$$\theta_1 = h_0(\eta - \eta') + h_1\eta^2 + 2(h_2 - h_1)\eta\eta' - h_2\eta'^2 + h_3\eta^3 + (2h_4 - 3h_3)\eta^2\eta'$$
$$+ (3h_5 - 2h_4)\eta\eta'^2 - h_5\eta'^3,$$

$$\theta_2 = \frac{1}{4}h_0^2(\eta^2 + \eta'^2) + \frac{1}{2}h_0 h_1\eta^3 + \left[\frac{1}{4}h_0(3h_2 - h_1 + b_1 - b_2) + b_6\right]\eta^2\eta'$$
$$- \left[\frac{1}{4}h_0\,(h_2 - 3h_1 + b_1 - b_2) + b_6\right]\eta\eta'^2 + \frac{1}{2}h_0 h_2\eta'^3.$$

We have still to ascertain the expression for f. By the substitution of the preceding values in the expression for $\frac{df}{d\phi}$ we obtain

$$\frac{df}{d\varphi} = f_0 + \nu\,\left\{\frac{1}{2}h_0\eta' + \left[b_1 + \frac{1}{2}b_0 c_1\right]\eta\eta' + \frac{1}{2}b_2\eta'^2 + \left[\frac{3}{2}b_3 + \frac{1}{2}h_1 c_1 + h_0(c_3 - c_1^2)\right]\eta^2\eta'\right.$$
$$\left. + \left[b_4 + \frac{1}{2}h_2 c_1 + \frac{1}{2}h_0(c_4 - 2c_1 c_2)\right]\eta\eta'^2 + \frac{1}{2}b_5\eta'^3\right\}\cos\theta'$$
$$+ \nu\,\left\{-\frac{1}{4}h_0^2\eta'^2 + \left[\frac{1}{4}h_0(h_2 - 2h_1 - b_2) + b_6\right]\eta\eta'^2 - \frac{1}{4}h_0(h_2 + b_2)\eta'^3\right\}\cos 2\theta'.$$

After integration this becomes

$$f = (f) + f_0\,(\phi + c)$$
$$+ \left[\frac{1}{2}h_0\eta' + h_1\eta\eta' + \frac{1}{2}h_2\eta'^2 + \frac{3}{2}h_3\eta^2\eta' + h_4\eta\eta'^2 + \frac{1}{2}h_5\eta'^3\right]\sin\theta'$$
$$+ \left\{-\frac{1}{8}h_0^2\eta'^2 + \left[\frac{1}{8}h_0(h_2 - 3h_1 + b_1 - b_2) + \frac{1}{2}b_6\right]\eta\eta'^2 - \frac{1}{2}h_0 h_2\eta'^3\right\}\sin 2\theta',$$

where (f) is the constant introduced by the integration and f_0 is of the same nature as θ_0, but of which the value is not given because we have no need of it.

From the expressions for θ and f it is easy to conclude the equation $f' = (f') + (f_0')(\phi + c)$

$$+\left[\frac{1}{2} h\,\eta + \frac{1}{2} h_1\eta^2 + h_2\eta\eta' + \frac{1}{2} h_3\eta^3 + h_4\eta^2\eta' + \frac{3}{2} h_5\eta\eta'^2\right]\sin\theta'$$

$$+\left\{\frac{1}{8} h_0^2\eta^2 + \frac{1}{4} h_0 h_1\eta^3 + \left[\frac{1}{8} h_0(3h_2 - h + b_1 - b_2) + \frac{1}{2} b_6\right]\eta^2\eta'\right\}\sin 2\theta'.$$

In the formulæ of Delaunay (Vol. I, p. 89) we must make $n' = 1$, $i''' = 1$, $i = -2$, $L = \eta$. Then the equations which connect the elements conjugate severally to f and f' with those just used are

$$\eta = \eta + \frac{1}{2}\theta_1\eta_1, \qquad \eta' = \eta' - \frac{1}{2}\theta_1\eta_1.$$

Consequently, in order to have the canonical form for the differential equations, we must further transform by making

$$\eta = \eta - \frac{1}{2}\left[h_0^2(\eta - \eta') + 2h_0h_1\eta^2 + 3h_0(h_2 - h_1)\eta\eta' - 2h_0h_2\eta'^2\right]\eta\eta',$$

$$\eta' = \eta' + \frac{1}{2}\left[h_0^2(\eta - \eta') + 2h_0h_1\eta^2 + 3h_0(h_2 - h_1)\eta\eta' - 2h_0h_2\eta'^2\right]\eta\eta'.$$

Thus result the following

Formulæ of Transformation.

Replace η by $\eta - \frac{1}{2}\left[h_0^2(\eta - \eta') + 2h_0h_1\eta^2 + 3h_0(h_2 - h_1)\eta\eta' - 2h_0h_2\eta'^2\right]\eta\eta'$

$$+ \left[h_0 + h_1\eta + h_2\eta' + h_3\eta^2 + h_4\eta\eta' + h_5\eta'^2\right]\eta\eta'\cos(\varphi - 2f + 2f')$$

$$+ \left[\frac{1}{4} h_0^2(c_2 - c_1) - b_6\right]\eta^2\eta'^2\cos 2(\varphi - 2f + 2f').$$

Replace η' by $\eta' + \frac{1}{2}\left[h_0^2(\eta - \eta') + 2h_0h_1\eta^2 + 3h_0(h_2 - h_1)\eta\eta' - 2h_0h_2\eta'^2\right]\eta\eta'$

$$- \left[h_0 + h_1\eta + h_2\eta' + h_3\eta^2 + h_4\eta\eta' + h_5\eta'^2\right]\eta\eta'\cos(\varphi - 2f + 2f')$$

$$- \left[\frac{1}{4} h_0^2(c_2 - c_1) - b_6\right]\eta^2\eta'^2\cos 2(\varphi - 2f + 2f').$$

Replace f by $f + \left[\frac{1}{2} h_0\eta' + h_1\eta\eta' + \frac{1}{2} h_2\eta'^2 + \frac{3}{2} h_3\eta^2\eta' + h_4\eta\eta'^2 + \frac{1}{2} h_5\eta'^3\right]$

$$\sin(\varphi - 2f + 2f')$$

$$+ \left\{-\frac{1}{8} h_0^2\eta'^2 + \left[\frac{1}{8} h_0(h_2 - 3h_1 + b_1 - b_2) + \frac{1}{2} b_6\right]\eta\eta'^2 - \frac{1}{2} h_0h_2\eta'^3\right]$$

$$\sin 2(\varphi - 2f + 2f').$$

Replace f' by $f' + \left[\frac{1}{2} h_0\eta + \frac{1}{2} h_1\eta^2 + h_2\eta\eta' + \frac{1}{2} h_3\eta^3 + h_4\eta^2\eta + \frac{3}{2} h_5\eta\eta'^2\right]$

$$\sin(\varphi - 2f + 2f')$$

$$+ \left\{\frac{1}{8} h_0^2\eta^2 + \frac{1}{4} h_0h_1\eta^3 + \left[\frac{1}{8} h_0(3h_2 - h_1 + b_1 - b_2) + \frac{1}{2} b_6\right]\eta^2\eta'\right\}$$

$$\sin 2(\varphi - 2f + 2f').$$

To obtain the new W suitable for the following operation we must make the preceding transformation in the former W and (Delaunay, Vol. I, p. 89) add the following expression

$$\frac{1}{2}(\eta_1 \cos \theta' + \eta_2 \cos 2\theta') - \frac{1}{4}\theta_1 \eta_1,$$

equivalent to

$$\begin{aligned}
&-\frac{1}{4}\left[h_0^2(\eta - \eta') + 2h_0h_1\eta^2 + 3h_0(h_2 - h_1)\eta\eta' - 2h_0h_2\eta'^2\right]\eta\eta' \\
&+\frac{1}{2}\left[h_0 + h_1\eta + h_2\eta' + h_3\eta^2 + h_4\eta\eta' + h_5\eta'^2\right]\eta\eta' \cos(\varphi - 2f + 2f') \\
&+ \quad \left[\frac{1}{8}h_0^2(c_2 - c_1) - \frac{1}{2}b_6\right]\eta^2\eta'^2 \cos 2(\varphi - 2f + 2f').
\end{aligned}$$

Case II.

In our example for illustration the truncated W has the form

$$\begin{aligned}
W = {} & a_1\eta + a_2\eta' + a_3\eta^2 + a_4\eta\eta' + a_5\eta'^2 + a_6\eta^3 + a_7\eta^2\eta' + a_8\eta\eta'^2 + a_9\eta'^3 \\
& + a_{10}\eta^4 + a_{11}\eta^3\eta' + a_{12}\eta^2\eta'^2 + a_{13}\eta\eta'^3 + a_{14}\eta'_4 \\
& + [a_{15} + a_{16}\eta + a_{17}\eta' + a_{18}\eta^2 + a_{19}\eta\eta' + a_{20}\eta'^2]\, \eta\eta'^{\frac{1}{2}} \cos(3\varphi + 2f - f') \\
& + [a_{21} + a_{22}\eta + a_{23}\eta']\, \eta^2\eta' \cos 2(3\varphi + 2f - f').
\end{aligned}$$

The differential equations being

$$\frac{d\eta}{d\varphi} = \frac{\partial W}{\partial f}, \quad \frac{d\eta'}{d\varphi} = \frac{\partial W}{\partial f'}, \quad \frac{df}{d\varphi} = -\frac{\partial W}{\partial \eta}, \quad \frac{df'}{d\varphi} = -\frac{\partial W}{\partial \eta'},$$

we evidently have the relation

$$\frac{d\eta}{d\varphi} + 2\frac{d\eta'}{d\varphi} = 0.$$

The integral is

$$\eta = \text{const.} - 2\eta'.$$

Instead of using this to eliminate η from the equations, as Delaunay does, we allow η to stand, keeping in mind, however, that when any function as H is to be partially differentiated with respect to η', we must take the operation

$$\frac{\partial H}{\partial \eta'} - 2\frac{\partial H}{\partial \eta}.$$

In accordance with this explanation we get two differential equations *virtually* involving but two variables, viz., η' and θ, from the relations

$$\frac{d\eta'}{d\varphi} = \frac{\partial W}{\partial f'}, \quad \frac{d\theta}{d\varphi} = 3 + \frac{\partial W}{\partial \eta'}.$$

Written at length they are

$$\frac{d\eta'}{d\varphi}=\left[a_{15}+a_{16}\eta+a_{17}\eta'+a_{18}\eta^2+a_{19}\eta\eta'+a_{20}\eta'^2\right]\eta\eta'^{1/2}\sin\theta$$
$$+2\left[a_{21}+a_{22}\eta+a_{23}\eta'\right]\eta^2\eta'\sin 2\theta\,;$$

$$\frac{d\theta}{d\varphi}=3+a_2-2a_1+(a_4-4a_3)\eta+(2a_5-2a_4)\,\eta'+(a_7-6a_6)\eta^2+(2a_8-4a_7)\eta\eta'$$
$$+(3a_9-2a_8)\eta'^2+(a_{11}-8a_{10})\eta^3+(2a_{12}-6a_{11})\eta^2\eta'+(3a_{13}-4a_{12})\eta\eta'^2-2a_{13}\eta'^3$$
$$+\left[\frac{1}{2}a_{15}\eta-2a_{15}\eta'+\frac{1}{2}a_{16}\eta^2+\left(\frac{3}{2}a_{17}-4a_{16}\right)\eta\eta'-2a_{17}\eta'^2+\frac{1}{2}a_{18}\eta^3\right.$$
$$\left.+\left(\frac{3}{2}a_{19}-6a_{18}\right)\eta^2\eta'+\left(\frac{5}{2}a_{20}-4a_{19}\right)\eta\eta'^2-2a_{20}\eta'^3\right]\eta'^{-1/2}\cos\theta$$
$$+\left[a_{21}\eta^2-4a_{21}\eta\eta'+a_{22}\eta^3+(2a_{23}-6a_{22})\eta^2\eta'-4a_{23}\eta\eta'^2\right]\cos 2\theta.$$

After η' and θ have been derived from the integration of these two equations, η will also be known, and it suffices for the completion of the problem that f should be found by a quadrature from the equation

$$\frac{df}{d\varphi}=-\left[a_1+2a_3\eta+a_4\eta'+3a_6\eta^2+2a_7\eta\eta'+a_8\eta'^2\right.$$
$$\left.+4a_{10}\eta^3+3a_{11}\eta^2\eta'+2a_{12}\eta\eta'^2+a_{13}\eta'^3\right]$$
$$-\left[a_{15}+2a_{16}\eta+a_{17}\eta'+3a_{18}\eta^2+2a_{19}\eta\eta'+a_{20}\eta'^2\right]\eta'^{1/2}\cos\theta$$
$$-\left[2a_{21}+3a_{22}\eta+2a_{23}\eta'\right]\eta\eta'\cos 2\theta.$$

The differential equations, to be first integrated, may be written

$$\frac{d\eta'}{d\varphi}=Q_1\sin\theta+2\,Q_2\sin 2\theta,$$
$$\frac{d\theta}{d\varphi}=P+\frac{\partial Q_1}{\partial\eta'}\cos\theta+\frac{\partial Q_2}{\partial\eta'}\cos 2\theta.$$

We may assume

$$\eta'=\eta'+\eta_1'\cos\theta'+\eta_2'\cos 2\,\theta',$$
$$\theta\;=\theta'+\theta_1\sin\theta'+\theta_2\sin 2\,\theta',$$

where $\theta'=\theta_0\,(\phi+c)$ and η' (of the right member of the first equation) and c are the arbitrary constants; θ_0 is constant and a function of the constants η and η'. In the terms of two dimensions with regard to planetary masses the approximation is carried to a dimension less in regard to η and η' than in the other terms.

Similarly, as in Case I, we have

$$\eta_1'=-\frac{Q_1}{P},\quad \eta_2'=\frac{1}{P}\left[\frac{1}{4}\eta_1'^2\frac{\partial P}{\partial\eta'}-Q_2\right],$$
$$\theta_1=-\frac{\partial\eta_1'}{\partial\eta'},\quad \theta_2=\frac{1}{P}\left[\frac{1}{4}\left(\eta_1'\frac{\partial^2Q_1}{\partial\eta'^2}+\theta_1\frac{\partial Q_1}{\partial\eta'}\right)+\frac{1}{2}\frac{\partial Q_2}{\partial\eta'}\right],$$
$$\theta_0=P.$$

Putting $\nu = 3 - 2a_1 + a_2$ we make

$$\nu b_0 = -a_{15},\ \nu b_1 = -a_{16},\ \nu b_2 = -a_{17},\ \nu b_3 = -a_{18},\ \nu b_4 = -a_{19},$$
$$\nu b_5 = -a_{20},\ \nu b_6 = -a_{21},\ \nu b_7 = -a_{22},\ \nu b_8 = -a_{23},$$
$$\nu c_1 = 4a_3 - a_4,\ \nu c_2 = 2a_4 - 2a_5,\ \nu c_3 = 6a_6 - a + \nu c_1^2,$$
$$\nu c_4 = 4a_6 - 2a_8 + 2\nu c_1 c_2,\ \nu c_5 = 2a_8 - 3a_9 + \nu c_2^2,$$
$$h_0 = b_0,\ h_1 = b_1 + b_0 c_1,\ h_2 = b_2 + b_0 c_2,\ h_3 = b_3 + b_0 c_3 + b_1 c_1,$$
$$h_4 = b_4 + b_0 c_4 + b_1 c_2 + b_2 c_1,\ h_5 = b_5 + b_0 c_5 + b_2 c_2,$$
$$h_6 = \frac{1}{4} h_0^2 (2c_1 - c_2) + b_6,\ h_7 = b_7 + b_6 c_1,\ h_8 = b_8 + b_6 c_2.$$

Then

$$\frac{\nu}{P} = 1 + c_1\eta + c_2\eta' + c_3\eta^2 + c_4\eta\eta' + c_5\eta'_2,$$
$$-\frac{Q_1}{\nu} = \left[b_0 + b_1\eta + b\,\eta' + b_3\eta^2 + b_4\eta\eta' + b_5\eta'^2\right]\eta\eta'^{\frac{1}{2}},$$
$$\eta'_1 = \left[h_0 + h_1\eta + h_2\eta' + h_3\eta^2 + h_4\eta\eta' + h_5\eta'^2\right]\eta\eta'^{\frac{1}{2}},$$
$$\eta'_2 = \left[h_6 + h_7\eta + h_8\eta'\right]\eta^2\eta',$$
$$\theta_1 = \Big[-\frac{1}{2}h_0\eta + 2h_0\eta' - \frac{1}{2}h_1\eta^2 + \left(4h_1 - \frac{3}{2}h_2\right)\eta\eta' + 2h_2\eta'^2$$
$$-\frac{1}{2}h_3\eta^3 + \left(6h_3 - \frac{3}{2}h_4\right)\eta^2\eta' + \left(4h_4 - \frac{5}{2}h_5\right)\eta\eta'^2 + 2h_5\eta'^3\Big]\eta'^{-\frac{1}{2}},$$
$$\theta_2 = \Big\{\frac{1}{8}h_0^2\eta^2 + h_0^2\eta'^2 + \frac{1}{4}h_0h_1\eta^3 - \left[\frac{1}{8}h_0\left(2h_1 - 3h_2 - 2b_1 + b_2\right) + \frac{1}{2}b_6\right]\eta^2\eta'$$
$$+\left[\frac{1}{2}h_0\left(6h_1 - h_2 - 2b_1 + b_2\right) + 2b_6\right]\eta\eta'^2 + 2h_0h_2\eta'^3\Big\}\frac{1}{\eta'}.$$

We have still to ascertain the expression for f. By the substitution of the preceding values in the expression for $\frac{df}{d\phi}$ we obtain the equation

$$\frac{df}{d\varphi} = f_0 + \nu\Big\{h_0 + (h_1 + b_1)\eta + b_2\eta' + (3b_3 + 2b_0c_3 - b_0c_1^2 + b_1c_1)\eta^2 + (2b_4 + b_0c_4 + b_2c_1)\eta\eta'$$
$$+ b_5\eta'^2\Big\}\eta'^{\frac{1}{2}}\cos\theta'$$
$$+\nu\Big\{h_0\eta' + \left[\frac{1}{2}h_0\left(4h_1 - h_2 + b_2\right) + 2b_6\right]\eta\eta' + h_0\left(h_2 + b_2\right)\eta'^2 + 3b_7\eta^2\eta'$$
$$+ 2b_8\eta\eta'^2\Big\}\cos 2\theta'.$$

After integration this becomes

$$f = (f) + f_0(\varphi + c) + \left[h_0 + 2h_1\eta + h_2\eta' + 3h_3\eta^2 + 2h_4\eta\eta' + h_5\eta'^2\right]\eta'^{\frac{1}{2}}\sin\theta'$$
$$+\Big\{\frac{1}{2}h_0^2\eta' + \left[\frac{1}{4}h_0\left(6h_1 - h_2 - 2b_1 + b_2\right) + b_6\right]\eta\eta' + h_0h_2\eta'^2 + 3b_7\eta^2\eta'$$
$$+ 2b_8\eta\eta'^2\Big]\sin 2\theta'.$$

From the expressions for θ and f it is easy to conclude the equation

$$f' = (f') + f_0'(\varphi + c) + \Big[\frac{1}{2}h_0\eta + \frac{1}{2}h_1\eta^2 + \frac{3}{2}h_2\eta\eta' + \frac{1}{2}h_3\eta^3 + \frac{3}{2}h_4\eta^2\eta' + \frac{5}{2}h_5\eta\eta'^2\Big]\eta'^{-\frac{1}{2}}\sin\theta'$$

$$+ \Big\{-\frac{1}{8}h_0^2\eta^2 - \frac{1}{4}h_0h_1\eta^3 + \Big[\frac{1}{8}h_0(2h_1 - 3h_2 - 2b_1 + b_2) + \frac{1}{2}b_6\Big]\eta^2\eta' + 6b_7\eta^2\eta'^2 + 4b_8\eta\eta'^3\Big\}\frac{1}{\eta'}\sin 2\theta'.$$

In the formulæ of Delaunay (Vol. I, p. 89) we must make $n' = 1$, $i''' = 3$, $i = -1$. Then the equations which connect the elements conjugate severally to f and f' with those just used are

$$\eta = \eta - \theta_1\eta_1', \qquad \eta' = \eta' + \frac{1}{2}\theta_1\eta_1'.$$

Consequently, in order to have the canonical form for the differential equations, we must further transform by making

$$\eta = \eta - \frac{1}{2}h_0^2\eta^2 + 2h_0^2\eta\eta' - h_0h_1\eta^3 + h_0(6h_1 - 2h_2)\eta^2\eta' + 4h_0h_2\eta\eta'^2,$$

$$\eta' = \eta' + \frac{1}{4}h_0^2\eta^2 - h_0^2\eta\eta' + \frac{1}{2}h_0h_1\eta^3 - h_0(3h_1 - h_2)\eta^2\eta' - 2h_0h_2\eta\eta'^2.$$

Thus result the following

Formulæ of Transformation.

Replace η by $\eta - \frac{1}{2}h_0^2\eta^2 + 2h_0^2\eta\eta' - h_0h_1\eta^3 + h_0(6h_1 - 2h_2)\eta^2\eta' + 4h_0h_2\eta\eta'^2$

$$- 2\Big[h_0 + h_1\eta + h_2\eta' + h_3\eta^2 + h_4\eta\eta' + h_5\eta'^2\Big]\eta\eta'^{\frac{1}{2}}\cos(3\varphi + 2f - f')$$

$$- 2\Big[h_6 + h_7\eta + h_8\eta'\Big]\eta^2\eta'\cos 2(3\varphi + 2f - f'),$$

Replace η' by $\eta' + \frac{1}{4}h_0^2\eta^2 - h_0^2\eta\eta' + \frac{1}{2}h_0h_1\eta^3 - h_0(3h_1 - h_2)\eta^2\eta' - 2h_0h_2$

$$+ \Big[h_0 + h_1\eta + h_2\eta' + h_3\eta^2 + h_4\eta\eta' + h_5\eta'^2\Big]\eta\eta'^{\frac{1}{2}}\cos(3\varphi + 2f - f')$$

$$+ \Big[h_6 + h_7\eta + h_8\eta'\Big]\eta^2\eta'\cos 2(3\varphi + 2f - f'),$$

Replace f by $f + \Big[h_0 + 2h_1\eta + h_2\eta' + 3h_3\eta^2 + 2h_4\eta\eta' + h_5\eta'^2\Big]\eta'^{\frac{1}{2}}\sin(3\varphi + 2f - f')$

$$+ \Big\{\frac{1}{2}h_0^2\eta' + \Big[\frac{1}{4}h_0(6h_1 - h_2 - 2b_1 + b_2) + b_6\Big]\eta\eta' + h_0h_2\eta'^2 + 3b_7\eta^2\eta' + 2b_8\eta\eta'^2\Big\}\sin 2(3\varphi + 2f - f'),$$

Replace f' by $f' + \frac{1}{2}\Big[h_0\eta + h_1\eta^2 + 3h_2\eta\eta' + h_3\eta^3 + 3h_4\eta^2\eta' + 5h_5\eta\eta'^2\Big]\eta'^{-\frac{1}{2}}$

$$\sin(3\varphi + 2f - f')$$

$$+ \Big\{-\frac{1}{8}h_0^2\eta^2 - \frac{1}{4}h_0h_1\eta^3 + \Big[\frac{1}{8}h_0(2h_1 - 3h_2 - 2b_1 + b_2) + \frac{1}{2}b_6\Big]\eta^2\eta' + 6b_7\eta^2\eta'^2 + 4b_8\eta\eta'^3\Big\}\frac{1}{\eta'}\sin 2(3\varphi + 2f - f').$$

To obtain the new W suitable for the following operation we must make the preceding transformation in the former W and (Delaunay, Vol. I, p. 89) add the following expression

$$3\,(\eta_1'\cos\theta' + \eta_2'\cos 2\theta') - \frac{3}{2}\,\theta_1\eta_1',$$

equivalent to

$$\frac{3}{4}\,h_0^2\eta^2 - 3h_0^2\eta\eta' + \frac{3}{2}\,h_0h_1\eta^3 - 3h_0(3h_1 - h_2)\eta^2\eta' - 6h_0h_2\eta\eta'^2$$
$$+ 3\left[h_0 + h_1\eta + h_2\eta' + h_3\eta^2 + h_4\eta\eta' + h_5\eta'^2\right]\eta\eta'^{\frac{1}{2}}\cos(3\varphi + 2f - f')$$
$$+ 3\left[h_6 + h_7\eta + h_8\eta'\right]\eta^2\eta'\cos 2\,(3\varphi + 2f - f').$$

Case III.

The difficulties inherent in the elaboration of this transformation arise from the circumstance that the mean motion of the argument is of the order of the planetary masses. Thus the resulting inequalities are of the zero order in the same respect, and it is necessary to push the approximation one order farther than would suffice without this circumstance. Supposing that the second order terms ought to be considered in the latter case, the third order terms ought to be taken into account here. This would, however, involve very heavy labor, even if no terms above four dimensions in regard to η and η' were considered in W.

Some modification must be made in the previous method, and the truncated W will be more serviceable when expressed in powers of the cosine of the argument than as in the preceding cases. Thus

$$\begin{aligned}
W = {} & a_1\eta + a_2\eta' + a_3\eta^2 + a_4\eta\eta' + a_5\eta'^2 + a_6\eta^3 + a_7\eta^2\eta' + a_8\eta\eta'^2 + a_9\eta'^3 \\
& + a_{10}\eta^4 + a_{11}\eta^3\eta' + a_{12}\eta^2\eta'^2 + a_{13}\eta\eta'^3 + a_{14}\eta'^4 \\
& + [a_{15} + a_{16}\eta + a_{17}\eta' + a_{18}\eta^2 + a_{19}\eta\eta' + a_{20}\eta'^2 + a_{21}\eta^3 + a_{22}\eta^2\eta' + a_{23}\eta\eta'^2 + a_{24}\eta'^3] \\
& \times \sqrt{\eta\eta'}\cos(\varphi - f + f') \\
& + [a_{25} + a_{26}\eta + a_{27}\eta' + a_{28}\eta^2 + a_{29}\eta\eta' + a_{30}\eta'^2]\eta\eta'\cos^2(\varphi - f + f') \\
& + [a_{31} + a_{32}\eta + a_{33}\eta']\,[\sqrt{\eta\eta'}\cos(\varphi - f + f']^3 \\
& + a\ \ \eta^2\eta'^2\cos^4(\varphi - f + f').
\end{aligned}$$

We have the relation

$$\frac{d\eta}{d\varphi} + \frac{d\eta'}{d\varphi} = 0$$

It is satisfied by supposing that

$$\eta = K(1 - x), \qquad \eta' = K(1 + x),$$

where K is the arbitrary constant and x a new variable to replace η and η'.

Also we put θ for $\phi - f + f'$. When x is introduced into W it becomes divisible by K; thus we adopt a new function

$$W' = \frac{1}{K} W - x.$$

In terms of this the differential equations are

$$\frac{dx}{d\varphi} = \frac{\partial W'}{\partial \theta}, \qquad \frac{d\theta}{d\varphi} = -\frac{\partial W'}{\partial x},$$

By adopting the following notation

$$\begin{array}{lll}
b_1 = a_1 + a_2, & b_2 = a_3 + a_4 + a_5, & b_3 = a_6 + a_7 + a_8 + a_9, \\
b_4 = a_{10} + a_{11} + a_{12} + a_{13} + a_{14}, & b_5 = -1 - a_1 + a_2, & b_6 = -2a_3 + 2a_5, \\
b_7 = -3a_6 - a_7 + a_8 + 3a_9, & b_8 = -4a_{10} - 2a_{11} + 2a_{13} + 4a_{14}, & b_9 = a_3 - a_4 + a_5, \\
b_{10} = 3a_6 - a_7 - a_8 + 3a_9, & b_{11} = 6a_{10} - 2a_{12} + 6a_{14}, & b_{12} = -a_6 + a_7 - a_8 + a_9, \\
b_{13} = -4a_{10} + 2a_{11} - 2a_{13} & b_{14} = a_{10} - a_{11} + a_{12} - a_{13} + a_{14}, & b_{15} = a_{15}, \\
\qquad + 4a_{14}, & & \\
b_{16} = a_{16} + a_{17}, & b_{17} = a_{18} + a_{19} + a_{20}, & b_{18} = a_{21} + a_{22} + a_{23} + a_{24}, \\
b_{19} = -a_{16} + a_{17}, & b_{20} = -2a_{18} + 2a_{20}, & b_{21} = -3a_{21} - a_{22} + a_{23} + 3a_{24}, \\
b_{22} = a_{18} - a_{19} + a_{20}, & b_{23} = 3a_{21} - a_{22} - a_{23} + 3a_{24}, & b_{24} = -a_{21} + a_{22} - a_{23} + a_{24}, \\
b_{25} = a_{25}, & b_{26} = a_{26} + a_{27}, & b_{27} = a_{28} + a_{29} + a_{30}, \\
b_{28} = -a_{26} + a_{27}, & b_{29} = -2a_{28} + 2a_{30}, & b_{30} = a_{28} - a_{29} + a_{30}, \\
b_{31} = a_{31}, & b_{32} = a_{32} + a_{33}, & b_{33} = -a_{32} + a_{33}, \\
b_{34} = a_{34}, & &
\end{array}$$

we have

$$\begin{aligned}
W' = {} & b_1 + b_2 K + b_3 K^2 + b_4 K^3 + [b_5 + b_8 K + b_7 K^2 + b_8 K^3]\, x \\
& + [b_9 K + b_{10} K^2 + b_{11} K^3]\, x^2 + [b_{12} K^2 + b_{13} K^3]\, x^3 + b_{14} K^3 x^4 \\
& + \{b_{15} + b_{16} K + b_{17} K^2 + b_{18} K^3 + [b_{19} K + b_{20} K^2 + b_{21} K^3]\, x \\
& + [b_{22} K^2 + b_{23} K^3]\, x^2 + b_{24} x^3\} \sqrt{1 - x^2} \cos\theta \\
& + \{b_{25} + b_{26} K + b_{27} K^2 + [b_{28} K + b_{29} K^2]\, x + b_{30} K^2 x^2\} K (1 - x^2) \cos^2\theta \\
& + \{b_{31} + b_{32} K + b_{33} K x\} K^2 (1 - x^2)^{\frac{3}{2}} \cos^3\theta + b_{34} K^3 (1 - x^2)^2 \cos^4\theta.
\end{aligned}$$

The differential equations form a system to be integrated by themselves, and they admit the integral $W' =$ a constant. The equations chosen for use in the solution are

$$W' = \text{a constant}, \qquad \frac{d\phi}{dx} = + \frac{1}{\dfrac{\partial W'}{\partial \theta}}.$$

Putting y for $\sqrt{1 - x^2} \cos\theta$, we have the form for W',

$$W' = J_0 + J_1 y + J_2 y^2 + J_3 y^3 + J_4 y^4,$$

whence may be derived

$$\frac{\partial W'}{\partial \theta} = -[J_1 + 2J_2 y + 3J_3 y^2 + 4J_4 y^3] \sqrt{1 - x^2 - y^2}.$$

It is not necessary that the constant value of W' should be represented by a single symbol. Having to derive the value of y from the equation $W' =$ a constant, we assign to it a form which will facilitate future operations; we merge in the constant the terms $b_1 + b_2 K + b_3 K^2 + b_4 K^3$, and then divide both members of the equation by $b_{15} + b_{16} K + b_{17} K^2 + b_{18} K^3$. Then we arrive at an equation of the form

$$\begin{aligned}
\text{const.} = {} & [c_1 + c_2 K + c_3 K^2 + c_4 K^3]\, x + [c_5 K + c_6 K^2 + c_7 K^3]\, x^2 + [c_8 K^2 + c_9 K^3]\, x^3 + c_{10} K^3 x^4 \\
& + \{1 + [c_{11} K + c_{12} K^2 + c_{13} K^3]\, x + [c_{14} K^2 + c_{15} K^3]\, x^2 + c_{16} K^3 x^3\}\, y \\
& + \{c_{17} K + c_{18} K^2 + c_{19} K^3 + [c_{20} K^2 + c_{21} K^3]\, x + c_{22} K^3\}\, y^2 \\
& + \{c_{23} K^2 + c_{24} K^3 + c_{25} K^3 x\}\, y^3 + c_{26} K^3 y^4.
\end{aligned}$$

The 26 numbers c are independent of the two arbitrary constants; their values may be got from the b by formulæ of recursion of so simple a character that it will suffice to note the four following:

$$b_{15} c_1 = b_5, \quad b_{15} c_2 + b_{16} c_1 = b_6, \quad b_{15} c_3 + b_{16} c_2 + b_{17} c_1 = b_7, \quad b_{15} c_4 + b_{16} c_3 + b_{17} c_2 + b_{18} c_1 = b_8.$$

C denoting an arbitrary constant, let the solution of our equation be

$$\begin{aligned}
y = \sqrt{1 + e_1^2}\, C & + \{e_1 + (e_2 + e_3 C) K + (e_4 + e_5 C + e_6 C^2) K^2 + (e_7 + e_8 C + e_9 C^2 + e_{10} C^3) K^3\}\, x \\
& + \{e_{11} K + (e_{12} + e_{13} C) K^2 + (e_{14} + e_{15} C + e_{16} C^2) K^3\}\, x^2 \\
& + \{e_{17} K^2 + (e_{18} + e_{19} C) K^3\}\, x^3 + e_{20} K^3 x^4.
\end{aligned}$$

To compute the coefficients e from the c we have the following equations (for brevity g_0 is written instead of $\sqrt{1 + e_1^2}$):

$$\begin{aligned}
e_1 &= c_1, \quad e_2 = c_2, \quad e_3 = g_0[c_{11} - 2c_{17} e_1], \quad e_4 = c_3, \quad e_5 = g_0[c_{12} - 2c_{17} e_2 - 2c_{18} e_1], \\
e_6 &= g_0^2 c_{20} - 2g_0 c_{17} e_3 - 3g_0^2 c_{23} e_1, \quad e_7 = c_4, \quad e_8 = g_0[c_{13} - 2c_{17} e_4 - 2c_{18} e_2 - 2c_{19} e_1], \\
e_9 &= g_0^2 c_{21} - 2g_0 c_{17} e_5 - 2g_0 c_{18} e_3 - 3g_0^2 c_{23} e_2 - 3g_0^2 c_{24} e_1, \quad e_{10} = g_0^3 c_{25} - 2g_0 c_{17} e_6 - 3g_0^2 c_{23} e_3 - 4g_0^3 c_{26} e_1, \\
e_{11} &= c_5 + c_{11} e_1 + c_{17} e_1^2, \quad e_{12} = e_8 + c_{11} e_2 + c_{12} e_1 + 2c_{17} e_1 e_2 + c_{18} e_1^2, \\
e_{13} &= g_0 c_{14} - 2g_0 c_{17} e_{11} + c_{11} e_3 + 2g_0 c_{20} e_1 + 2c_{17} e_1 e_3 + 3g_0 c_{23} e_1^2, \\
e_{14} &= c_7 + c_{11} e_4 + c_{12} e_2 + c_{13} e_1 + c_{17}(e_2^2 + 2e_1 e_4) + 2c_{18} e_1 e_2 + c_{18} e_1^2, \\
e_{15} &= g_0 c_{15} - 2g_0 c_{17} e_{12} - 2g_0 c_{18} e_{11} + c_{11} e_5 + c_{12} e_3 + 2g_0 c_{20} e_2 + 2g_0 c_{21} e_1 + 2c_{17}(e_1 e_5 + e_2 e_3) + 2c_{18} e_1 e_3 \\
&\qquad + 6g_0 c_{23} e_1 e_2 + 3g_0 c_{24} e_1^2, \\
e_{16} &= g_0^2 c_{22} - 2g_0^2 c_{17} e_{13} - 2g_0^2 c_{23} e_{11} + c_{11} e_8 + 2g_0 c_{20} e_3 + 3g_0^2 c_{25} e_1 + c_{17}(e_3^2 + 2e_1 e_8) + 6g_0 c_{23} e_1 e_3 \\
&\qquad + 6g_0^2 c_{26} e_1^2, \\
e_{17} &= c_8 + c_{14} e_1 + c_{11} e_{11} + 2c_{17} e_1 e_{11} + c_{20} e_1^2 + c_{23} e_1^2, \\
e_{18} &= c_9 + c_{14} e_2 + c_{15} e_1 + c_{11} e_{12} + c_{12} e_{11} + 2c_{17}(e_1 e_{12} + e_2 e_{11}) + 2c_{18} e_1 e_{11} + 2c_{20} e_1 e_2 + 2c_{21} e_1^2 \\
&\qquad + 3c_{23} e_1^2 e_2 + c_{24} e_1^3, \\
e_{19} &= g_0 c_{18} - 2g_0 c_{17} e_{17} + c_{14} e_3 + 2g_0 c_{22} e_1 + c_{11} e_{13} + g_0 c_{20} e_{11} + 2c_{17}(e_1 e_{13} + e_3 e_{11}) + 6g_0 c_{23} e_1 e_{11} \\
&\qquad + 2c_{20} e_1 e_3 + 3g_0 c_{25} e_1^2 + 3c_{23} e_1^2 e_3 + 4g_0 c_{26} e_1^3, \\
e_{20} &= c_{10} + c_{16} e_1 + c_{15} e_{11} + c_{11} e_{17} + c_{22} e_1^2 + 2c_{20} e_1 e_{11} + c_{25} e_1^3 + 3c_{23} e_1^2 e_{11} + 2c_{17} e_1 e_{17}.
\end{aligned}$$

By substituting the value of y just found we obtain the expression

$$\frac{\partial W'}{\partial \theta} = -[b_{15}+b_{16}K+b_{17}K^2+b_{18}K^3]\{1+f_1CK+(f_2C+f_3C^2)K^2+(f_4C+f_5C^2+f_6C^3)K^3$$
$$+[f_7K+(f_8+f_9C)K^2+(f_{10}+f_{11}C+f_{12}C^2)K^3]x$$
$$+[f_{13}K^2+(f_{14}+f_{15}C)K^3]x^2+f_{16}K^3x^3\}\sqrt{1-x^2-y^2},$$

where the f have the following equivalents:

$$f_1=2g_0c_{17},\quad f_2=2g_0c_{18},\quad f_3=3g_0^2c_{23},\quad f_4=2g_0c_{19},\quad f_5=3g_0^2c_{24},$$
$$f_6=3g_0^3c_{26},\quad f_7=c_{11}+2c_{17}e_1^2,\quad f_8=c_{12}+2c_{17}e_2+2c_{18}e_1,$$
$$f_9=2c_{17}e_3+2g_0c_{20}+6g_0c_{23}e_1,\quad f_{10}=c_{13}+2c_{17}e_4+2c_{18}e_2+2c_{19}e_1,$$
$$f_{11}=2c_{17}e_5+2c_{18}e_3+2g_0c_{21}+6g_0c_{23}e_2+6g_0c_{24}e_1,$$
$$f_{12}=2c_{17}e_6+6g_0c_{23}e_3+3g_0^2c_{25}+12g_0^2c_{26}e_1,\quad f_{13}=2c_{17}c_{11}+2c_{20}e_1+3c_{23}e_1^2,$$
$$f_{14}=c_{15}+2c_{17}e_{12}+2c_{18}e_{11}+2c_{20}e_2+2c_{21}e_1+6c_{23}e_1e_2+3c_{24}e_1^2,$$
$$f_{15}=2c_{17}e_{13}+2c_{20}e_3+2g_0c_{22}+6c_{23}(g_0e_{11}+2e_1e_3)+6g_0c_{25}e_1+12g_0c_{26}e_1^2,$$
$$f_{16}=c_{16}+2c_{17}e_{17}+2c_{22}e_1+6c_{23}e_1e_{11}+3c_{25}c_1^2+4c_{26}e_1^3.$$

On substituting the expression for y we find that

$$1-x^2-y^2=1-g_0^2C^2-2g_0\{e_1C+(e_2C+e_3C^2)K+(e_4C+e_5C^2+e_8C^3)K^2$$
$$+(e_7C+e_8C^2+e_9C^3+e_{10}C^4)K^3\}x$$
$$-\{g_0^2+(g_1+g_2C)K+(g_3+g_4C+g_5C^2)K^2+(g_6+g_7C+g_8C^2+g_9C^3)K^3\}x^2$$
$$-\{g_{10}K+(g_{11}+g_{12}C)K^2+(g_{13}+g_{14}C+g_{15}C^2)K^3\}x^3$$
$$-\{g_{16}K^2+(g_{17}+g_{18}C)K^3\}x^4-g_{19}K^3x^5,$$

where the coefficients g are determined by the equations

$$g_0=\sqrt{1+e_1^2},\qquad g_{10}=2e_1e_{11},$$
$$g_1=2e_1e_2,\qquad g_{11}=2e_1c_{12}+2e_2e_{11},$$
$$g_2=2g_0e_{11}+2e_1e_3.\qquad g_{12}=2g_0c_{17}+2e_1e_{13}+2e_3e_{11},$$
$$g_3=2e_1e_4+e_2^2,\qquad g_{13}=2e_1e_{14}+2e_2e_{12}+2e_4e_{11},$$
$$g_4=2g_0e_{12}+2e_1e_5+2e_2e_3,\qquad g_{14}=2g_0e_{18}+2e_1e_{15}+2e_2e_{13}+2e_3e_{12}+2e_5e_{11},$$
$$g_5=2g_0e_{13}+2e_1c_6+e_3^2,\qquad g_{15}=2g_0e_{19}+2e_1e_{16}+2e_3e_{13}+2e_6e_{11},$$
$$g_6=2c_1e_7+2e_2e_4,\qquad g_{16}=e_{11}^2+2e_1e_{17},$$
$$g_7=2g_0e_{14}+2e_1e_8+2e_2e_5+2e_3e_4,\qquad g_{17}=2e_{11}e_{12}+2c_1e_{18}+2e_2e_{17},$$
$$g_8=2g_0e_{15}+2e_1e_9+2e_2e_6+2e_3e_5,\qquad g_{18}=2g_0e_{20}+2c_{11}e_{13}+2c_1e_{19}+2c_3e_{17},$$
$$g_9=2g_0e_{16}+2e_1e_{10}+2e_3e_6,\qquad g_{19}=2e_1e_{20}+2e_{11}e_{17}.$$

Suppose, for brevity, we put

$$1-x^2-y^2=A_0+A_1x+A_2x^2+KA_3x^2+K^2A_4x^4+K^3A_5x^5,$$

where the A are all of the zero order with respect to K. Let the right member of this be multiplied by the factor

$$B_0+KB_1x+K^2B_2x^2+K^3B_3x^3,$$

the coefficients B having the same quality as the A. The product is

$$A_0B_0+[A_1B_0+KA_0B_1]x+[A_2B_0+KA_1B_1+K^2A_0B_2]x^2$$
$$+[A_3B_0+A_2B_1+KA_1B_2+K^2A_0B_3]Kx^3$$
$$+[A_4B_0+A_3B_1+A_2A_2+KA_1B_3]K^2x^4$$
$$+[A_5B_0+A_4B_1+A_3B_2+A_2B_3]K^3x^5.$$

Let the B be determined by the relations

$$
\begin{aligned}
&A_2B_0 + KA_1B_1 + K^2A_0B_2 = -1\,,\\
&A_3B_0 + A_2B_1 + KA_1B_2 + K^2A_0B_3 = 0\,,\\
&A_4B_0 + A_3B_1 + A_2B_2 + KA_1B_3 = 0\,,\\
&A_5B_0 + A_4B_1 + A_3B_2 + A_2B_3 = 0\,.
\end{aligned}
$$

If we assume that

$$
\begin{aligned}
&B_0 = h_1 + (h_2 + h_3C)\,K + (h_4 + h_5C + h_6C^2)\,K^2 + (h_7 + h_8C + h_9C^2 + h_{10}C^3)\,K^3,\\
&B_1 = h_{11} + (h_{12} + h_{13}C)\,K + (h_{14} + h_{15}C + h_{16}C^2)\,K^2,\\
&B_2 = h_{17} + (h_{18} + h_{19}C)\,K\,,\\
&B_3 = h_{20},
\end{aligned}
$$

the coefficients h may be derived through the following formulæ of recursion:

$$
\begin{aligned}
&g_0^2h_1 = 1\,,\\
&g_0^2h_{11} + g_{10}h_1 = 0\,,\\
&g_0^2h_{17} + g_{10}h_{11} + g_{16}h_1 = 0\,,\\
&g_0^2h_{20} + g_{10}h_{17} + g_{16}h_{11} + g_{19}h_1 = 0\,,\\
&g_0^2h_2 + g_1h_1 = 0\,,\\
&g_0^2h_{12} + g_1h_{11} + g_{11}h_1 = 0\,,\\
&g_0^2h_{18} + g_1h_{17} + g_{10}h_{12} + g_{11}h_{11} + g_{16}h_2 + g_{17}h_1 = 0\,,\\
&g_0^2h_4 + g_1h_2 + g_3h_1 - h_{17} = 0\,,\\
&g_0^2h_{14} + g_1h_{12} + g_3h_{11} + g_{10}h_4 + g_{11}h_2 + g_{13}h_1 + h_{20} = 0\,,\\
&g_0^2h_7 + g_1h_4 + g_3h_2 + g_6h_1 - h_{18} = 0\,,\\
&g_0^2h_3 + g_2h_1 + 2g_0e_1h_{11} = 0\,,\\
&g_0^2h_{13} + g_2h_{11} + g_{10}h_3 + g_{12}h_1 + 2g_0e_1h_{17} = 0\,,\\
&g_0^2h_{19} + g_2h_{17} + g_{10}h_{13} + g_{12}h_{11} + g_{16}h_3 + g_{18}h_1 + 2g_0e_1h_{20} = 0\,,\\
&g_0^2h_5 + g_1h_3 + g_2h_2 + g_4h_1 + 2g_0e_1h_{12} + 2g_0e_2h_{11} = 0\,,\\
&g_0^2h_{15} + g_1h_{13} + g_2h_{12} + g_4h_{11} + g_{10}h_5 + g_{11}h_3 + g_{12}h_2 + g_{14}h_1 + 2g_0e_1h_{18} + 2g_0e_2h_{17} = 0\,,\\
&g_0^2h_8 + g_1h_5 + g_3h_3 + g_2h_4 + g_4h_2 + g_7h_1 + 2g_0e_1h_{14} + 2g_0e_2h_{12} + 2g_0e_4h_{11} - h_{19} = 0\,,\\
&g_0^2h_6 + g_2h_3 + g_5h_1 + 2g_0e_3h_{11} + 2g_0e_1h_{13} + g_0^2h_{17} = 0\,,\\
&g_0^2h_{16} + g_5h_{11} + g_{10}h_6 + g_{12}h_3 + g_{15}h_1 + 2g_0e_1h_{19} + 2g_0e_3h_{17} + g_0^2h_{20} = 0\,,\\
&g_0^2h_9 + g_1h_6 + g_2h_5 + g_4h_3 + g_5h_2 + g_8h_1 + 2g_0e_3h_{12} + 2g_0e_5h_{11} + 2g_0e_1h_{15} + g_0^2h_{18} = 0\,,\\
&g_0^2h_{10} + g_2h_6 + g_5h_3 + g_9h_1 + 2g_0e_1h_{16} + 2g_0e_3h_{13} + 2g_0e_6h_{11} + g_0^2h_{19} = 0\,.
\end{aligned}
$$

If we put the reciprocal of the product of the first and second factors (the negative being disregarded) equal to

$$
\begin{aligned}
&l_1 + (l_2 + l_3C)\,K + (l_4 + l_5C + l_6C^2)\,K^2 + (l_7 + l_8C + l_9C^2 + l_{10}C^3)\,K^3\\
&+ [l_{11}K + (l_{12} + l_{13}C)\,K^2 + (l_{14} + l_{15}C + l_{16}C^2)\,K^3]\,x\\
&+ [l_{17}K^2 + (l_{18} + l_{19}C)\,K^3]\,x^2 + l_{20}K^3x^3,
\end{aligned}
$$

the equations to recursion, for determining the values of the coefficients, are

$$
\begin{array}{ll}
b_{15}l_1 = 1, & b_{15}l_2 + b_{16}l_1 = 0, \\
b_{15}l_4 + b_{16}l_2 + b_{17}l_1 = 0, & b_{15}l_7 + b_{16}l_4 + b_{17}l_2 + b_{18}l_1 = 0, \\
l_3 + f_1 l_1 = 0, & l_5 + f_1 l_2 + f_2 l_1 = 0, \\
l_8 + f_1 l_4 + f_2 l_2 + f_3 l_1 = 0, & l_6 + f_1 l_3 + f_3 l_1 = 0, \\
l_9 + f_1 l_5 + f_2 l_3 + f_3 l_2 + f_5 l_1 = 0, & l_{10} + f_1 l_6 + f_3 l_3 + f_6 l_1 = 0, \\
l_{11} + f_7 l_1 = 0, & l_{12} + f_7 l_2 + f_8 l_1 = 0, \\
l_{14} + f_7 l_4 + f_8 l_2 + f_{10} l_1 = 0, & l_{13} + f_1 l_{11} + f_7 l_3 + f_9 l_1 = 0, \\
l_{15} + f_1 l_{12} + f_2 l_{11} + f_7 l_6 + f_{12} l_1 = 0, & l_{16} + f_1 l_{13} + f_3 l_{11} + f_7 l_6 + f_{12} l_1 = 0, \\
l_{17} + f_7 l_{11} + f_{13} l_1 = 0, & l_{18} + f_7 l_{12} + f_8 l_{11} + f_{13} l_2 + f_{14} l_1 = 0, \\
l_{19} + f_1 l_{17} + f_7 l_{13} + f_9 l_{11} + f_{13} l_3 + f_{15} l_1 = 0, & l_{20} + f_7 l_{17} + f_{13} l_{11} + f_{16} l_1 = 0,
\end{array}
$$

In order to have the numerator of the value of $\frac{d\Phi}{dx}$ expressed in powers of x it is still necessary to multiply the preceding expression by

$$\sqrt{B_0 + KB_1 x + K^2 B_2 x^2 + K^3 B_3 x^3}$$

This factor, expanded in powers of x, C and K, has the expression

$$
\begin{aligned}
F = m_1 &+ (m_2 + m_3 C) K + (m_4 + m_5 C + m_6 C^2) K^2 + (m_7 + m_8 C + m_9 C^2 + m_{10} C^3) K^3 \\
&+ [m_{11} K + (m_{12} + m_{13} C) K^2 + (m_{14} + m_{15} C + m_{16} C^2) K^3] x \\
&+ [m_{17} K^2 + (m_{18} + m_{19} C) K^3] x^2 + m_{20} K^3 x^3,
\end{aligned}
$$

where the m are determined by the relations

$$
\begin{array}{l}
g_0 m_1 = 1, \quad m_2 = \tfrac{1}{2} g_0 h_2, \quad m_3 = \tfrac{1}{2} g_0 h_3, \quad m_4 = \tfrac{1}{2} g_0 h_4 - \tfrac{1}{2} g_0 m_2^2, \\
m_5 = \tfrac{1}{2} g_0 h_5 - g_0 m_2 m_3, \quad m_6 = \tfrac{1}{2} g_0 h_6 - \tfrac{1}{2} g_0 m_3^2, \quad m_7 = -\tfrac{1}{2} g_0 h_7 - g_0 m_2 m_4, \\
m_8 = \tfrac{1}{2} g_0 h_8 - g_0 m_3 m_4 - g_0 m_2 m_5, \quad m_9 = \tfrac{1}{2} g_0 h_9 - g_0 m_2 m_6 - g_0 m_3 m_5, \\
m_{10} = \tfrac{1}{2} g_0 h_{10} - g_0 m_3 m_6, \quad m_{11} = \tfrac{1}{2} g_0 h_{11}, \quad m_{12} = \tfrac{1}{2} g_0 h_{12} - g_0 m_2 m_{11}, \\
m_{13} = \tfrac{1}{2} g_0 h_{13} - g_0 m_3 m_{11}, \quad m_{14} = \tfrac{1}{2} g_0 h_{14} - g_0 m_2 m_{12} - g_0 m_4 m_{11}, \\
m_{15} = \tfrac{1}{2} g_0 h_{15} - g_0 m_2 m_{13} - g_0 m_5 m_{11} - g_0 m_3 m_{12}, \quad m_{16} = \tfrac{1}{2} g_0 h_{16} - g_0 m_6 m_{11} - g_0 m_3 m_{13}, \\
m_{17} = \tfrac{1}{2} g_0 h_{17} - \tfrac{1}{2} g_0 m_{11}^2, \quad m_{18} = \tfrac{1}{2} g_0 h_{18} - g_0 m_2 m_{17} - g_0 m_{11} m_{12}, \\
m_{19} = \tfrac{1}{2} g_0 h_{19} - g_0 m_3 m_{17} - g_0 m_{11} m_{13}, \quad m_{20} = \tfrac{1}{2} g_0 h_{20} - g_0 m_{11} m_{17}.
\end{array}
$$

Denote by X the negative of the numerator in the expression of $\frac{d\Phi}{dx}$, it will have the form

$$
\begin{aligned}
X = n_1 &+ (n_2 + n_3 C) K + (n_4 + n_5 C + n_6 C^2) K^2 + (n_7 + n_8 C + n_9 C^2 + n_{10} C^3) K^3 \\
&+ [n_{11} K (n_{12} + n_{13} C) K^2 + (n_{14} + n_{15} C + n_{16} C^2) K^3] x \\
&+ [n_{17} K^2 + (n_{18} + n_{19} C) K^3] x^2 + n_{20} K^3 x^3;
\end{aligned}
$$

where the n are determined by the relations

$$
\begin{aligned}
n_1 &= m_1 l_1, \qquad n_2 = m_1 l_2 + m_2 l_1, \qquad n_3 = m_1 l_3 + m_3 l_1,\\
n_4 &= m_1 l_4 + m_2 l_2 + m_4 l_1, \qquad n_5 = m_1 l_5 + m_2 l_3 + m_3 l_2 + m_5 l_1,\\
n_6 &= m_1 l_6 + m_3 l_3 + m_6 l_1, \qquad n_7 = m_1 l_7 + m_2 l_4 + m_4 l_2 + m_7 l_1,\\
n_8 &= m_1 l_8 + m_2 l_5 + m_4 l_3 + m_3 l_4 + m_5 l_2 + m_8 l_1,\\
n_9 &= m_1 l_9 + m_2 l_6 + m_3 l_5 + m_5 l_3 + m_6 l_2 + m_9 l_1,\\
n_{10} &= m_1 l_{10} + m_3 l_6 + m_6 l_3 + m_{10} l_1, \qquad n_{11} = m_1 l_{11} + m_{11} l_1,\\
n_{12} &= m_1 l_{12} + m_2 l_{11} + m_{11} l_2 + m_{12} l_1, \qquad n_{13} = m_1 l_{13} + m_3 l_{11} + m_{11} l_3 + m_{13} l_1,\\
n_{14} &= m_1 l_{14} + m_2 l_{12} + m_4 l_{11} + m_{11} l_4 + m_{12} l_2 + m_{14} l_1,\\
n_{15} &= m_1 l_{15} + m_2 l_{13} + m_3 l_{12} + m_5 l_{11} + m_{13} l_2 + m_{15} l_1,\\
n_{16} &= m_1 l_{16} + m_3 l_{13} + m_6 l_{11} + m_{11} l_6 + m_{13} l_3 + m_{16} l_1, \qquad n_{17} = m_1 l_{17} + m_{11} l_{11} + m_{17} l_1,\\
n_{18} &= m_1 l_{18} + m_2 l_{17} + m_{11} l_{12} + m_{12} l_{11} + m_{17} l_2 + m_{18} l_1,\\
n_{19} &= m_1 l_{19} + m_3 l_{17} + m_{11} l_{13} + m_{13} l_{11} + m_{17} l_3 + m_{19} l_1,\\
n_{20} &= m_1 l_{20} + m_{11} l_{17} + m_{17} l_{11} + m_{20} l_1.
\end{aligned}
$$

We now have

$$\frac{d\phi}{dx} = -\frac{X}{\sqrt{A_0 B_0 + (A_1 B_0 + K A_0 B_1)\, x - x^2}}.$$

Let the quantity under the radical sign be resolved into the factors $\mathrm{a} - x$ and $\mathrm{b} + x$, and adopt an auxiliary variable ψ such that

$$x = \tfrac{1}{2}(\mathrm{a} - \mathrm{b}) - \tfrac{1}{2}(\mathrm{a} + \mathrm{b}) \cos\psi.$$

Then

$$-\frac{dx}{\sqrt{(\mathrm{a} - x)(\mathrm{b} + x)}} = d\psi,$$

The values of the two constants in the expression for x are

$$
\begin{aligned}
\tfrac{1}{2}(\mathrm{a} - \mathrm{b}) &= h = \tfrac{1}{2}(A_1 B_0 + K A_0 B_1),\\
\tfrac{1}{2}(\mathrm{a} + \mathrm{b}) &= k = \sqrt{h^2 + A_0 B_0}.
\end{aligned}
$$

The expression for h can be ordered thus

$$h = o_1 C + [o_2 + o_3 C + o_4 C^2] K + [o_5 + o_6 C + o_7 C^2 + o_8 C^3] K^2 + [o_9 + o_{10} C + \ldots + o_{13} C^4] K^3,$$

where

$$
\begin{aligned}
o_1 &= -\frac{e_1}{g_0}, \qquad o_2 = \tfrac{1}{2} h_{11}, \qquad o_3 = -g_0 (e_1 h_2 + e_2 h_1), \qquad o_4 = -g_0 (e_1 h_3 + e_3 h_1 + \tfrac{1}{2} g_0 h_{11}),\\
o_5 &= \tfrac{1}{2} h_{12}, \quad o_6 = \tfrac{1}{2} h_{13} - g_0 (e_1 h_4 + e_2 h_2 + e_4 h_1), \quad o_7 = -g_0 (e_1 h_5 + e_2 h_3 + e_3 h_2 + e_5 h_1 + \tfrac{1}{2} g_0 h_{12}),\\
o_8 &= -g_0 (e_1 h_6 + e_3 h_3 + e_6 h_1 + \tfrac{1}{2} g_0 h_{13}), \quad o_9 = \tfrac{1}{2} h_{14}, \quad o_{10} = \tfrac{1}{2} h_{15} - g_0 (e_1 h_7 + e_2 h_4 + e_4 h_2 + e_7 h_1),\\
o_{11} &= \tfrac{1}{2} h_{16} - g_0 (e_1 h_8 + e_2 h_5 + e_3 h_4 + e_4 h_3 + e_5 h_2 + e_8 h_1 + \tfrac{1}{2} g_0 h_{14}),\\
o_{12} &= -g_0 (e_1 h_9 + e_2 h_6 + e_3 h_5 + e_5 h_3 + e_6 h_2 + e_9 h_1 + \tfrac{1}{2} g_0 h_{15}),\\
o_{13} &= -g_0 (e_1 h_{10} + e_3 h_6 + e_6 h_3 + e_{10} h_1 + \tfrac{1}{2} g_0 h_{16}).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
g_0 k = \Big\{ 1 &+ [k_1 + k_2 C + k_3 C^2 + k_4 C^3] \frac{K}{1 - C^2} + [k_5 + k_6 C + \ldots + k_{11} C^6] \frac{K^2}{(1 - C^2)^2}\\
&+ [k_{12}' + k_{13} C + \ldots + k_{21} C^9] \frac{K^3}{(1 - C^2)^3} \Big\} \sqrt{1 - C^2},
\end{aligned}
$$

where

$$k_1 = \tfrac{1}{2} g_0^2 h_2, \qquad k_2 = g_0^2 (o_1 o_2 + \tfrac{1}{2} h_3), \qquad k_3 = g_0^2 (o_1 o_3 - \tfrac{1}{2} g_0^2 h_2), \qquad k_4 = g_0^2 (o_1 o_4 - \tfrac{1}{2} g_0^2 h_3),$$
$$k_5 = g_0^2 (\tfrac{1}{2} o_2^2 + \tfrac{1}{2} h_4) - \tfrac{1}{2} k_1^2, \qquad k_6 = g_0^2 (o_1 o_5 + o_2 o_3 + \tfrac{1}{2} h_5) - k_1 k_2,$$
$$k_7 = g_0^2 (o_1 o_6 + o_2 o_4 + \tfrac{1}{2} o_3^2 - \tfrac{1}{2} o_2^2 + \tfrac{1}{2} h_6 - \tfrac{1}{2} (1 + g_0^2) h_4) - \tfrac{1}{2} k_2^2 - k_1 k_3,$$
$$k_8 = g_0^2 (o_1 o_7 + o_3 o_4 - o_1 o_5 - o_2 o_3 - \tfrac{1}{2} (1 + g_0^2) h_5) - k_1 k_4 - k_2 k_3,$$
$$k_9 = g_0^2 (o_1 o_8 - o_1 o_6 - o_2 o_4 + \tfrac{1}{2} o_4^2 - \tfrac{1}{2} o_3^2 - \tfrac{1}{2} (1 + g_0^2) h_8 + \tfrac{1}{2} g_0^2 h_4 - \tfrac{1}{2} k_3^2 - k_2 k_4,$$
$$k_{10} = - g_0^2 (o_1 o_7 + o_3 o_4 - \tfrac{1}{2} g_0^2 h_5) - k_3 k_4, \qquad k_{11} = - g_0^2 (o_1 o_8 + \tfrac{1}{2} o_4^2 - \tfrac{1}{2} g_0^2 h_6) - \tfrac{1}{2} k_4^2,$$
$$k_{12} = g_0^2 (o_2 o_5 + \tfrac{1}{2} h_7) - k_1 k_5, \qquad k_{13} = g_0^2 (o_1 o_9 + o_2 o_6 + o_3 o_5 + \tfrac{1}{2} h_8) - k_1 k_6 - k_2 k_5,$$
$$k_{14} = g_0^2 (o_1 o_{10} + o_2 o_7 + o_3 o_6 + o_4 o_5 - 2 o_2 o_5 + \tfrac{1}{2} h_9 - h_7 - \tfrac{1}{2} g_0^2 h_7) - k_1 k_7 - k_2 k_6 - k_3 k_5,$$
$$\begin{aligned} k_{15} = g_0^2 (o_1 o_{11} + o_2 o_8 + o_3 o_7 + o_4 o_6 - 2 o_1 o_9 - 2 o_2 o_6 - 2 o_3 o_5 + \tfrac{1}{2} h_{10} - h_8 - \tfrac{1}{2} g_0^2 h_8) \\ - k_1 k_8 - k_2 k_7 - k_3 k_6 - k_4 k_5, \end{aligned}$$
$$\begin{aligned} k_{18} = g_0^2 (o_1 o_{12} + o_3 o_8 + o_4 o_7 - 2 o_1 o_{10} - 2 o_2 o_7 - 2 o_3 o_6 - 2 o_4 o_5 - h_9 - \tfrac{1}{2} g_0^2 (h_9 - 2 h_7)) \\ - k_1 k_9 - k_2 k_8 - k_3 k_7 - k_4 k_5, \end{aligned}$$
$$\begin{aligned} k_{17} = g_0^2 (o_1 o_{13} + o_4 o_8 - 2 o_1 o_{11} - 2 o_2 o_8 - 2 o_3 o_7 - 2 o_4 o_6 - h_{10} - \tfrac{1}{2} g_0^2 (h_{10} - 2 h_8)) \\ - k_1 k_{10} - k_2 k_9 - k_3 k_8 - k_4 k_7, \end{aligned}$$
$$\begin{aligned} k_{18} = g_0^2 (o_1 o_{10} + o_2 o_7 + o_3 o_8 + o_4 o_5 - 2 o_1 o_{12} - 2 o_3 o_8 - 2 o_4 o_7 + \tfrac{1}{2} h_9 - \tfrac{1}{2} g_0^2 (h_7 - 2 h_9)) \\ - k_1 k_{11} - k_2 k_{10} - k_3 k_9 - k_4 k_8, \end{aligned}$$
$$\begin{aligned} k_{19} = g_0^2 (o_1 o_{11} + o_2 o_8 + o_3 o_7 + o_4 o_8 - 2 o_1 o_{13} - 2 o_4 o_8 + \tfrac{1}{2} h_{10} - \tfrac{1}{2} g_0^2 h_8 - 2 h_{10})) \\ - k_2 k_{11} - k_3 k_{10} - k_4 k_9, \end{aligned}$$
$$k_{20} = g_0^2 (o_1 o_{12} + o_3 o_8 + o_4 o_7 - \tfrac{1}{2} g_0^2 h_9) - k_3 k_{11} - k_4 k_{10},$$
$$k_{21} = g_0^2 (o_1 o_{13} + o_4 o_8 - \tfrac{1}{2} g_0^2 h_{10}) - k_4 k_{11}.$$

Let us abbreviate by putting

$$X = X_0 + X_1 x + X_2 x^2 + X_3 x^3.$$

By making the substitution

$$x = h - k \cos \psi,$$

the differential equation becomes

$$\begin{aligned} \frac{d\varphi}{d\psi} = X_0 + hX_1 + (h^2 + \tfrac{1}{2} k^2) X_2 + (h^2 + \tfrac{3}{2} k^2) hX_3 \\ - [X_1 + 2hX_2 + (3h^2 + \tfrac{3}{4} k^2) X_3] k \cos \psi \\ + \tfrac{1}{2} [X_2 + 3hX_3] k^2 \cos 2\psi - \tfrac{1}{4} X\ k^3 \cos 3\psi. \end{aligned}$$

Let us write this

$$\theta_0 \frac{d\varphi}{d\psi} = 1 + \theta_1 \cos \psi + 2\theta_2 \cos 2\psi + 3\theta_3 \cos 3\psi.$$

By integration

$$\theta_0 (\varphi + c) = \psi + \theta_1 \sin \psi + \theta_2 \sin 2\psi + \theta_3 \sin 3\psi.$$

Putting θ' for $\theta_0(\phi + c)$, the inversion of this equation is

$$\psi = \theta' - [\theta_1 + \tfrac{1}{2} \theta_1\theta_2 - \tfrac{1}{8} \theta_1^3] \sin \theta' - [\theta_2 - \tfrac{1}{2} \theta_1^2] \sin 2\theta' - [\theta_3 - \tfrac{3}{2}\theta_1\theta_2 + \tfrac{3}{8}\theta_1^3] \sin 3\theta'.$$

From this may be derived

$$\cos\psi = \tfrac{1}{2}\theta_1 + [1 + \tfrac{1}{2}\theta_2 - \tfrac{7}{8}\theta_1]\cos\theta' + [-\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_3 - \theta_1\theta_2 + \tfrac{5}{24}\theta_1^3]\cos 2\theta'.$$
$$+ [-\tfrac{1}{2}\theta_2 + \tfrac{3}{8}\theta_1^2]\cos 3\theta' + [-\tfrac{1}{2}\theta_3 + \theta_1\theta_2 - \tfrac{5}{24}\theta_1^3]\cos 4\theta'.$$

By substituting the just obtained values we have expressions in terms of ψ for the three variables η, η' and θ; thus

$$\eta = K(1 - h + k\cos\psi), \qquad \eta' = K(1 + h - k\cos\psi),$$
$$\sqrt{\eta\eta'}\cos\theta = Ky = g_0CK + \{e_1 + (e_2 + e_3C)K + (e_4 + e_5C + e_6C^2)K^2 + (e_7 + e_8C$$
$$+ e_9C^2 + e_{10}C^3)K^3\}\, K(h - k\cos\psi)$$
$$+ \{e_{11} + (e_{12} + e_{13}C)K + (e_{14} + e_{15}C + e_{16}C^2)K^2\}\, K^2(h - k\cos\psi)^2$$
$$+ \{e_{17} + (e_{18} + e_{19}C)K\}\, K^3(h - k\cos\psi)^3 + e_{20}K^4(h - k\cos\psi)^4,$$
$$\sqrt{\eta\eta'}\sin\theta = K\sqrt{1 - x^2 - y^2} = \frac{Kk\sin\psi.}{X}$$

We also find that

$$\frac{1}{\theta_0} = p_1 + (p_2 + p_3C)K + p_4 + p_5C + p_6C^2)K^2 + (p_7 + p_8C + p_9C^2 + p_{10}C^3)K^3,$$

where we have

$$p_1 = n_1, \qquad p_2 = n_2, \qquad p_3 = n_3 + o_1 n_{11}, \qquad p_4 = n_4 + o_2 n_{11} + \tfrac{1}{2} h_1 n_{17},$$
$$p_5 = n_5 + o_1 n_{12} + o_3 n_{11} + (3o_1o_2 + \tfrac{1}{2}h_3)\, n_{17}, \qquad p_6 = n_6 + \tfrac{1}{2}(3o_1^2 - 1)\, n_{17},$$
$$p_7 = n_7 + o_5 n_{11} + o_2 n_{12} + \tfrac{1}{2} h_1 n_{18} + \tfrac{1}{2} h_2 n_{17} + \tfrac{3}{2} h_1 n_{20},$$
$$p_8 = n_8 + o_1 n_{14} + o_2 n_{13} + o_6 n_{12} + \tfrac{1}{2} h_1 n_{19} + (3o_1o_2 + \tfrac{1}{2}h_3)\, n_{17},$$
$$p_9 = n_9 + o_1 n_{15} + o_3 n_{12} + o_4 n_{12} + o_7 n_{11} + \tfrac{1}{2}(3o_1^2 - 1)\, n_{18} + (3o_1o_3 - \tfrac{1}{2}g_0^2h_2)\, n_{17} + \tfrac{1}{2}(5o_1^2 - 3)\, n_{20},$$
$$p_{10} = n_{10} + o_1 n_{16} + o_4 n_{13} + o_8 n_{11} + \tfrac{1}{2}(3o_1^2 - 1)\, n_{19} + (3o_1o_4 - \tfrac{1}{2}g_0^2h_6)\, n_{17}.$$

The reciprocal of this is

$$\theta_0 = q_1 + (q_2 + q_3C)K + (q_4 + q_5C + q_6C^2)K^2 + (q_7 + q_8C + q_9C^2 + q_{10}C^3)K^3,$$

where the q are determined by

$$p_1q_1 = 1, \; p_1q_2 + p_2q_1 = 0, \; p_1q_3 + p_3q_1 = 0, \; p_1q_4 + p_2q_2 + p_4q_1 = 0,$$
$$p_1q_5 + p_2q_3 + p_3q_2 + p_5q_1 = 0, \; p_1q_6 + p_3q_3 + p_6q_1 = 0,$$
$$p_1q_7 + p_2q_4 + p_4q_2 + p_7q_1 = 0, \; p_1q_8 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_8q_1 = 0,$$
$$p_1q_9 + p_2q_6 + p_3q_5 + p_5q_3 + p_6q_2 + p_9q_1 = 0, \; p_1q_{10} + p_3q_6 + p_6q_3 + p_{10}q_1 = 0.$$

For the sake of brevity we write z for $\cos\theta'$, and get

$$\cos\psi = \left[r_1K + \frac{r_2 + r_3C + r_4C^2 + r_5C^3}{1 - C^2}K^2 + \frac{r_6 + r_7C + \ldots + r_{12}C^6}{(1 - C^2)^2}K^3\right]\sqrt{1 - C^2}$$
$$+ \left[1 + r_{13}(1 - C^2)K^2 + (r_{14} + r_{15}C + r_{16}C^2 + r_{17}C^3 + r_{18}C^4)K^3\right] z$$
$$+ \left[-r_1K - \frac{r_2 + r_3C + r_4C^2 + r_5C^3}{1 - C^2}K^2 + \frac{r_{19} + r_{20}C + \ldots + r_{25}C^6}{(1 - C^2)^2}K^3\right]\sqrt{1 - C^2}\, z^2$$
$$+ \left[r_{26}(1 - C^2)K^2 + (r_{27} + r_{28}C + r_{29}C^2 + r_{30}C^3 + r_{31}C^4)K^3\right] z^3 + r_{32}(1 - C^2)^{\frac{3}{2}}K^3z^4,$$

where

$$g_0 r_1 = -q_1 n_{11},\ g_0 r_2 = -(q_1 n_{12} + q_2 n_{11}) + g_0 k_1 r_1,\ g_0 r_3 = -(q_1 n_{13} + 2o_1 q_1 n_{17} + q_3 n_1 + g_0 k_2 + r_1,$$

$$r_4 = -r_2 + (k_1 + k_3) r_1, \quad r_5 = -r_3 + (k_2 + k_4) r_1,$$

$$g_0 r_6 = -(q_1 n_{14} + 2 o_2 q_1 n_{17} + \tfrac{1}{4} h_1 q_1 n_{20} + q_2 n_{12} + q_4 n_{11}) + g_0 k_1 r_3 + g_0 (k_5 - k_1 k_2) r_1 + \tfrac{1}{4} r_{32},$$

$$g_0 r_7 = -(q_1 n_{15} + 2o_1 q_1 n_{18} + 2 o_3 q_1 n_{17} + 2 q_2 n_{13} + 2o_1 q_2 n_{17} + q_3 n_{12} + q_5 n_{11}) + g_0 k_1 r_3 + g_0 k_2 r_2 \\ + g_0 (k_6 - 2 k_1 k_2) r_1,$$

$$g_0 r_8 = -(q_1 n_{16} + o_1 q_1 n_{19} + o_4 q_1 n_{17} + (3 e_1^2 - \tfrac{1}{4}) h_1 q_1 n_{20} + q_3 n_{13} + 2 o_1 q_3 n_{17} + q_6 n_{11}) - 2 g_0 r_8 \\ + g_0 (2 k_1 + k_2) r_3 + g_0 (k_3 - k_1) r_2 + g_0 (k_7 + 2k_6 - k_2^2 + k_1^2 - k_1 k_3 - 2 k_1 k_2) r_1 - \tfrac{1}{4} r_{32},$$

$$r_9 = -2 r_7 + (k_1 + k_3) r_3 + (k_2 + k_4) r_2 + (k_8 + 2 k_8 - k_2 k_3 - k_1 k_4 - 2 k_1 k_2) r_1,$$

$$r_{10} = -r_8 - r_6 + (k_1 + k_4) r_3 - k_1 r_2 + (k_9 + k_7 + k_5 + k_1^2 - k_1 k_2 - k_2 k_4) r_1 + \tfrac{3}{4} r_{32},$$

$$r_{11} = r_7 - (k_1 + k_3) r_3 - k_2 r_2 + (k_{10} - k_6 + k_1 k_4 + k_2 k_3 + 2 k_1 k_2) r_1,$$

$$r_{12} = r_8 + 2 r_6 + (-2k_1 + k_2 - k_4) r_3 + (k_1 - k_3) r_2 + (k_{11} - k_7 - 2 k_5 + k_2^2 - k_1^2 + k_2 k_4 \\ + k_1 k_3 + 2 k_1 k_2) r_1 - \tfrac{1}{2} r_{32},$$

$$r_{13} = -r_{26} - \tfrac{1}{2} r_1^2, \qquad r_{14} = -r_{27} - r_1 r_2, \qquad r_{15} = -r_{28} - r_1 r_3, \qquad r_{16} = -r_{29} - r_1 r_4,$$

$$r_{17} = -r_{30} - r_1 r_5, \qquad r_{18} = -r_{31}, \qquad r_{19} = -r_6 - r_{32}, \qquad r_{20} = -r_7,$$

$$r_{21} = -r_8 + 3 r_{32}, \qquad r_{22} = -r_9, \qquad r_{23} = -r_{10} - 3 r_{32}, \qquad r_{24} = -r_{11},$$

$$r_{25} = -r_{12} + r_{32}, \qquad r_{26} = -\tfrac{1}{2} h_1 q_1 (n_{17} - 3 q_1 n_{11}^2),$$

$$r_{27} = -\tfrac{1}{2} h_1 q_1 (n_{18} - 6 q_1 n_{11} n_{13} - 3 q_2 n_{11}^2) - \tfrac{1}{2} h_1 (q_2 + 2 k_1 q_1) (n_{17} - 3q_1 n_{11}^2),$$

$$r_{28} = -\tfrac{1}{2} h_1 q_1 (n_{19} + 3 o_1 n_{20} - 6 q_1 n_{11} n_{13} - 12 o_1 q_1 n_{11} n_{17} - 3 q_3 n_{11}^2) - \tfrac{1}{2} h_1 (q_3 + 2k_2 q_1) (n_{17} - 3 q_1 n_{11}^2)$$

$$r_{29} = -r_{27} + \tfrac{1}{2} h_1 [q_3 - q_2 - 2 (k_1 + k_3) q_1] (n_{17} - 3 q_1 n_{11}^2) + r_{31},$$

$$r_{30} = -r_{28} - h_1 q_1 (k_2 + k_4) (n_{17} - 3q_1 n_{11}^2),$$

$$r_{31} = -\tfrac{1}{2} h_1 q_1 (6 q_3 n_{11} n_{13} + 12 o_1 q_3 n_{11} n_{17}), \qquad r_{32} = h_2^2 q_1 (\tfrac{1}{3} n_{20}^2 - 2 q_1 n_{11} n_{17} + \tfrac{5}{3} q_1^2 n_{11}^2).$$

By substituting the foregoing value of $\cos\psi$ in $x = h - k\cos\psi$ we obtain an expression of the form

$$x = s_1 C + (s_2 + s_3 C + s_4 C^2) K + (s_5 + s_6 C + s_7 C^2 + s_8 C^3) K^2 + \frac{s_9 + s_{10} C + \ldots + s_{15} C^6}{1 - C^2} K^3 \\ + \left\{ s_{16} + \frac{s_{17} + s_{18} C + \ldots + s_{20} C^3}{1 - C^2} K + \frac{s_{21} + s_{22} C + \ldots + s_{27} C_6}{(1 - C^2)^2} K^2 \right. \\ \left. + \frac{s_{28} + s_{29} C + \ldots + s_{37} C^9 + s_{38} C^{10}}{(1 - C^2)^3} K^3 \right\} \sqrt{1 - C^2}\, z \\ + \left\{ s_{39} (1 - C^2) K + (s_{40} + s_{41} C + \ldots + s_{43} C^3) K^2 + \frac{s_{44} + s_{45} C \ldots + s_{50} C^6}{1 - C^2} K^3 \right\} z^2 \\ + \left\{ s_{51} (1 - C^2) K^2 + (s_{52} + s_{53} C + \ldots + s_{55} C^3 + s_{56} C^4 K^3 \right\} \sqrt{1 - C^2}\, z^3 + s_{57} (1 - C^2) K^3 z^4,$$

where

$$s_1 = o_1, \quad g_0 s_2 = g_0 o_2 - r_1, \quad s_3 = o_8, \quad g_0 s_4 = g_0 o_4 + r_1, \quad g_0 s_5 = g_0 o_5 - k_1 r_1 - r_2,$$

$$g_0 s_6 = g_0 o_6 - k_2 r_1 - r_3, \quad g_0 s_7 = g_0 o_7 - k_3 r_1 - r_4, \quad g_0 s_8 = g_0 o_8 - k_4 r_1 - r_5,$$

$$g_0 s_9 = g_0 o_9 - k_5 r_1 - k_1 r_2 - r_6, \quad g_0 s_{10} = g_0 o_{10} - k_6 r_1 - k_2 r_2 - k_1 r_8 - r_7,$$

$$g_0 s_{11} = g_0 (o_{11} - o_9) - k_7 r_1 - k_3 r_2 - k_2 r_3 - k_1 r_4 - r_8,$$

$$g_0 s_{12} = g_0 (o_{12} - o_{10}) - k_8 r_1 - k_4 r_2 - k_3 r_3 - k_2 r_4 - k_1 r_5 - r_9,$$

$$g_0 s_{13} = g_0 (o_{13} - o_{11}) - k_9 r_1 - k_4 r_3 - k_3 r_4 - k_2 r_5 - r_{10}, \quad g_0 s_{14} = -g_0 o_{12} - k_{10} r_1 - k_4 r_4 - k_3 r_5 - r_{11},$$

$$g_0 s_{15} = -g_0 o_{13} - k_{11} r_1 - k_4 r_5 - r_{12}, \quad g_0 s_{16} = -1, \quad g_0 s_{17} = -k_1, \quad g_0 s_{18} = -k_2,$$
$$g_0 s_{19} = -k_3, \quad g_0 s_{20} = -k_4, \quad g_0 s_{21} = -k_5 - r_{13}, \quad g_0 s_{22} = -k_6, \quad g_0 s_{23} = -k_7 + 3r_{13},$$
$$g_0 s_{24} = -k_8, \quad g_0 s_{25} = -k_9 - 3r_{13}, \quad g_0 s_{26} = -k_{10}, \quad g_0 s_{27} = -k_{11} + r_{13},$$
$$g_0 s_{28} = -k_{12} - k_1 r_{13} - r_{14}, \quad g_0 s_{29} = -k_{13} - k_2 r_{13} - r_{15}, \quad g_0 s_{30} = -k_{14} - (k_3 - 3k_1) r_{13} + 3r_{14} - r_{16},$$
$$g_0 s_{31} = -k_{15} - (k_4 - 3k_2) r_{13} + 3r_{15} - r_{17}, \quad g_0 s_{32} = -k_{16} + 3(k_3 - k_1) r_{13} - 3r_{14} + 3r_{16} - r_{18},$$
$$g_0 s_{33} = -k_{17} + (k_4 - k_2) r_{13} - 3r_{15} + 3r_{17}, \quad g_0 s_{34} = -k_{18} - (3k_3 - k_1) r_{13} + r_{14} - 3r_{16} + 3r_{18},$$
$$g_0 s_{35} = -k_{19} - (3k_4 - k_2) r_{13} + r_{15} - 3r_{17}, \quad g_0 s_{36} = -k_{20} + k_3 r_{13} + r_{16} - 3r_{18},$$
$$g_0 s_{37} = -k_{21} + k_4 r_{13} + r_{17}, \quad g_0 s_{38} = r_{18}, \quad g_0 s_{39} = r_1, \quad g_0 s_{40} = k_1 r_1 + r_2,$$
$$g_0 s_{41} = k_2 r_1 + r_3, \quad g_0 s_{42} = k_3 r_1 + r_4, \quad g_0 s_{43} = k_4 r_1 + r_5, \quad g_0 s_{44} = k_5 r_1 + k_1 r_2 - r_{19},$$
$$g_0 s_{45} = k_6 r_1 + k_2 r_2 + k_1 r_3 - r_{20}, \quad g_0 s_{46} = k_7 r_1 + k_3 r_2 + k_2 r_3 + k_1 r_4 - r_{21},$$
$$g_0 s_{47} = k_8 r_1 + k_4 r_2 + k_3 r_3 + k_2 r_4 + k_1 r_5 - r_{22}, \quad g_0 s_{48} = k_9 r_1 + k_4 r_3 + k_3 r_4 + k_2 r_5 - r_{23},$$
$$g_0 s_{49} = k_{10} r_1 + k_4 r_4 + k_3 r_5 - r_{24}, \quad g_0 s_{50} = k_{11} r_1 + k_4 r_5 - r_{25}, \quad g_0 s_{51} = -r_{26},$$
$$g_0 s_{52} = -k_1 r_{26} - r_{27}, \quad g_0 s_{53} = -k_2 r_{26} - r_{28}, \quad g_0 s_{54} = -k_3 r_{26} - r_{29},$$
$$g_0 s_{55} = -k_4 r_{26} - r_{30}, \quad g_0 s_{56} = -r_{31}, \quad g_0 s_{57} = -r_{32}.$$

From the expression of W, given at the beginning of this case, we may derive the following values for $\dfrac{df}{d\phi}$ and $\dfrac{df'}{d\phi}$:

$$\begin{aligned}\frac{df}{d\phi} = & -\left[a_1 + 2a_3\eta + a_4\eta' + 3a_6\eta^2 + 2a_7\eta\eta' + a_8\eta'^2 + 4a_{10}\eta^3 + 3a_{11}\eta^2\eta' + 2a_{12}\eta\eta'^2 + a_{13}\eta'^3\right] \\ & - \tfrac{1}{2}\left[a_{15} + 3a_{18}\eta + a_{17}\eta' + 5a_{18}\eta^2 + 3a_{19}\eta\eta' + a_{20}\eta'^2 + 7a_{21}\eta^3 + 5a_{22}\eta^2\eta' + 3a_{23}\eta\eta'^2 + a_{24}\eta'^3\right]\frac{Ky}{\eta} \\ & - \left[a_{25} + 2a_{26}\eta + a_{27}\eta' + 3a_{28}\eta^2 + 2a_{29}\eta\eta' + a_{30}\eta'^2\right]\frac{K^2y^2}{\eta} \\ & - \tfrac{1}{2}\left[3a_{31} + 5a_{32}\eta + 3a_{33}\eta'\right]\frac{K^3y^3}{\eta} - 2a_{34}\frac{K^4y^4}{\eta},\end{aligned}$$

$$\begin{aligned}\frac{df'}{d\phi} = & -\left[a_2 + a_4\eta + 2a_5\eta' + a_7\eta^2 + 2a_8\eta\eta' + 3a_9\eta'^2 + a_{11}\eta^3 + 2a_{12}\eta^2\eta' + 3a_{13}\eta\eta'^2 + 4a_{14}\eta'^3\right] \\ & - \tfrac{1}{2}\left[a_{15} + a_{16}\eta + 3a_{17}\eta' + a_{18}\eta^2 + 3a_{19}\eta\eta' + 5a_{20}\eta'^2 + a_{21}\eta^3 + 3a_{22}\eta^2\eta' + 5a_{23}\eta\eta'^2 + 7a_{24}\eta'^3\right]\frac{Ky}{\eta'} \\ & - \left[a_{25} + a_{26}\eta + 2a_{27}\eta' + a_{28}\eta^2 + 2a_{29}\eta\eta' + 3a_{30}\eta'^2\right]\frac{K^2y^2}{\eta'} \\ & - \tfrac{1}{2}\left[3a_{31} + 3a_{32}\eta + 5a_{33}\eta'\right]\frac{K^3y^3}{\eta'} - 2a_{34}\frac{K^4y^4}{\eta'}.\end{aligned}$$

In the formulæ $\sqrt{1-x^2}\cos\theta$ has been replaced by y. Substitute the values

$$\eta = K(1-x), \qquad \eta' = K(1+x);$$

then

$$\begin{aligned}\frac{df}{d\phi} = & \; d_1 + d_2K + d_3K^2 + d_4K^3 + (d_5K + d_6K^2 + d_7K^3)\,x + (d_8K^2 + d_9K^3)\,x^2 + d_{10}K^3x^3 \\ & + \Big[d_{11} + d_{12}K + d_{13}K^2 + d_{14}K^3 + (d_{15}K + d_{16}K^2 + d_{17}K^3)\,x + (d_{18}K^2 + d_{19}K^3)\,x^2 \\ & \qquad + d_{20}K^3x^3\Big]\frac{y}{1-x} \\ & + \left[d_{21}K + d_{22}K^2 + d_{23}K^3 + (d_{24}K^2 + d_{25}K^3)\,x + d_{26}K^3x^2\right]\frac{y^2}{1-x} \\ & + \left[d_{27}K^2 + d_{28}K^3 + d_{29}K^3x\right]\frac{y^3}{1-x} + d_{30}K^4\frac{y^4}{1-x},\end{aligned}$$

with a similar equation for the motion of f', to be obtained from that just written by accenting the d and substituting the divisor $1+x$ for $1-x$. The coefficients d and d' are determined by the following equations:

$$
\begin{aligned}
&d_1 = -a_1,\ d_2 = -2a_3 - a_4,\ d_3 = -3a_6 - 2a_7 - a_8,\ d_4 = -4a_{10} - 3a_{11} - 2a_{12} - a_{13},\\
&d_5 = 2a_3 - a_4,\ d_6 = 6a_6 - 2a_3,\ d_7 = 12a_{10} + 3a_{11} - 2a_{12} - 3a_{13},\\
&d_8 = -3a_6 + 2a_7 - a_8,\ d_9 = -12a_{10} + 3a_{11} + 2a_{12} - 3a_{13},\ d_{10} = 4a_{10} - 3a_{11} + 2a_{12} - a_{13},\\
&d_{11} = -\tfrac{1}{2}a_{15},\ d_{12} = -\tfrac{3}{2}a_{16} - \tfrac{1}{2}a_{17},\ d_{13} = -\tfrac{5}{2}a_{18} - \tfrac{3}{2}a_{19} - \tfrac{1}{2}a_{20},\\
&d_{14} = -\tfrac{7}{2}a_{21} - \tfrac{5}{2}a_{22} - \tfrac{3}{2}a_{23} - \tfrac{1}{2}a_{24},\ d_{15} = \tfrac{3}{2}a_{16} - \tfrac{1}{2}a_{17},\ d_{16} = 5a_{18} - a_{20},\\
&d_{17} = \tfrac{21}{2}a_{21} + \tfrac{6}{2}a_{22} - \tfrac{3}{2}a_{23} - \tfrac{3}{2}a_{24},\ d_{18} = -\tfrac{5}{2}a_{18} + \tfrac{3}{2}a_{19} - \tfrac{1}{2}a_{20},\\
&d_{19} = -\tfrac{21}{2}a_{21} + \tfrac{5}{2}a_{22} + \tfrac{3}{2}a_{23} - \tfrac{3}{2}a_{24},\ d_{20} = \tfrac{7}{2}a_{21} - \tfrac{6}{2}a_{22} + \tfrac{3}{2}a_{23} - \tfrac{1}{2}a_{24},\\
&d_{21} = -a_{25},\ d_{22} = -2a_{26} - a_{27},\ d_{23} = -3a_{28} - 2a_{29} - a_{30},\ d_{24} = 2a_{26} - a_{27},\\
&d_{25} = 6a_{28} - 2a_{30},\ d_{26} = -3a_{28} + 2a_{29} - a_{30},\ d_{27} = -\tfrac{3}{2}a_{31},\ d_{28} = -\tfrac{5}{2}a_{32} - \tfrac{3}{2}a_{33},\\
&d_{29} = \tfrac{5}{2}a_{32} - \tfrac{3}{2}a_{33},\ d_{30} = -2a_{34};\\
&d'_1 = -a_2,\ d'_2 = -a_4 - 2a_5,\ d'_3 = -a_7 - 2a_8 - 3a_9,\ d'_4 = -a_{11} - 2a_{12} - 3a_{13} - 4a_{14},\\
&d'_5 = a_4 - 2a_5,\ d'_6 = 2a_7 - 6a_9,\ d'_7 = 3a_{11} + 2a_{12} - 3a_{13} - 12a_{14},\\
&d'_8 = -a_7 + 2a_3 - 3a_9,\ d'_9 = -3a_{11} + 2a_{12} + 3a_{13} - 12a_{14},\ d'_{10} = a_{11} - 2a_{12} + 3a_{13} - 4a_{14},\\
&d'_{11} = -\tfrac{1}{2}a_{15},\ d'_{12} = -\tfrac{1}{2}a_{16} - \tfrac{3}{2}a_{17},\ d'_{13} = -\tfrac{1}{2}a_{18} - \tfrac{3}{2}a_{19} - \tfrac{5}{2}a_{20},\\
&d'_{14} = -\tfrac{1}{2}a_{21} - \tfrac{3}{2}a_{22} - \tfrac{5}{2}a_{23} - \tfrac{7}{2}a_{24},\ d'_{15} = \tfrac{1}{2}a_{16} - \tfrac{3}{2}a_{17},\ d'_{16} = a_{18} - 5a_{20},\\
&d'_{17} = \tfrac{3}{2}a_{21} + \tfrac{3}{2}a_{22} - \tfrac{6}{2}a_{23} - \tfrac{21}{2}a_{24},\ d'_{18} = -\tfrac{1}{2}a_{18} + \tfrac{3}{2}a_{19} - \tfrac{5}{2}a_{20},\\
&d'_{19} = -\tfrac{3}{2}a_{21} + \tfrac{3}{2}a_{22} + \tfrac{5}{2}a_{23} - \tfrac{21}{2}a_{24},\ d'_{20} = \tfrac{1}{2}a_{21} - \tfrac{3}{2}a_{22} + \tfrac{5}{2}a_{23} - \tfrac{7}{2}a_{24},\\
&d'_{21} = -a_{25},\ d'_{22} = -a_{26} - 2a_{27},\ d'_{23} = -a_{23} - 2a_{29} - 3a_{30},\ d'_{24} = a_{26} - 2a_{27},\\
&d'_{25} = 2a_{28} - 6a_{30},\ d'_{26} = -a_{28} + 2a_{29} - 3a_{30},\ d'_{27} = -\tfrac{3}{2}a_{31},\ d'_{28} = -\tfrac{3}{2}a_{32} - \tfrac{5}{2}a_{33},\\
&d'_{29} = \tfrac{3}{2}a_{32} - \tfrac{5}{2}a_{33},\ d'_{30} = -2a_{34}.
\end{aligned}
$$

By means of the previously given value we eliminate y from the just written expressions of $\frac{df}{d\phi}$ and $\frac{df'}{d\phi}$, and, by division, reduce the numerators belonging to the divisors $1-x$ and $1+x$ to constants. Thus we arrive at expressions having the form

$$
\begin{aligned}
\frac{df}{d\phi} = {} & j_1 + (j_2 + j_3 C)K + (j_4 + j_5 C + j_6 C^2)K^2 + (j_7 + j_8 C + j_9 C^2 + j_{10} C^3)K^3\\
&+ [j_{11}K + (j_{12} + j_{13}C)K^2 + (j_{14} + j_{15}C + j_{16}C^2)K^3]x + [j_{17}K^2 + (j_{18} + j_{19}C)K^3]x^2\\
&+ j_{20}K^3x^3 + [j_{21} + j_{22}C + (j_{23} + j_{24}C + j_{25}C^2)K + (j_{26} + j_{27}C + j_{28}C^2 + j_{29}C^3)K^2\\
&+ (j_{30} + j_{31}C + j_{32}C^2 + j_{33}C^3 + j_{34}C^4)K^3]\frac{1}{1-x}.
\end{aligned}
$$

For $\frac{df'}{d\phi}$ we simply accent the j and change the divisor $1-x$ into $1+x$.

The j and j' are determined by the relations

$$j_1 = d_1 - e_1 d_{11},\ j_2 = d_2 - (e_2 + e_{11}) d_{11} - e_1(d_{12} + d_{15}) - e_1^2 d_{21},\ j_3 = - e_3 d_{11} - g_0 d_{15} - 2g_0 e_1 d_{21},$$

$$\begin{aligned} j_4 &= d_3 - (e_4 + e_{12} + e_{17}) d_{11} - (e_2 + e_{11}) d_{12} - e_1(d_{13} + d_{16} + d_{18}) - (e_2 + e_{11}) d_{19} - (g_1 + g_{10}) d_{21} \\ &\quad - e_1^2 (d_{22} + d_{24}) - e_1^3 d_{27}, \end{aligned}$$

$$\begin{aligned} j_5 &= -(e_5 + e_{13}) d_{11} - e_3 (d_{12} + d_{15}) - g_0 (d_{16} + d_{18}) - (2g_0 e_2 + g_2) d_{21} - 2g_0 e_1 d_{24} \\ &\quad - 3g_0 (e_1^2 + 2e_1 e_2) d_{27}, \end{aligned}$$

$$j_6 = - e_6 d_{11} - 2g_0 e_3 d_{21} - g_0^2 d_{24} - 3g_0^2 e_1 d_{27},$$

$$\begin{aligned} j_7 &= d_4 - (e_7 + e_{14} + e_{18} + e_{20}) d_{11} - (e_4 + e_{12} + e_{17}) d_{12} - (e_2 + e_{11}) d_{13} - e_1(d_{14} + d_{17} + d_{19} + d_{20}) \\ &\quad - (e_4 + e_{12} + e_{17}) d_{15} - (e_2 + e_{11}) (d_{16} + d_{18}) - (g_3 + g_{11} + g_{16}) d_{21} - (g_1 + g_{10}) (d_{22} + d_{24}) \\ &\quad - (2g_0 e_1 + e_1^2) d_{23} - e_1^2 (d_{25} + d_{26}) - 3e_1^2 (e_2 + e_{11}) d_{27} - e_1^3 d_{29} - e_1^4 d_{30}, \end{aligned}$$

$$\begin{aligned} j_8 &= -(e_8 + e_{15} + e_{19}) d_{11} - (e_5 + e_{13}) (d_{12} + d_{15}) - e_3 (d_{13} + d_{18}) - g_0 (d_{17} + d_{19} + d_{20}) \\ &\quad - (g_4 + g_{12}) d_{21} - g_2 (d_{22} + d_{24}) - 4g_0 e_2 d_{24} - 2g_0 e_1 (d_{25} + d_{26}) - (6g_0 e_1 e_{11} + 3e_1^2 e_3) d_{27} \\ &\quad - 3g_0 e_1^2 d_{28} - 4g_0 e_1^3 d_{30}, \end{aligned}$$

$$\begin{aligned} j_9 &= -(e_9 + e_{16}) d_{11} - e_6 (d_{12} + d_{15}) - (2g_0 e_5 + g_5) d_{21} - 2g_0 e_3 d_{22} - g_0^2 (d_{25} + d_{26}) \\ &\quad - (3g_0^2 e_2 + 3g_0^2 e_{11} + 6g_0 e_1 e_3) d_{27} - 3g_0^2 e_1 (d_{28} + d_{29}) - 6g_0^2 e_1^2 d_{30}, \end{aligned}$$

$$j_{10} = - e_{10} d_{11} - 2g_0 e_6 d_{21} - 3g_0^2 e_3 d_{27} - g_0^3 d_{29} - 4g_0^3 e_1 d_{30}, \quad j_{11} = d_5 - e_{11} d_{11} - e_1 d_{15} - e_1^2 d_{21},$$

$$\begin{aligned} j_{12} &= d_6 - (e_{12} + e_{17}) d_{11} - e_{11} d_{12} - (e_2 + e_{11}) d_{15} - e_1 (d_{16} + d_{18}) - (g_1 + g_{10}) d_{21} \\ &\quad - e_1^2 (d_{22} + d_{24}) - e_1^3 d_{27}, \end{aligned}$$

$$j_{13} = - e_{13} d_{11} - e_3 d_{15} - g_0 d_{18} - g_2 d_{21} - 2g_0 e_1 d_{24} - (6g_0 e_1 e_2 + 3g_0 e_1^2) d_{27},$$

$$\begin{aligned} j_{14} &= d_7 - (e_{14} + e_{20}) d_{11} - (e_{12} + e_{17}) d_{12} - e_{11} d_{13} - (e_4 + e_{12} + e_{17}) d_{15} - (e_2 + e_{11}) (d_{16} + d_{18}) \\ &\quad - e_1 (d_{17} + d_{19} + d_{20}) - (g_3 + g_{11} + g_{16}) d_{21} - (g_1 + g_{10}) (d_{22} + d_{24}) - e_1^2 (d_{23} + d_{25} + d_{26}) \\ &\quad - 3e_1^2 (e_2 + e_{11}) d_{27} - e_1^3 (d_{28} + d_{29}) - e_1^4 d_{30}, \end{aligned}$$

$$\begin{aligned} j_{15} &= -(e_{15} + e_{19}) d_{11} - e_{13} d_{12} - (e_5 + e_{13}) d_{15} - e_3 (d_{18} + d_{18}) - g_0 (d_{19} + d_{20}) - (g_4 + g_{12}) d_{21} \\ &\quad - g_2 (d_{22} + d_{24}) - 4g_0 e_2 d_{24} - 2g_0 e_1 (d_{25} + d_{26}) - (6g_0 e_1 e_{11} + 3e_1^2 e_3) d_{27} - 3g_0 e_1^2 d_{28} - 4g_0 e_1^3 d_{30}, \end{aligned}$$

$$j_{16} = - e_{16} d_{11} - e_6 d_{15} - g_5 d_{21} - g_0^2 d_{26} - (6g_0 e_1 e_3 + 3g_0^2 e_{11}) d_{27} - 3g_0^2 e_1 d_{29} - 6g_0^2 e_1^2 d_{30},$$

$$j_{17} = d_8 - e_{17} d_{11} - e_{11} d_{15} - e_1 d_{18} - g_{10} d_{21} - e_1^2 d_{24} - e_1^3 d_{27},$$

$$\begin{aligned} j_{18} &= d_9 - (e_{18} + e_{20}) d_{11} - e_{17} d_{12} - (e_{12} + e_{17}) d_{15} - e_{11} d_{16} - (e_2 + e_{11}) d_{18} - e_1(d_{19} + d_{20}) \\ &\quad - (g_{11} + g_{16}) d_{21} - g_{10} d_{22} - (g_1 + g_{10}) d_{24} - e_1^2 (d_{25} + d_{26}) - 3e_1^2 (e_2 + e_{11}) d_{27} \\ &\quad - e_1^3 (d_{28} + d_{29}) - e_1^4 d_{30}, \end{aligned}$$

$$j_{19} = - e_{19} d_{11} - e_{13} d_{15} - e_3 d_{18} - g_0 d_{20} - g_{12} d_{21} - g_2 d_{24} - 2g_0 e_1 d_{26} - (6g_0 e_1 e_{11} + 3e_1^2 e_3 + 3e_1^2 e_2) d_{27},$$

$$j_{20} = d_{10} - e_{20} d_{11} - e_{17} d_{15} - e_{11} d_{18} - e_1 d_{20} - g_{18} d_{21} - g_{10} d_{24} - e_1^2 d_{28} - 3e_1^2 e_{11} d_{27} - e_1^3 d_{29} - e_1^4 d_{30},$$

$$j_{21} = e_1 d_{11},\ j_{22} = d_{11},\ j_{23} = (e_2 + e_{11}) d_{11} + e_1 (d_{12} + d_{15}) + e_1^2 d_{21},$$

$$j_{24} = e_3 d_{11} + d_{12} + g_0 d_{15} + 2g_0 e_1 d_{21}, \quad j_{25} = d_{21}, \quad j_{26} = d_3 + j_4, \quad j_{27} = d_{13} - j_5,$$

$$j_{28} = d_{22} - j_6,\ j_{29} = d_{27},\ j_{30} = d_4 - j_7,\ j_{31} = d_{14} - j_8,\ j_{32} = d_{23} - j_9,$$

$$j_{33} = d_{28} - j_{10},\ j_{34} = d_{30};$$

$$j'_1 = d'_1 + e_1 d'_{11}, \quad j'_2 = d'_2 + (e_2 - e_{11}) d'_{11} + e_1 (d'_{12} - d'_{15}) - e_1^2 d'_{21}, \quad j'_3 = e_3 d'_{11} + g_0 d'_{16} + 2g_0 e_1 d'_{21},$$

$$\begin{aligned} j'_4 &= d'_3 + (e_4 - e_{12} + e_{17}) d'_{11} + (e_2 - e_{11}) (d'_{12} - d'_{15}) + e_1 (d'_{13} - d'_{16} + d'_{18}) - (g_1 - g_{10}) d'_{21} \\ &\quad - e_1^2 (d'_{22} - d'_{24}) + e_1^3 d'_{27}, \end{aligned}$$

$$\begin{aligned} j'_5 &= (e_5 - e_{13}) d'_{11} + e_3 (d'_{12} - d'_{15}) + g_0 (d'_{16} - d'_{18}) + (2g_0 e_2 - g_2) d'_{21} + 2g_0 e_1 (d'_{22} - d'_{24}) \\ &\quad - 3g_0 (e_1^2 + 2e_1 e_2) d'_{27}, \end{aligned}$$

$$j'_6 = e_6 d'_{11} + 2g_0 e_3 d'_{21} + g_0^2 d'_{24} + 3g_0^2 e_1 d'_{27},$$

$$\begin{aligned} j'_7 &= d'_4 + (e_7 - e_{14} + e_{18} - e_{20}) d'_{11} + (e_4 - e_{12} + e_{17}) (d'_{12} - d'_{17}) + (e_2 - e_{11}) (d'_{13} - d'_{18} + d'_{18}) \\ &\quad + e_1 (d'_{14} - d'_{17} + d'_{19} - d'_{20}) + (g_3 - g_{11} + g_{16}) d'_{21} + (g_1 - g_{10}) (d'_{22} - d'_{24}) \\ &\quad + (2g_0 e_1 - e_1^2) d'_{23} - e_1^2 (d'_{25} - d'_{26}) + 3e_1^2 (e_2 - e_{11}) d'_{27} - e_1^3 (d'_{28} - d'_{29}) - e_1^4 d'_{30}, \end{aligned}$$

$$
\begin{aligned}
j'_8 &= (e_8 - e_{15} + e_{19})\, d'_{11} + (e_6 - e_{13})\,(d'_{12} - d'_{15}) + e_3\,(d'_{13} - d'_{18}) + g_0\,(d'_{17} - d'_{19} + d'_{20}) \\
&\quad - (g_4 - g_{12})\, d'_{21} - g_2\,(d'_{22} - d'_{24}) - 4g_0\, e_2\, d'_{24} - 2g_0\, e_1\,(d'_{25} - d'_{26}) + (6g_0\, e_1\, e_{11} + 3e_1^2\, e_3)\, d'_{27} \\
&\quad - 3g_0\, e_1^2\, d'_{28} + 4g_0\, e_1^3\, d'_{30}, \\
j'_9 &= (e_9 - e_{16})\, d'_{11} + e_6\,(d'_{12} - d'_{15}) + (2g_0\, e_5 - g_5)\, d'_{21} + 2g_0\, e_3\, d'_{22} + g_0^2\,(d'_{25} - d'_{26}) \\
&\quad + (3g_0^2\, e_2 - 3g_0^2\, e_{11} - 6g_0\, e_1\, e_3)\, d'_{27} + 3g_0^2\, e_1\,(d'_{28} - d'_{29}) - 6g_0^2\, e_1^2\, d'_{30}, \\
j'_{10} &= e_{10}\, d'_{11} + 2g_0\, e_6\, d'_{21} + 3g_0^2\, e_3\, d'_{27} + g_0^3\, d'_{29} + 4g_0^3\, e_1\, d'_{30}, \quad j'_{11} = d'_6 + e_{11}\, d'_{11} + e_1\, d'_{15} + e_1^2\, d'_{21}, \\
j'_{12} &= d'_6 + (e_{12} - e_{17})\, d'_{11} + e_{11}\, d'_{12} + (e_2 - e_{11})\, d'_{15} + e_1\,(d'_{16} - d'_{18}) + (g_1 - g_{10})\, d'_{21} \\
&\quad + e_1^2\,(d'_{22} - d'_{24}) - e_1^3\, d'_{27}, \\
j'_{13} &= e_{13}\, d'_{11} + e_3\, d'_{16} + g_0\, d'_{18} + g_2\, d'_{21} + 2g_0\, e_1\, d'_{24} + (6g_0\, e_1\, e_2 + 3g_0\, e_1^2)\, d'_{27}, \\
j'_{14} &= d'_7 + (e_{14} - e_{18})\, d'_{11} + (e_{12} - e_{17})\, d'_{12} + e_{11}\, d'_{13} + (e_4 - e_{12} + e_{17})\, d'_{15} + (e_2 - e_{11})\,(d'_{16} - d'_{18}) \\
&\quad + e_1\,(d'_{17} - d'_{19} + d'_{20}) + (g_3 - g_{11} + g_{16})\, d'_{21} + (g_1 - g_{10})\,(d'_{22} - d'_{24}) + e_1^2\,(d'_{23} - d'_{25} + d'_{26}) \\
&\quad - 3e_1^2\,(e_2 - e_{11})\, d'_{27} - e_1^3\,(d'_{28} - d'_{29}) + e_1^4\, d'_{30}, \\
j'_{15} &= (e_{15} - e_{19})\, d'_{11} + e_{13}\, d'_{12} + (e_5 - e_{13})\, d'_{15} + e_3\,(d'_{16} - d'_{18}) + g_0(d'_{19} - d'_{20}) + (g_4 - g_{12})\, d'_{21} \\
&\quad + g_2(d'_{22} - d'_{24}) + 4g_0 e_2 d'_{24} + 2g_0 e_1(d'_{25} - d'_{26}) - (6g_0 e_1 e_{11} + 3e_1^2 e_3)\, d'_{27} + 3g_0 e_1^2 d'_{28} - 4g_0 e_1^3 d'_{30}, \\
j'_{16} &= e_{16}\, d'_{11} + e_6\, d'_{15} + g_5\, d'_{21} + g_0^2\, d'_{26} + (6g_0\, e_1\, e_3 + 3g_0^2\, e_{11})\, d'_{27} + 3g_0^2\, e_1\, d'_{29} + 6g_0^2\, e_1^2\, d'_{30}, \\
j'_{17} &= d'_8 + e_{17}\, d'_{11} + e_{11}\, d'_{15} + e_1\, d'_{18} + g_{10}\, d'_{21} + e_1^2\, d'_{24} + e_1^3\, d'_{27}, \\
j'_{18} &= d'_9 + (e_{18} - e_{20})\, d'_{11} + e_{17}\, d'_{12} + (e_{12} - e_{17})\, d'_{15} + e_{11}\, d'_{16} + (e_2 - e_{11})\, d'_{18} + e_1\,(d'_{19} - d'_{20}) \\
&\quad + (g_{11} - g_{16})\, d'_{21} + g_{10}\, d'_{22} + (g_1 - g_{10})\, d'_{24} + e_1^2\,(d'_{25} - d'_{26}) + 3e_1^2\,(e_2 - e_{11})\, d'_{27} \\
&\quad + e_1^3\,(d'_{28} - d'_{29}) - e_1^4\, d'_{30}, \\
j'_{19} &= e_1 d'_{11} + e_{13} d'_{15} + e_3 d'_{18} + g_0 d'_{20} + g_{12} d'_{21} + g_2 d'_{24} + 2g_0\, e_1 d'_{26} + (6g_0 e_1 e_{11} + 3e_1^2 e_3)\, d'_{27} + 4g_0 e_1^3 d'_{30}, \\
j'_{20} &= d'_{10} + e_{20} d'_{11} + e_{17} d'_{15} + e_{11} d'_{18} + e_1\, d'_{20} + g_{16} d'_{21} + g_{10}\, d'_{24} + e_1^2\, d'_{26} + 3e_1^2 e_{11}\, d'_{27} + e_1^3\, d'_{29} + e_1^4\, d'_{30}, \\
j'_{21} &= -e_1\, d'_{11}, \quad j'_{22} = d'_{11}, \quad j'_{23} = -(e_2 - e_{11})\, d'_{11} - e_1\,(d'_{12} - d'_{15}) + e_1^2\, d'_{21}, \\
j'_{24} &= d'_{12} - e_3\, d'_{11} - g_0\, d'_{15} - 2g_0\, e_1\, d'_{21}, \quad j'_{25} = d'_{21}, \quad j'_{26} = d'_3 - j'_4, \quad j'_{27} = d'_{13} - j'_5, \\
j'_{28} &= d'_{22} - j'_6, \quad j'_{29} = d'_{27}, \quad j'_{30} = d'_4 - j'_7, \quad j'_{31} = d'_{14} - j'_8, \quad j'_{32} = d'_{23} - j'_9, \\
j'_{33} &= d'_{28} - j'_{10}, \quad j'_{34} = d'_{30}.
\end{aligned}
$$

It is now necessary to change the two preceding expressions from functions of x into functions of z. First consider the two terms of the fractional form $\dfrac{S}{1-x}$ and $\dfrac{S'}{1+x}$. Let the expression of x in terms of z be abbreviated to

$$x = B_0 + B_1 z + B_2 z^2 + B_3 z^3 + B_4 z^4;$$

then we have the fractions

$$\frac{S}{1 - B_0 - B_1 z - B_2 z^2 - B_3 z^3 - B_4 z^4} \text{ and } \frac{S'}{1 + B_0 + B_1 z + B_2 z^2 + B_3 z^3 + B_4 z^4}.$$

If we multiply numerator and denominator of both fractions by

$$1 + D_1 z + D_2 z^2 + D_3 z^3,$$

in which we suppose that D_1, D_2, D_3 are severally of the first, second and third orders with respect to K, these coefficients may be determined by the

conditions that the resulting denominators contain no powers of z above the first. In the first case this gives us the equations

$$\begin{aligned}
&(1-B_0)D_2 - B_1D_1 - B_2 &&= 0,\\
&(1-B_0)D_3 - B_1D_2 - B_2D_1 - B_3 &&= 0,\\
&-B_1D_3 - B_2D_2 - B_3D_1 - B_4 &&= 0,
\end{aligned}$$

with the resulting denominator

$$1 - B_0 - [B_1 - (1-B_0)D_1]\,z;$$

and, in the second case, the equations

$$\begin{aligned}
&(1+B_0)D_2' + B_1D_1' + B_2 &&= 0,\\
&(1+B_0)D_3' + B_1D_2' + B_2D_1' + B_3 &&= 0,\\
&B_1D_3' + B_2D_2' + B_3D_1' + B_4 &&= 0,
\end{aligned}$$

with the resulting denominator

$$1 + B_0 + [B_1 + (1-B_0)D_1']\,z.$$

We solve these equations by approximation, rejecting all terms of an order above the third with reference to K. In the first case, the equations are equivalent to the three formulæ to recursion:

$$\begin{aligned}
D_3 &= -\frac{B_4}{B_1} + 2\frac{B_2B_3}{B_1^2} - \left(\frac{B_2}{B_1}\right)^3,\\
D_2 &= -\frac{B_3}{B_1} + \left(\frac{B_2}{B_1}\right)^2 + \frac{1-B_0}{B_1}D_3,\\
D_1 &= -\frac{B_2}{B_1} + \frac{1-B_0}{B_1}D_2;
\end{aligned}$$

in the second case, the equations are equivalent to

$$\begin{aligned}
D_3' &= -\frac{B_4}{B_1} + 2\frac{B_2B_3}{B_1^2} - \left(\frac{B_2}{B_1}\right)^3,\\
D_2' &= -\frac{B_3}{B_1} + \left(\frac{B_2}{B_1}\right)^2 - \frac{1+B_0}{B_1}D_3',\\
D_1' &= -\frac{B_2}{B_1} - \frac{1+B_0}{B_1}D_2'.
\end{aligned}$$

For brevity let the new denominators of the fractions be represented by

$$1 - B_0 - Dz \quad\text{and}\quad 1 + B_0 + D'z.$$

The fractions then are

$$\frac{S + SD_1z + SD_2z^2 + SD_3z^3}{1 - B_0 - Dz} \quad\text{and}\quad \frac{S' + S'D_1'z + S'D_2'z^2 + S'D_3'z^3}{1 + B_0 + D'z}.$$

They can be reduced to the forms

$$H_0 + H_1z + H_2z^2 + \frac{H_3}{1 - B_0 - Dz} \quad\text{and}\quad H_0' + H_1'z + H_2'z^2 + \frac{H_3'}{1 + B_0 + D'z},$$

where the H are constants. The latter are determined by the equations

$$\left.\begin{aligned} -DH_2 &= SD_3 \\ -DH_1 + (1-B_0)H_2 &= SD_2 \\ -DH_0 + (1-B_0)H_1 &= SD_1 \\ H_3 + (1-B_0)H_0 &= S \end{aligned}\right\}, \qquad \left.\begin{aligned} D'H_2' &= S'D_3' \\ D'H_1' + (1+B_0)H_2' &= S'D_2' \\ D'H_0' + (1+B_0)H_1' &= S'D_1' \\ H_3' + (1+B_0)H_0' &= S' \end{aligned}\right\}.$$

Rejecting quantities above the third order, these equations are equivalent to the following formulæ to recursion:

$$\left.\begin{aligned} H_2 &= -S\frac{D_3}{B_1} \\ H_1 &= -S\frac{D_2}{D} + \frac{1-B_0}{B_1}H_2 \\ H_0 &= -S\frac{D_1}{D} + \frac{1-B_0}{B_1}H_1 \\ H_3 &= S-(1-B_0)H_0 \end{aligned}\right\}, \qquad \left.\begin{aligned} H_2' &= S'\frac{D_3'}{B_1} \\ H_1' &= S'\frac{D_2'}{D'} - \frac{1+B_0}{B_1}H_2' \\ H_0' &= S'\frac{D_1'}{D'} - \frac{1+B_0}{D'}H_1' \\ H_3' &= S'-(1+B_0)H_0' \end{aligned}\right\}.$$

All the constants involved in these expressions must be exhibited in powers and products of C and K. By putting

$$\frac{B_2}{B_1} = \left\{t_1(1-C^2)K + \frac{t_2+t_3C+\ldots+t_7C^5}{1-C^2}K^2 + \frac{t_8+t_9C+\ldots+t_{16}C^8}{(1-C^2)^2}K^3\right\}\frac{1}{\sqrt{1-C^2}}$$

we have for determining the coefficients t the relations

$s_{16}t_1 = s_{39}$, $\quad s_{16}t_2 + s_{17}t_1 = s_{40}$, $\quad s_{16}t_3 + s_{18}t_1 = s_{41}$, $\quad s_{16}t_4 + s_{19}t_1 = s_{42}$,
$s_{16}t_5 + (s_{20}-s_{17})t_1 = s_{43}$, $\quad s_{16}t_6 - s_{19}t_1 = 0$, $\quad s_{16}t_7 - s_{20}t_1 = 0$,
$s_{16}t_8 + s_{17}t_2 + s_{21}t_1 = s_{44}$, $\quad s_{16}t_9 + s_{17}t_3 + s_{18}t_2 + s_{22}t_1 = s_{45}$,
$s_{16}t_{10} + s_{17}t_4 + s_{18}t_3 + s_{19}t_2 + (s_{23}-s_{21})t_1 = s_{46}$,
$s_{16}t_{11} + s_{17}t_5 + s_{18}t_4 + s_{19}t_3 + s_{20}t_2 + (s_{24}-s_{22})t_1 = s_{47}$,
$s_{16}t_{12} + s_{17}t_6 + s_{18}t_5 + s_{19}t_4 + s_{20}t_3 + (s_{25}-s_{23})t_1 = s_{48}$,
$s_{16}t_{13} + s_{17}t_7 + s_{16}t_8 + s_{19}t_5 + s_{20}t_4 + (s_{26}-s_{24})t_1 = s_{49}$,
$s_{16}t_{14} + s_{18}t_7 + s_{19}t_6 + s_{20}t_5 + (s_{27}-s_{25})t_1 = s_{50}$, $\quad s_{16}t_{15} + s_{19}t_7 + s_{20}t_6 - s_{26}t_1 = 0$,
$s_{16}t_{16} + s_{20}t_7 - s_{27}t_1 = 0$.

Putting also

$$\frac{B_3}{B_1} = t_{17}(1-C^2)K^2 + \frac{t_{18}+t_{19}C+\ldots+t_{24}C^6}{1-C^2}K^3,$$

the coefficients are determined by the relations

$s_{16}t_{17} = s_{51}$, $\quad s_{16}t_{18} + s_{17}t_{17} = s_{52}$, $\quad s_{16}t_{19} + s_{18}t_{17} = s_{53}$,
$s_{16}t_{20} + (s_{19}-s_{17})t_{17} = s_{54} - s_{52}$, $\quad s_{16}t_{21} + (s_{20}-s_{18})t_{17} = s_{55} - s_{53}$, $\quad s_{16}t_{22} - s_{19}t_{17} = s_{56} - s_{54}$,
$s_{16}t_{23} - s_{20}t_{17} = s_{55}$, $\quad s_{16}t_{24} = -s_{56}$.

Also

$$\frac{B_4}{B_1} = t_{25}(1-C)^{\frac{3}{2}}K^3, \qquad s_{16}t_{25} = s_{57}.$$

We have

$$D_3 = D_3' = t_{26}(1 - C^2)^{\frac{3}{2}} K^3,$$

where

$$t_{26} = t_{25} + 2t_1 t_{17} - t_1^3.$$

We have

$$D_2 = t_{27}(1 - C^2) K^2 + \frac{t_{28} + t_{29} C + t_{30} C^2 + t_{31} C^3 + t_{32} C^4 + t_{33} C^5 + t_{34} C^6}{1 - C^2} K^3,$$

$$D_2' = t_{27}(1 - C^2) K^2 + \frac{t_{35} + t_{29} C + t_{36} C^2 + t_{31} C^3 + t_{37} C^4 + t_{33} C^5 + t_{34} C^6}{1 - C^2} K^3,$$

where

$$t_{27} = -t_{17} + t_1^3, \qquad t_{28} = -t_{18} + 2t_1 t_2 + \frac{t_{26}}{s_{16}}, \qquad t_{29} = -t_{19} + 2t_1 t_3 - \frac{s_1 t_{26}}{s_{16}},$$

$$t_{30} = -t_{20} + 2t_1 t_4 - 2\frac{t_{26}}{s_{16}}, \qquad t_{31} = -t_{21} + 2t_1 t_5 + 2\frac{s_1 t_{26}}{s_{16}}, \qquad t_{32} = -t_{22} + 2t_1 t_6 + \frac{t_{26}}{s_{16}},$$

$$t_{33} = -t_{23} + 2t_1 t_7 - \frac{s_1 t_{26}}{s_{16}}, \qquad t_{34} = -t_{24},$$

$$t_{35} = -t_{18} + 2t_1 t_2 - \frac{t_{26}}{s_{16}}, \qquad t_{36} = -t_{20} + 2t_1 t_4 + 2\frac{t_{26}}{s_{16}}, \qquad t_{37} = -t_{22} + 2t_1 t_6 - \frac{t_{26}}{s_{16}}.$$

Again

$$D_1 = -t_1 \sqrt{1 - C^2}\, K + \frac{t_{38} + t_{39} C + \ldots + t_{43} C^6}{(1 - C^2)^{\frac{3}{2}}} K^2 + \frac{t_{44} + t_{45} C + \ldots + t_{53} C^9}{(1 - C^2)^{\frac{5}{2}}} K^3,$$

$$D_1' = -t_1 \sqrt{1 - C^2}\, K + \frac{t_{54} + t_{55} C + \ldots + t_{59} C^5}{(1 - C^2)^{\frac{3}{2}}} K^2 + \frac{t_{60} + t_{61} C + \ldots + t_{69} C^9}{(1 - C^2)^{\frac{5}{2}}} K^3,$$

where

$$t_{38} = -t_2 + \frac{t_{27}}{s_{16}}, \qquad t_{39} = -t_3 - s_1 \frac{t_{27}}{s_{16}}, \qquad t_{40} = -t_4 - 2\frac{t_{27}}{s_{16}}, \qquad t_{41} = -t_5 + 2 s_1 \frac{t_{27}}{s_{16}},$$

$$t_{42} = -t_6 + \frac{t_{27}}{s_{16}}, \qquad t_{43} = -t_7 - s_1 \frac{t_{27}}{s_{16}}, \qquad t_{44} = -t_8 - s_2 \frac{t_{27}}{s_{16}} + \frac{t_{28}}{s_{16}} - s_{17} \frac{t_{27}}{s_{16}^2},$$

$$t_{45} = -t_9 - s_3 \frac{t_{27}}{s_{16}} + (t_{29} - s_1 t_{28}) \frac{1}{s_{16}} - (s_{16} - s_1 s_{17}) \frac{t_{27}}{s_{16}^2},$$

$$t_{46} = -t_{10} - (s_4 - 3 s_2) \frac{t_{27}}{s_{16}} + (t_{30} - s_1 t_{29} - t_{28}) \frac{1}{s_{16}} - (s_{19} - s_1 s_{16} - 2 s_{17}) \frac{t_{27}}{s_{16}^2},$$

$$t_{47} = -t_{11} + 3 s_3 \frac{t_{27}}{s_{26}} + (t_{31} - s_1 t_{30} - t_{29} + s_1 t_{28}) \frac{1}{s_{16}} - (s_{20} - s_1 s_{19} - 2 s_{18} + 2 s_1 s_{17}) \frac{t_{27}}{s_{16}^2},$$

$$t_{48} = -t_{12} + (3 s_4 - 3 s_2) \frac{t_{27}}{s_{16}} + (t_{32} - s_1 t_{31} - t_{30} + s_1 t_{29}) \frac{1}{s_{16}} + (s_1 s_{20} + 2s_{19} - 2s_1 s_{16} - s_{17}) \frac{t_{27}}{s_{16}^2},$$

$$t_{49} = -t_{13} - 3 s_3 \frac{t_{27}}{s_{16}} + (t_{33} - s_1 t_{32} - t_{31} + s_1 t_{30}) \frac{1}{s_{16}} + (2 s_{20} - 2 s_1 s_{19} - s_{16} + s_1 s_{17}) \frac{t_{27}}{s_{16}^2},$$

$$t_{50} = -t_{14} - (3 s_4 - s_2) \frac{t_{27}}{s_{16}} + (t_{34} - s_1 t_{33} - t_{32} + s_1 t_{31}) \frac{1}{s_{16}} - (2 s_1 s_{20} + s_{19} - s_1 s_{16}) \frac{t_{27}}{s_{16}^2},$$

$$t_{51} = -t_{15} + s_3 \frac{t_{27}}{s_{16}} - (s_1 t_{34} + t_{33} - s_1 t_{32}) \frac{1}{s_{16}} - (s_{20} - s_1 s_{19}) \frac{t_{27}}{s_{16}^2},$$

$$t_{52} = -t_{16} + s_4 \frac{t_{27}}{s_{16}} - (t_{34} - s_1 t_{33}) \frac{1}{s_{16}} + s_1 s_{20} \frac{t_{27}}{s_{16}^2}, \qquad t_{53} = \frac{s_1 t_{34}}{s_{16}},$$

$$t_{54} = -t_2 - \frac{t_{27}}{s_{16}}, \qquad t_{55} = -t_3 - s_1 \frac{t_{27}}{s_{16}}, \qquad t_{56} = -t_4 + 2\frac{t_{27}}{s_{16}}, \qquad t_{57} = -t_6 + 2 s_1 \frac{t_{27}}{s_{16}},$$

$$t_{58} = -t_6 - \frac{t_{27}}{s_{16}}, \qquad t_{59} = -t_7 - s_1 \frac{t_{27}}{s_{16}}, \qquad t_{60} = -t_8 - s_2 \frac{t_{27}}{s_{18}} - \frac{t_{28}}{s_{16}} - s_{17} \frac{t_{27}}{s_{16}^2},$$

$$t_{61} = -t_9 - s_3 \frac{t_{27}}{s_{16}} - (t_{29} + s_1 t_{28}) \frac{1}{s_{16}} + (s_{18} + s_1 s_{17}) \frac{t_{27}}{s_{16}^2},$$

$$t_{62} = -t_{10} - (s_4 - 3 s_2) \frac{t_{27}}{s_{16}} - (t_{30} + s_1 t_{29} - t_{28}) \frac{1}{s_{16}} + (s_{19} + s_1 s_{18} - 2 s_{17}) \frac{t_{27}}{s_{16}^2},$$

$$t_{63} = -t_{11} + 3 s_3 \frac{t_{27}}{s_{16}} - (t_{31} + s_1 t_{30} - t_{29} - s_1 t_{28}) \frac{1}{s_{16}} + (s_{20} + s_1 s_{19} - 2 s_{18} - 2 s_1 s_{17}) \frac{t_{27}}{s_{16}^2},$$

$$t_{64} = -t_{12} + (3 s_4 - 3 s_2) \frac{t_{27}}{s_{16}} - (t_{32} + s_1 t_{31} - t_{30} - s_1 t_{29}) \frac{1}{s_{16}} + (s_1 s_{20} - 2 s_{19} - 2 s_1 s_{18} + s_{17}) \frac{t_{27}}{s_{16}^2},$$

$$t_{65} = -t_{13} - 3 s_3 \frac{t_{27}}{s_{16}} - (t_{33} + s_1 t_{32} - t_{31} - s_1 t_{30}) \frac{1}{s_{16}} - (2 s_{20} + 2 s_1 s_{19} - s_{18} - s_1 s_{17}) \frac{t_{27}}{s_{16}^2},$$

$$t_{66} = -t_{14} - (3 s_4 - s_2) \frac{t_{27}}{s_{16}} - (t_{34} + s_1 t_{33} - t_{32} - s_1 t_{31}) \frac{1}{s_{16}} - (2 s_1 s_{20} - s_{19} - s_1 s_{18}) \frac{t_{27}}{s_{16}^2},$$

$$t_{67} = -t_{15} + s_3 \frac{t_{27}}{s_{16}} - (s_1 t_{34} - t_{33} - s_1 t_{32}) \frac{1}{s_{16}} + (s_{20} + s_1 s_{19}) \frac{t_{27}}{s_{16}^2},$$

$$t_{68} = -t_{16} + s_4 \frac{t_{27}}{s_{16}} + (t_{34} + s_1 t_{33}) \frac{1}{s_{16}} + s_1 s_{20} \frac{t_{27}}{s_{16}^2}, \qquad t_{69} = \frac{s_1 t_{34}}{s_{16}}.$$

From the foregoing expressions it is found that the reduced denominator for $\frac{S}{1-x}$ is

$$1 - s_1 C - (s_2 + s_3 C + s_4 C^2) K - (s_5 + s_6 C + s_7 C^2 + s_8 C^3) K^2 - \frac{s_9 + s_{10} C + \ldots + s_{15} C^6}{1 - C^2} K^3$$

$$- \left\{ s_{16} \sqrt{1 - C^2} + \frac{t_{70} + t_{71} C + \ldots + t_{73} C^3}{(1 - C^2)^{\frac{1}{2}}} K + \frac{t_{74} + t_{75} C + \ldots + t_{80} C^8}{(1 - C^2)^{\frac{3}{2}}} K^2 \right.$$

$$\left. + \frac{t_{81} + t_{82} C + \ldots + t_{90} C^9 + t_{91} C^{10}}{(1 - C^2)^{\frac{5}{2}}} K^3 \right\} z,$$

where the coefficients t result from the equations

$$t_{70} = s_{17} + t_1, \qquad t_{71} = s_{18} - s_1 t_1, \qquad t_{72} = s_{19} - 1, \qquad t_{73} = s_{20} + s_1,$$

$$t_{74} = s_{21} - t_{38} - s_2 t_1, \qquad t_{75} = s_{22} - t_{39} + s_1 t_{38} - s_3 t_1, \qquad t_{76} = s_{23} - t_{40} + s_1 t_{39} - (s_4 - 2 s_2) t_1,$$

$$t_{77} = s_{24} - t_{41} + s_1 t_{40} + 2 s_3 t_1, \qquad t_{78} = s_{25} - t_{42} + s_1 t_{41} + (2 s_4 - s_2) t_1,$$

$$t_{79} = s_{26} - t_{43} + s_1 t_{42} - s_3 t_1, \qquad t_{80} = s_{27} + s_1 t_{43} - s_4 t_1, \qquad t_{81} = s_{28} - t\ \ + s_2 t_{38} - s_5 t_1,$$

$$t_{82} = s_{29} - t_{45} + s_1 t_{44} + s_2 t_{39} + s_3 t_{38} - s_6 t_1,$$

$$t_{83} = s_{30} - t_{46} + s_1 t_{45} + s_2 t_{40} + s_3 t_{39} + (s_4 - s_2) t_{38} - (s_7 - 3 s_5) t_1,$$

$$t_{84} = s_{31} - t_{47} + s_1 t_{46} + s_2 t_{41} + s_3 t_{40} + (s_4 - s_2) t_{39} - s_3 t_{38} - (s_8 - 3 s_6) t_1,$$

$$t_{85} = s_{32} - t_{48} + s_1 t_{47} + s_2 t_{42} + s_3 t_{41} + (s_4 - s_2) t_{40} - s_3 t_{39} - s_4 t_{38} + (3 s_7 - 3 s_5) t_1,$$

$$t_{86} = s_{33} - t_{49} + s_1 t_{48} + s_2 t_{43} + s_3 t_{42} + (s_4 - s_2) t_{41} - s_3 t_{40} - s_4 t_{39} + (3 s_8 - 3 s_6) t_1,$$

$$t_{87} = s_{34} - t_{50} + s_1 t_{49} + s_3 t_{43} + (s_4 - s_2) t_{42} - s_3 t_{41} - s_4 t_{40} - (3 s_7 - s_5) t_1,$$

$$t_{88} = s_{35} - t_{51} + s_1 t_{50} + (s_4 - s_2) t_{43} - s_3 t_{42} - s_4 t_{41} - (3 s_8 - s_6) t_1,$$

$$t_{89} = s_{36} - t_{52} + s_1 t_{51} - s_3 t_{43} - s_4 t_{42} + s_7 t_1, \qquad t_{90} = s_{37} - t_{53} + s_1 t_{52} - s_4 t_{43} + s_8 t_1, \qquad t_{91} = s_1 t_{53}.$$

And the reduced denominator for $\dfrac{S'}{1+x}$ is

$$1 + s_1 C + (s_2 + s_3 C + s_4 C^2) K + (s_5 + s_6 C + s_7 C^2 + s_8 C^3) K^2 + \frac{s_9 + s_{10} C + \ldots + s_{15} C^6}{1 - C^2} K^3$$
$$+ \left\{ s_{16} \sqrt{1 - C^2} + \frac{t_{92} + t_{93} C + \ldots + t_{95} C^3}{\sqrt{1 - C^2}} K + \frac{t_{96} + t_{97} C + \ldots + t_{102} C^6}{(1 - C^2)^{\frac{3}{2}}} K^2 \right.$$
$$\left. + \frac{t_{103} + t_{104} C + \ldots + t_{112} C^9 + t_{113} C^{10}}{(1 - C^2)^{\frac{5}{2}}} K^3 \right\} z,$$

where the coefficients t result from the equations

$$t_9 = s_{17} - t_1, \qquad t_{93} = s_{18} - s_1 t_1, \qquad t_{94} = s_{19} + 1, \qquad t_{95} = s_{20} + s_1,$$
$$t_{96} = s_{21} + t_{54} - s_2 t_1, \quad t_{97} = s_{22} + t_{55} + s_1 t_{54} - s_3 t_1, \quad t_{98} = s_{23} + t_{56} + s_1 t_{55} + (2 s_2 - s_4) t_1,$$
$$t_{99} = s_{24} + t_{57} + s_1 t_{56} + 2 s_3 t_1, \qquad t_{100} = s_{25} + t_{58} + s_1 t_{57} + (2 s_4 - s_2) t_1,$$
$$t_{101} = s_{26} + t_{59} + s_1 t_{58} - s_3 t_1, \qquad t_{102} = s_{27} + s_1 t_{59} - s_4 t_1, \qquad t_{103} = s_{28} + t_{60} + s_2 t_{54} - s_5 t_1,$$
$$t_{104} = s_{29} + t \quad + s_1 t_{60} + s_2 t_{55} + s_3 t_{54} - s_6 t_1,$$
$$t_{105} = s_{30} + t_{62} + s_1 t_{61} + s_2 t_{56} + s_3 t_{55} + (s_4 - s_2) t_{54} - (s_7 - 3 s_5) t_1,$$
$$t_{106} = s_{31} + t_{63} + s_1 t_{62} + s_2 t_{57} + s_3 t_{56} + (s_1 - s_2) t_{55} - s_3 t_{54} - (s_8 - 3 s_6) t_1,$$
$$t_{107} = s_{32} + t_{64} + s_1 t_{63} + s_2 t_{58} + s_3 t_{57} + (s_4 - s_2) t_{56} - s_3 t_{55} - s_4 t_{54} + (3 s_7 - 3 s_5) t_1,$$
$$t_{108} = s_{33} + t_{65} + s_1 t_{64} + s_2 t_{59} + s_3 t_{58} + (s_4 - s_2) t_{57} - s_3 t_{56} - s_4 t_{55} + (3 s_8 - 3 s_6) t_1,$$
$$t_{109} = s_{34} + t_{66} + s_1 t_{65} + s_3 t_{59} + (s_4 - s_2) t_{58} - s_3 t_{57} - s_4 t_{56} - (3 s_7 - s_5) t_1,$$
$$t_{110} = s_{35} + t_{67} + s_1 t_{66} + (s_4 - s_2) t_{59} - s_3 t_{58} - s_4 t_{57} - (3 s_8 - s_6) t_1,$$
$$t_{111} = s_{36} + t_{68} + s_1 t_{67} - s_3 t_{59} - s_4 t_{58} + s_7 t_1, \quad t_{112} = s_{37} + t_{69} + s_1 t_{68} - s_4 t_{59} + s_8 t_1, \quad t_{113} = s_1 t_{69}.$$

Proceding now to the determination of the H we have

$$H_2 = -\frac{t_{26}}{s_{16}} (j_{21} + j_{22} C)(1 - C^2) K^3, \qquad H'_2 = \frac{t_{26}}{s_{16}} (j'_{21} + j'_{22} C)(1 - C^2) K^3.$$

Also

$$H_1 = -\frac{t_{27}}{s_{16}} (j_{21} + j_{22} C) \sqrt{1 - C^2} K^2 + \frac{t_{114} + t_{115} C + \ldots + t_{121} C^7}{(1 - C^2)^{\frac{3}{2}}} K^3,$$

where

$$t_{114} = -(j_{21} t_{28} + j_{23} t_{27}) \frac{1}{s_{16}} - j_{21} \frac{t_{26}}{s_{16}^2} + j_{21} t_{70} \frac{t_{27}}{s_{16}^2},$$
$$t_{115} = -(j_{21} t_{29} + j_{22} t_{28} + j_{24} t_{27}) \frac{1}{s_{16}} - (j_{22} - s_1 j_{21}) \frac{t_{26}}{s_{16}^2} + (j_{21} t_{71} + j_{22} t_{70}) \frac{t_{27}}{s_{16}^2},$$
$$t_{116} = -(j_{21} t_{30} + j_{22} t_{29} - (j_{25} - 2 j_{23}) t_{27}) \frac{1}{s_{16}} + (2 j_{21} + s_1 j_{22}) \frac{t_{26}}{s_{16}^2} + (j_{21} t_{72} + j_{22} t_{71} - j_{21} t_{70}) \frac{t_{27}}{s_{16}^2},$$
$$t_{117} = -(j_{21} t_{31} + j_{22} t_{30} + 2 j_{24} t_{27}) \frac{1}{s_{16}} + (2 j_{22} + 2 s_1 j_{21}) \frac{t_{26}}{s_{16}^2} + (j_{21} t_{73} + j_{22} t_{72} - j_{21} t_{71} - j_{22} t_{70}) \frac{t_{27}}{s_{16}^2},$$
$$t_{118} = -(j_{21} t_{32} + j_{22} t_{31} + (2 j_{25} - j_{23}) t_{27}) \frac{1}{s_{16}} - (j_{21} + 2 s_1 j_{22}) \frac{t_{26}}{s_{16}^2} + (j_{22} t_{73} - j_{21} t_{72} - j_{22} t_{71}) \frac{t_{27}}{s_{16}^2},$$
$$t_{119} = -(j_{21} t_{33} + j_{22} t_{32} - j_{24} t_{27}) \frac{1}{s_{16}} - (j_{22} - s_1 j_{21}) \frac{t_{26}}{s_{16}^2} - (j_{21} t_{73} + j_{22} t_{72}) \frac{t_{27}}{s_{16}^2},$$
$$t_{120} = -(j_{21} t_{34} + j_{22} t_{33} - j_{25} t_{27}) \frac{1}{s_{16}} + s_1 j_{22} \frac{t_{26}}{s_{16}^2} - j_{22} t_{73} \frac{t_{27}}{s_{16}^2}, \qquad t_{121} = -\frac{j_{22} t_{34}}{s_{16}}.$$

In like manner

$$H_1' = \frac{t_{27}}{s_{16}}(j_{21}' + j_{22}' C)\sqrt{1 - C^2}\,K^2 + \frac{t_{122} + t_{123} C + \ldots + t_{129} C^7}{(1 - C^2)^{\frac{3}{2}}} K^3,$$

where

$$t_{122} = (j_{21}' t_{35} + j_{23}' t_{27})\frac{1}{s_{16}} - j_{21}' \frac{t_{26}}{s_{16}^{\ 2}} - j_{21}' t_{70} \frac{t_{27}}{s_{16}^{\ 2}},$$

$$t_{123} = (j_{21}' t_{29} + j_{22}' t_{35} + j_{24}' t_{27})\frac{1}{s_{16}} - (j_{22}' + s_1 j_{21}')\frac{t_{26}}{s_{16}^{\ 2}} - (j_{21}' t_{71} + j_{22}' t_{70})\frac{t_{27}}{s_{16}^{\ 2}},$$

$$t_{124} = (j_{21}' t_{36} + j_{22}' t_{29} - (j_{25}' - 2j_{23}') t_{27})\frac{1}{s_{16}} + (2j_{21}' - s_1 j_{22}')\frac{t_{26}}{s_{16}^{\ 2}} - (j_{21}' t_{72} + j_{22}' t_{71} - j_{21}' t_{70})\frac{t_{27}}{s_{16}^{\ 2}},$$

$$t_{125} = (j_{21}' t_{31} + j_{22}' t_{36} + 2j_{24}' t_{27})\frac{1}{s_{16}} + (2j_{22}' - 2s_1 j_{21}')\frac{t_{26}}{s_{16}^{\ 2}} - (j_{21}' t_{73} + j_{22}' t_{72} - j_{21}' t_{71} - j_{22}' t_{70})\frac{t_{27}}{s_{16}^{\ 2}},$$

$$t_{126} = (j_{21}' t_{37} + j_{22}' t_{31} + (2j_{25}' - j_{23}') t_{27})\frac{1}{s_{16}} - (j_{21}' - 2s_1 j_{22}')\frac{t_{26}}{s_{16}^{\ 2}} - (j_{22}' t_{73} - j_{21}' t_{72} - j_{22}' t_{71})\frac{t_{27}}{s_{16}^{\ 2}},$$

$$t_{127} = (j_{21}' t_{33} + j_{22}' t_{37} - j_{24}' t_{27})\frac{1}{s_{16}} - (j_{22}' + s_1 j_{21}')\frac{t_{26}}{s_{16}^{\ 2}} + (j_{21}' t_{73} + j_{22}' t_{72})\frac{t_{27}}{s_{16}^{\ 2}},$$

$$t_{128} = (j_{21}' t_{34} + j_{22}' t_{33} - j_{25}' t_{27})\frac{1}{s_{16}} - s_1 j_{22}' \frac{t_{26}}{s_{16}^{\ 2}} + j_{22}' t_{73}\frac{t_{27}}{s_{16}^{\ 2}}, \qquad t_{129} = \frac{j_{22}' t_{34}}{s_{16}}.$$

It is easy to see that, for H_0, we have an expression of the form

$$H_0 = \frac{t_1}{s_{16}}(j_{21} + j_{22} C)K + \frac{t_{130} + t_{131} C + \ldots + t_{136} C^6}{(1 - C^2)^2} K^2 + \frac{t_{137} + t_{138} C + \ldots + t_{147} C^{10}}{(1 - C^2)^3} K^3,$$

where the coefficients t are determined by the following equations

$$s_{16} t_{130} = -\frac{j_{21}}{s_{16}}(t_1 t_{70} + t_{27}) - j_{21} t_{38} + j_{23} t_1,$$

$$s_{16} t_{131} = -\frac{j_{21}}{s_{16}}(t_1 t_{71} + t_{27}) - (j_{21} t_{39} + j_{22} t_{38}) + \left(\frac{j_{22}}{s_{16}} t_{70} + j_{24}\right) t_1,$$

$$s_{16} t_{132} = -[j_{21} t_{72} + j_{22} t_{71} - j_{21} t_{70}]\frac{t_1}{s_{16}} - (j_{21} t_{40} + j_{22} t_{39}) + (2j_{21} + s_1 j_{22})\frac{t_{27}}{s_{16}} + (j_{25} - 2j_{23}) t_1,$$

$$s_{16} t_{133} = -[j_{21}(t_{73} - t_{71}) + j_{22}(t_{72} - t_{70})]\frac{t_1}{s_{16}} - (j_{21} t_{41} + j_{22} t_{40}) + 2(j_{22} - s_1 j_{21})\frac{t_{27}}{s_{16}} - j_{24} t_1,$$

$$s_{16} t_{134} = -[-j_{21} t_{72} + j_{22}(t_{73} - t_{71})]\frac{t_1}{s_{16}} - (j_{21} t_{42} + j_{22} t_{41}) - (j_{21} + 2s_1 j_{22})\frac{t_{27}}{s_{16}} + (j_{23} - 2j_{25}) t_1,$$

$$s_{16} t_{135} = -[j_{21} t_{73} + j_{22} t_{72}]\frac{t_1}{s_{16}} - (j_{21} t_{43} + j_{22} t_{42}) - (j_{22} - s_1 j_{21})\frac{t_{27}}{s_{16}} + j_{24} t_1,$$

$$s_{16} t_{136} = j_{22} t_{73}\frac{t_1}{s_{16}} - j_{22} t_{43} + s_1 j_{22}\frac{t_{27}}{s_{16}} + j_{25} t_1,$$

$$s_{16} t_{137} = -t_{70} t_{130} + t_{114} - j_{21} t_{74}\frac{t_1}{s_{16}} - j_{21} t_{44} - j_{23} t_{48} + s_2 j_{21}\frac{t_{27}}{s_{16}} + j_{26} t_1,$$

$$s_{16} t_{138} = -t_{70} t_{131} - t_{71} t_{130} + t_{115} - s_1 t_{114} - (j_{21} t_{75} + j_{22} t_{74})\frac{t_1}{s_{16}} - (j_{21} t_{45} + j_{22} t_{44} + j_{23} t_{39} + j_{24} t_{38})$$
$$+ (s_2 j_{22} + s_3 j_{21})\frac{t_{27}}{s_{16}} + j_{27} t_1,$$

$$s_{16} t_{139} = -t_{70} t_{132} - t_{71} t_{131} - t_{72} t_{130} + t_{116} - s_1 t_{115} - t_{114} - [j_{21}(t_{76} - t_{74}) + j_{22} t_{75}]\frac{t_1}{s_{16}} - (j_{21} t_{46} + j_{22} t_{45}$$
$$+ j_{23} t_{40} + j_{24} t_{39} + (j_{26} - j_{23}) t_{38}) + [(s_4 - 3s_2) j_{21} + s_3 j_{22}]\frac{t_{27}}{s_{16}} + (j_{28} - 3j_{26}) t_1,$$

$$s_{16}t_{140} = -t_{70}t_{133} - t_{71}t_{132} - t_{72}t_{131} - t_{73}t_{130} + t_{117} - s_1 t_{116} - t_{115} + s_1 t_{114} - [j_{21}(t_{77} - t_{76}) + j_{22}(t_{76} - t_{74})]\frac{t_1}{s_{16}} - [j_{21}t_{47} + j_{22}t_{46} + j_{23}t_{41} + j_{24}t_{40} + (j_{25} - j_{23})t_{39} - j_{24}t_{38}] + [(s_4 - 3s_2)j_{22} - 3s_3 j_{21}]\frac{t_{27}}{s_{16}} + (j_{29} - 3j_{27})t_1,$$

$$s_{16}t_{141} = -t_{70}t_{134} - t_{71}t_{133} - t_{72}t_{132} - t_{73}t_{131} + t_{118} - s_1 t_{117} - t_{116} + s_1 t_{115} - [j_{21}(t_{78} - t_{76}) + j_{22}(t_{77} - t_{75})]\frac{t_1}{s_{16}} - [j_{21}t_{48} + j_{22}t_{47} + j_{23}t_{42} + j_{24}t_{41} + (j_{25} - j_{23})t_{40} - j_{24}t_{39} - j_{25}t_{38}] - 3[(s_4 - s_2)j_{21} + s_3 j_{22}]\frac{t_{27}}{s_{16}} - 3(j_{28} - j_{26})t_1,$$

$$s_{16}t_{142} = -t_{70}t_{135} - t_{71}t_{134} - t_{72}t_{133} - t_{73}t_{132} + t_{119} - s_1 t_{118} - t_{117} + s_1 t_{116} - [j_{21}(t_{79} - t_{77}) + j_{22}(t_{78} - t_{76})]\frac{t_1}{s_{16}} - [j_{21}t_{49} + j_{22}t_{48} + j_{23}t_{43} + j_{24}t_{42} + (j_{25} - j_{23})t_{41} - j_{24}t_{40} - j_{25}t_{39}] + 3[(-s_4 + s_2)j_{22} + s_3 j_{21}]\frac{t_{27}}{s_{16}} - 3(j_{29} - j_{27})t_1,$$

$$s_{16}t_{143} = -t_{70}t_{136} - t_{71}t_{135} - t_{72}t_{134} - t_{73}t_{133} + t_{120} - s_1 t_{119} - t_{118} + s_1 t_{117} - [j_{21}(t_{80} - t_{78}) + j_{22}(t_{79} - t_{77})]\frac{t_1}{s_{16}} - [j_{21}t_{50} + j_{22}t_{49} + j_{24}t_{43} + (j_{25} - j_{23})t_{42} - j_{24}t_{41} - j_{25}t_{40}] + [(3s_4 - s_2)j_{21} + 3s_3 j_{22}]\frac{t_{27}}{s_{16}} + (3j_{28} - j_{26})t_1,$$

$$s_{16}t_{144} = -t_{71}t_{136} - t_{72}t_{135} - t_{73}t_{134} + t_{121} - s_1 t_{120} - t_{119} + s_1 t_{118} - [-j_{21}t_{79} + j_{22}(t_{80} - t_{78})]\frac{t_1}{s_{16}} - [j_{21}t_{51} + j_{22}t_{50} + (j_{25} - j_{23})t_{43} - j_{24}t_{42} - j_{25}t_{41}] + [-s_3 j_{21} + (3s_4 - s_2)j_{22}]\frac{t_{27}}{s_{16}} + (3j_{29} - j_{27})t_1,$$

$$s_{16}t_{145} = -t_{72}t_{136} - t_{73}t_{135} - s_1 t_{121} - t_{120} + s_1 t_{119} + [j_{21}t_{80} + j_{22}t_{79}]\frac{t_1}{s_{16}} - [j_{21}t_{52} + j_{22}t_{51} - j_{24}t_{43} - j_{25}t_{42}] - (s_4 j_{21} + s_3 j_{22})\frac{t_{27}}{s_{16}} - j_{28}t_1,$$

$$s_{16}t_{146} = -t_{73}t_{136} - t_{121} + s_1 t_{120} + j_{22}t_{80}\frac{t_1}{s_{16}} - j_{21}t_{53} - j_{22}t_{52} + j_{25}t_{43} - s_4 j_{22}\frac{t_{27}}{s_{16}} - j_{29}t_1,$$

$$s_{16}t_{147} = s_1 t_{121} - j_{22}t_{53}.$$

In like manner we have

$$H'_0 = -\frac{t_1}{s_{16}}(j'_{21} + j'_{22}C)K + \frac{t_{148} + t_{149}C + \ldots + t_{154}C^6}{(1 - C^2)^2}K^2 + \frac{t_{155} + t_{156}C + \ldots + t_{165}C^{10}}{(1 - C^2)^3}K^3,$$

where the coefficients can be obtained from the preceding group of equations by applying the following rule:

Change the sign of the whole expression, accent all the j from j_{21} to j_{29}, for t_{38} to t_{53} write t_{54} to t_{69}, for t_{70} to t_{80} write t_{92} to t_{102}, change the signs of s_1, s_2, s_3, s_4, t_{27}, for t_{114} to t_{121} write t_{122} to t_{129}, for t_{130} to t_{136} write t_{148} to t_{154}. Thus we have

$$s_{16}t_{148} = \frac{j''_{21}}{s_{16}}(t_1 t_{92} + t_{28}) + j'_{21}t_{54} - j'_{23}t_1,$$

$$s_{16}t_{149} = \frac{j''_{21}}{s_{16}}(t_1 t_{93} + t_{28}) + (j'_{21}t_{55} + j'_{22}t_{54}) + \left(\frac{j''_{22}}{s_{16}}t_{92} + j'_{24}\right)t_1,$$

$$s_{16}\, t_{150} = [j''_{21}\, t_{94} + j''_{22}\, t_{93} - j''_{21}\, t_{92}] \frac{t_1}{s_{16}} + (j''_{21}\, t_{56} + j''_{22}\, t_{55}) - (2 j''_{21} - s_1 j''_{22}) \frac{t_{27}}{s_{16}} - (j''_{25} - 2 j''_{23})\, t_1 ,$$

$$s_{16}\, t_{151} = [j''_{21}\, (t_{95} - t_{93}) + j''_{22}\, (t_{94} - t_{92})] \frac{t_1}{s_{16}} + (j''_{21}\, t_{57} + j''_{22}\, t_{56}) - 2 (j''_{22} + s_1 j''_{21}) \frac{t_{27}}{s_{16}} + j''_{24}\, t_1 ,$$

$$s_{16}\, t_{152} = [- j''_{21}\, t_{94} + j''_{22}\, (t_{95} - t_{93})] \frac{t_1}{s_{16}} + (j''_{21}\, t_{58} + j''_{22}\, t_{57}) + (j''_{21} - 2 s_1 j''_{22}) \frac{t_{27}}{s_{16}} - (j''_{23} - 2 j''_{25})\, t_1 ,$$

$$s_{16}\, t_{153} = [j''_{21}\, t_{95} + j''_{22}\, t_{94}] \frac{t_1}{s_{16}} + (j''_{21}\, t_{59} + j''_{22}\, t_{58}) + (j''_{22} + s_1 j''_{21}) \frac{t_{27}}{s_{16}} - j''_{24}\, t_1 ,$$

$$s_{16}\, t_{154} = - j''_{22}\, t_{95} \frac{t_1}{s_{16}} + j''_{22}\, t_{69} + s_1 j''_{22} \frac{t_{27}}{s_{16}} - j''_{25}\, t_1 ,$$

$$s_{16}\, t_{156} = t_{92}\, t_{146} - t_{122} + j''_{21}\, t_{96} \frac{t_1}{s_{16}} + j''_{21}\, t_{60} + j''_{22}\, t_{61} - s_2 j''_{21} \frac{t_{27}}{s_{16}} - j''_{28}\, t_1 ,$$

$$\begin{aligned} s_{16}\, t_{156} = {} & t_{92}\, t_{149} + t_{93}\, t_{146} - t_{123} - s_1\, t_{122} + (j''_{21}\, t_{97} + j''_{22}\, t_{96}) \frac{t_1}{s_{16}} + (j''_{21}\, t_{61} + j''_{22}\, t_{60} + j''_{23}\, t_{55} + j''_{24}\, t_{54}) \\ & - (s_2 j''_{22} + s_3 j''_{21}) \frac{t_{27}}{s_{16}} - j''_{27}\, t_1 , \end{aligned}$$

$$\begin{aligned} s_{16}\, t_{157} = {} & t_{92}\, t_{150} + t_{93}\, t_{149} + t_{94}\, t_{148} - t_{124} - s_1\, t_{123} + t_{122} + [j''_{21}\, (t_{98} - t_{96}) + j''_{22}\, t_{97}] \frac{t_1}{s_{16}} \\ & + [j''_{21}\, t_{62} + j''_{22}\, t_{61} + j''_{23}\, t_{56} + j''_{24}\, t_{55} + (j''_{25} - j''_{23})\, t_{54}] - [(s_4 - 3 s_2) j''_{21} + s_3 j''_{22}] \frac{t_{27}}{s_{16}} \\ & - (j''_{28} - 3 j''_{26})\, t_1 , \end{aligned}$$

$$\begin{aligned} s_{16}\, t_{158} = {} & t_{92}\, t_{151} + t_{93}\, t_{150} + t_{94}\, t_{149} + t_{95}\, t_{148} - t_{125} - s_1\, t_{124} + t_{123} + s_1\, t_{122} + [j''_{21}\, (t_{99} - t_{97}) + j''_{22}\, (t_{98} - t_{96})] \frac{t_1}{s_{16}} \\ & + [j''_{21}\, t_{63} + j''_{22}\, t_{62} + j''_{23}\, t_{57} + j''_{24}\, t_{56} + (j''_{26} - j''_{23})\, t_{55} - j''_{24}\, t_{54}] - [(s_4 - 3 s_2) j''_{22} - 3 s_3 j''_{21}] \frac{t_{27}}{s_{16}} \\ & - (j''_{29} - 3 j''_{27})\, t_1 , \end{aligned}$$

$$\begin{aligned} s_{16}\, t_{159} = {} & t_{92} t_{152} + t_{93} t_{151} + t_{94} t_{150} + t_{95} t_{149} - t_{126} - s_1 t_{125} + t_{124} + s_1 t_{123} + [j''_{21} (t_{100} - t_{98}) + j''_{22} (t_{99} - t_{97})] \frac{t_1}{s_{16}} \\ & + [j''_{21} t_{64} + j''_{22} t_{63} + j''_{23} t_{58} + j''_{24} t_{57} + (j''_{26} - j''_{23}) t_{56} - j''_{24} t_{55} - j''_{26} t_{54}] + 3 [(s_4 - s_2) j''_{21} + s_3 j''_{22}] \frac{t_{27}}{s_{16}} \\ & + 3 (j''_{26} - j''_{26})\, t_1 , \end{aligned}$$

$$\begin{aligned} s_{16}\, t_{160} = {} & t_{92} t_{153} + t_{93} t_{152} + t_{94} t_{151} + t_{95} t_{150} - t_{127} - s_1 t_{126} + t_{125} + s_1 t_{124} + [j''_{21} (t_{101} - t_{99}) + j''_{22} (t_{100} - t_{98})] \frac{t_1}{s_{16}} \\ & + [j''_{21} t_{65} + j''_{22} t_{64} + j''_{23} t_{59} + j''_{24} t_{58} + (j''_{26} - j''_{23}) t_{67} - j''_{24} t_{56} - j''_{26}\, t_{55}] - 3 [(- s_4 + s_2) j''_{22} + s_3 j''_{21}] \frac{t_{27}}{s_{16}} \\ & + 3 (j''_{29} - j''_{27})\, t_1 , \end{aligned}$$

$$\begin{aligned} s_{16}\, t_{161} = {} & t_{92} t_{154} + t_{93}\, t_{153} + t_{94}\, t_{152} + t_{95}\, t_{151} - t_{126} - s_1\, t_{127} + t_{126} + s_1\, t_{125} + [j''_{21}\, (t_{102} - t_{100}) + j''_{22} (t_{101} - t_{99})] \frac{t_1}{s_{16}} \\ & + [j''_{21} t_{66} + j''_{22}\, t_{65} + j''_{24} t_{59} + (j''_{26} - j''_{23})\, t_{58} - j''_{24}\, t_{67} - j''_{26}\, t_{56}] - [(3 s_4 - s_2) j''_{21} + 3 s_3 j''_{22}] \frac{t_{27}}{s_{16}} \\ & - (3 j''_{28} - j''_{26})\, t_1 , \end{aligned}$$

$$\begin{aligned} s_{16}\, t_{162} = {} & t_{93}\, t_{154} + t_{94}\, t_{153} + t_{95}\, t_{152} - t_{129} - s_1\, t_{126} + t_{127} + s_1\, t_{126} + [j''_{21}\, t_{101} + j''_{22}\, (t_{102} - t_{100})] \frac{t_1}{s_{16}} \\ & + [j''_{21}\, t_{67} + j''_{22}\, t_{66} + (j''_{28} - j''_{23})\, t_{69} - j''_{24}\, t_{58} - j''_{26}\, t_{67}] - [- s_3 j''_{21} + (3 s_4 - s_2) j''_{22}] \frac{t_{27}}{s_{16}} \\ & - (3 j''_{29} - j''_{27})\, t_1 , \end{aligned}$$

$$s_{16}\,t_{163} = t_{94}\,t_{154} + t_{95}\,t_{153} - s_1\,t_{129} + t_{128} + s_1\,t_{127} - [j''_{21}\,t_{102} + j''_{22}\,t_{101}]\frac{t_1}{s_{16}} + [j''_{21}\,t_{68} + j''_{22}\,t_{67} - j''_{24}\,t_{59} - j''_{26}\,t_{58}]$$
$$+ (s_4\,j''_{21} + s_3\,j''_{22})\frac{t_{27}}{s_{16}} + j''_{28}\,t_1\,,$$
$$s_{16}\,t_{164} = t_{95}\,t_{154} + t_{129} + s_1\,t_{128} - j''_{22}\,t_{102}\frac{t_1}{s_{16}} + [j''_{21}\,t_{69} + j''_{22}\,t_{68}] + s_4\,j''_{22}\frac{t_{27}}{s_{16}} + j''_{29}\,t_1\,,$$
$$s_{16}\,t_{165} = s_1\,t_{129} + j''_{22}\,t_{69}\,.$$

In the next place

$$H_3 = j_{21} + j_{22}\,C + [t_{166} + t_{167}\,C + t_{168}\,C^2]\,K + \frac{t_{169} + t_{170}\,C + \ldots + t_{176}\,C^7}{(1 - C^2)^2}\,K^2$$
$$+ \frac{t_{177} + t_{178}\,C + \ldots + t_{188}\,C^{11}}{(1 - C^2)^3}\,K^3\,,$$

where the coefficients are determined by the formulae

$$t_{166} = j_{23} - j_{21}\frac{t_1}{s_{16}}\,, \qquad t_{167} = j_{24} - (j_{22} + s_1\,j_{21})\frac{t_1}{s_{16}}\,, \qquad t_{168} = j_{25} + s_1\,j_{22}\frac{t_1}{s_{16}}\,,$$
$$t_{169} = j_{26} + s_2\,j_{21}\frac{t_1}{s_{16}} - t_{130}\,, \qquad t_{170} = j_{27} + (s_3\,j_{21} + s_2\,j_{22})\frac{t_1}{s_{16}} - t_{131} + s_1\,t_{130}\,,$$
$$t_{171} = j_{28} - 2\,j_{26} + [(s_4 - 2\,s_2)\,j_{21} + s_3\,j_{22}]\frac{t_1}{s_{16}} - t_{132} + s_1\,t_{131}\,,$$
$$t_{172} = j_{29} - 2\,j_{27} - [(-s_4 + 2\,s_2)\,j_{22} + 2\,s_3\,j_{21}]\frac{t_1}{s_{16}} - t_{133} + s_1\,t_{132}\,,$$
$$t_{173} = -2\,j_{28} + j_{26} - [(2\,s_4 - s_2)\,j_{21} + s_3\,j_{22}]\frac{t_1}{s_{16}} - t_{134} + s_1\,t_{133}\,,$$
$$t_{174} = -2\,j_{29} + j_{27} - [-s_3\,j_{21} + (2\,s_4 + s_2)\,j_{22}]\frac{t_1}{s_{16}} - t_{135} + s_1\,t_{134}\,,$$
$$t_{175} = j_{28} + [s_4\,j_{21} + s_3\,j_{22}]\frac{t_1}{s_{16}} - t_{136} + s_1\,t_{135}\,,$$
$$t_{176} = j_{29} + s_4\,j_{22}\frac{t_1}{s_{16}} + s_1\,t_{136}\,, \qquad t_{177} = j_{30} + s_6\,j_{21}\frac{t_1}{s_{16}} + s_2\,t_{130} - t_{137}\,,$$
$$t_{178} = j_{31} + [s_6\,j_{21} + s_5\,j_{22}]\frac{t_1}{s_{16}} + s_2\,t_{131} + s_3\,t_{130} - t_{138} + s_1\,t_{137}\,,$$
$$t_{179} = j_{32} - 3\,j_{30} + [(s_7 - 3\,s_5)\,j_{21} + s_6\,j_{22}]\frac{t_1}{s_{16}} + s_2\,t_{132} + s_3\,t_{131} + (s_4 - s_2)\,t_{130} - t_{139} + s_1\,t_{138}\,,$$
$$t_{180} = j_{33} - 3\,j_{31} + [(s_8 - 3\,s_6)\,j_{21} + (s_7 - 3\,s_5)\,j_{22}]\frac{t_1}{s_{16}} + s_2\,t_{133} + s_3\,t_{132} + (s_4 - s_2)\,t_{131}$$
$$- s_3\,t_{130} - t_{140} + s_1\,t_{139}\,,$$
$$t_{181} = j_{34} - 3\,j_{32} + 3\,j_{30} - [(3\,s_7 - 3\,s_6)\,j_{21} + (-s_8 + 3\,s_6)\,j_{22}]\frac{t_1}{s_{16}} + s_2\,t_{134} + s_3\,t_{133}$$
$$+ (s_4 - s_2)\,t_{132} - s_3\,t_{131} - s_4\,t_{130} - t_{141} + s_1\,t_{140}\,,$$
$$t_{182} = -3\,j_{33} + 3\,j_{31} - [3\,(s_8 - s_6)\,j_{21} + 3\,(s_7 - s_6)\,j_{22}]\frac{t_1}{s_{16}} + s_2\,t_{135} + s_3\,t_{134}$$
$$+ (s_4 - s_2)\,t_{133} - s_3\,t_{132} - s_4\,t_{131} - t_{142} + s_1\,t_{141}\,,$$
$$t_{183} = -3\,j_{34} + 3\,j_{32} - [(-3\,s_7 + s_6)\,j_{21} + (3\,s_8 - 3\,s_6)\,j_{22}]\frac{t_1}{s_{16}} + s_2\,t_{136} + s_3\,t_{135}$$
$$+ (s_4 - s_2)\,t_{134} - s_3\,t_{133} - s_4\,t_{132} - t_{143} + s_1\,t_{142}\,,$$

$$t_{184} = 3 j_{33} - j_{31} - [(-3\, s_6 + s_5)\, j_{21} + (-3\, s_7 + s_5)\, j_{22}] \frac{t_1}{s_{16}} + s_3\, t_{136}$$
$$+ (s_4 - s_2)\, t_{135} - s_3\, t_{134} - s_4\, t_{133} - t_{144} + s_1\, t_{143}\,,$$

$$t_{185} = 3 j_{34} - j_{32} - [s_7 j_{21} + (-3\, s_8 + s_6)\, j_{22}] \frac{t_1}{s_{16}} + (s_4 - s_2)\, t_{136} - s_3\, t_{135} - s_4\, t_{134} - t_{145} + s_1\, t_{144}\,,$$

$$t_{186} = - j_{33} - [s_8\, j_{21} + s_7\, j_{22}] \frac{t_1}{s_{16}} - s_3\, t_{136} - s_4\, t_{135} - t_{146} + s_1\, t_{145}\,,$$

$$t_{187} = - j_{34} - s_8\, j_{22} \frac{t_1}{s_{16}} - s_4\, t_{136} - t_{147} + s_1\, t_{146}\,, \qquad t_{188} = s_1\, t_{147}\,.$$

Also

$$H_3' = j_{21}' + j_{22}' C + [t_{189} + t_{190} C + t_{191} C^2] K + \frac{t_{192} + t_{193} C + \ldots + t_{198} C^6 + t_{199} C^7}{(1 - C^2)^2} K^2$$
$$+ \frac{t_{200} + t_{201} C + \ldots + t_{211} C^{11}}{(1 - C^2)^3} K^3\,,$$

where the coefficients can be obtained from the preceding group by accenting all the j from j_{21} to j_{34}, changing the signs of the s from s_1 to s_8 and that of t_1, and adding 18 to the subscripts of the t from t_{130} to t_{147}: Thus

$$t_{189} = j_{23}' + j_{21}' \frac{t_1}{s_{15}}\,, \qquad t_{190} = j_{24}' + (j_{22}' - s_1 j_{21}') \frac{t_1}{s_{15}}\,, \qquad t_{191} = j_{25}' + s_1\, j_{22}' \frac{t_1}{s_{15}}\,,$$

$$t_{192} = j_{26}' + s_2\, j_{21}' \frac{t_1}{s_{15}} - t_{148}\,, \qquad t_{193} = j_{27}' + (s_3\, j_{21}' + s_2\, j_{22}') \frac{t_1}{s_{16}} - t_{149} - s_1\, t_{148}\,,$$

$$t_{194} = j_{28}' - 2 j_{25}' + [(s_4 - 2\, s_2)\, j_{21}' + s_3\, j_{22}'] \frac{t_1}{s_{16}} - t_{150} - s_1\, t_{149}\,,$$

$$t_{195} = j_{29}' - 2 j_{27}' - [(-s_4 + 2\, s_2)\, j_{22}' + 2\, s_3\, j_{21}'] \frac{t_1}{s_{15}} - t_{151} - s_1\, t_{150}\,,$$

$$t_{196} = -2 j_{28}' + j_{25}' - [(2\, s_4 - s_2)\, j_{21}' + s_3\, j_{22}'] \frac{t_1}{s_{15}} - t_{152} - s_1\, t_{151}\,,$$

$$t_{197} = -2 j_{29}' + j_{27}' - [-s_3\, j_{21}' + (2\, s_4 + s_2)\, j_{22}'] \frac{t_1}{s_{15}} - t_{153} - s_1\, t_{152}\,,$$

$$t_{198} = j_{28}' + [s_4\, j_{21}' + s_3\, j_{22}'] \frac{t_1}{s_{16}} - t_{154} - s_1\, t_{153}\,,$$

$$t_{199} = j_{29}' + s_4\, j_{22}' \frac{t_1}{s_{16}} - s_1\, t_{154}\,, \qquad t_{200} = j_{30}' + s_5\, j_{21}' \frac{t_1}{s_{15}} - s_2\, t_{148} - t_{155}\,,$$

$$t_{201} = j_{31}' + [s_6\, j_{21}' + s_5\, j_{22}'] \frac{t_1}{s_{16}} - s_2\, t_{149} - s_3\, t_{148} - t_{156} - s_1\, t_{155}\,,$$

$$t_{202} = j_{32}' - 3 j_{30}' + [(s_7 - 3\, s_5)\, j_{21}' + s_5\, j_{22}'] \frac{t_1}{s_{15}} - s_2\, t_{150} - s_3\, t_{149} - (s_4 - s_2)\, t_{148} - t_{157} - s_1\, t_{156}\,,$$

$$t_{203} = j_{33}' - 3 j_{31}' + [(s_8 - 3\, s_6)\, j_{21}' + (s_7 - 3\, s_5)\, j_{22}' \frac{t_1}{s_{16}} - s_2\, t_{161} - s_3\, t_{150} - (s_4 - s_2)\, t_{149}$$
$$+ s_3\, t_{148} - t_{158} - s_1\, t_{157}\,,$$

$$t_{204} = j_{34}' - 3 j_{32}' + 3 j_{30}' - [(3\, s_7 - 3\, s_5)\, j_{21}' + (-s_8 + 3\, s_6)\, j_{22}'] \frac{t_1}{s_{16}} - s_2\, t_{162} - s_3\, t_{151}$$
$$- (s_4 - s_2)\, t_{150} + s_3\, t_{149} + s_4\, t_{148} - t_{159} - s_1\, t_{158}\,,$$

$$t_{205} = -3j''_{33} + 3j''_{31} - [3(s_8 - s_6)j''_{21} + 3(s_7 - s_5)j''_{22}]\frac{t_1}{s_{16}} - s_2 t_{153} - s_3 t_{152}$$
$$- (s_4 - s_2)t_{151} + s_3 t_{150} + s_4 t_{149} - t_{160} - s_1 t_{169},$$
$$t_{206} = -3j''_{34} + 3j''_{32} - [(-3s_7 + s_5)j''_{21} + 3(s_8 - s_6)j''_{22}]\frac{t_1}{s_{16}} - s_2 t_{154} - s_3 t_{153}$$
$$- (s_4 - s_2)t_{152} + s_3 t_{151} + s_4 t_{150} - t_{161} - s_1 t_{160},$$
$$t_{207} = 3j''_{33} - j''_{31} - [(-3s_8 + s_6)j''_{21} + (-3s_7 + s_5)j''_{22}]\frac{t_1}{s_{16}} - s_3 t_{154}$$
$$- (s_4 - s_2)t_{153} + s_3 t_{152} + s_4 t_{161} - t_{162} - s_1 t_{161},$$
$$t_{208} = 3j''_{34} - j''_{32} - [s_7 j''_{21} + (-3s_8 + s_6)j''_{22}]\frac{t_1}{s_{16}} - (s_4 - s_2)t_{154} + s_3 t_{153} + s_4 t_{152} - t_{163} - s_1 t_{162},$$
$$t_{209} = -j''_{33} - [s_8 j''_{21} + s_7 j''_{22}]\frac{t_1}{s_{16}} + s_3 t_{164} + s_4 t_{153} - t_{164} - s_1 t_{163},$$
$$t_{210} = -j''_{34} - s_8 j''_{22}\frac{t_1}{s_{16}} + s_4 t_{154} - t_{165} - s_1 t_{164}, \qquad t_{211} = -s_1 t_{166}.$$

It remains to turn the portions of $\frac{df}{d\phi}$ and $\frac{df'}{d\phi}$ which are proportional to integral powers of x into terms proportional to powers of z. The mentioned portion of $\frac{df}{d\phi}$ will then be

$$\frac{df}{d\phi} = u_1 + (u_2 + u_3 C)K + (u_4 + u_5 C + u_6 C^2)K^2 + (u_7 + u_8 C + u_9 C^2 + u_{10} C^3)K^3$$
$$+ \left\{ u_{11}\sqrt{1 - C^2}\,K + \frac{u_{12} + u_{13} C + u_{14} C^2 + u_{15} C^3}{(1 - C^2)^{\frac{1}{2}}} K^2 + \frac{u_{16} + u_{17} C + .. + u_{22} C^6}{(1 - C^2)^{\frac{3}{2}}} K^3 \right\} z$$
$$+ \left\{ u_{23} + u_{24} C)(1 - C^2)K^2 + (u_{25} + u_{26} C + .. + u_{29} C^4)K^3 \right\} z^2$$
$$+ u_{30}(1 - C^2)^{\frac{3}{2}} K^3 z^3,$$

where the coefficients are determined by the equations

$$u_1 = j_{11}, \qquad u_2 = j_2, \qquad u_3 = j_3 + s_1 j_{11}, \qquad u_4 = j_4 + s_2 j_{11}, \qquad u_5 = j_5 + s_3 j_{11} + s_1 j_{12},$$
$$u_6 = j_6 + s_4 j_{11} + s_1 j_{13} + s_1^2 j_{17}, \qquad u_7 = j_7 + s_5 j_{11} + s_2 j_{12},$$
$$u_8 = j_8 + s_6 j_{11} + s_3 j_{12} + s_2 j_{13} + s_1 j_{14} + 2s_1 s_2 j_{17},$$
$$u_9 = j_9 + s_7 j_{11} + s_4 j_{12} + s_3 j_{13} + s_1 j_{15} + 2s_1 s_3 j_{17} + s_1^2 j_{18},$$
$$u_{10} = j_{10} + s_8 j_{11} + s_4 j_{13} + s_1 j_{16} + 2s_1 s_4 j_{17} + s_1^2 j_{19} + s_1^3 j_{20}, \qquad u_{11} = s_{16} j_{11},$$
$$u_{12} = s_{17} j_{11} + s_{16} j_{12}, \qquad u_{13} = s_{18} j_{11} + s_{16} j_{13} + 2s_1 s_{16} j_{17}, \qquad u_{14} = s_{19} j_{11} - s_{16} j_{12},$$
$$u_{15} = s_{20} j_{11} - s_{16} j_{13} - 2s_1 s_{16} j_{17}, \qquad u_{16} = s_{21} j_{11} + s_{17} j_{12} + s_{16} j_{14} + 2s_2 s_{16} j_{17},$$
$$u_{17} = s_{22} j_{11} + s_{18} j_{12} + s_{17} j_{13} + s_{16} j_{15} + 2(s_3 s_{16} + s_1 s_{17})j_{17} + 2s_1 s_{16} j_{18},$$
$$u_{18} = s_{23} j_{11} + s_{19} j_{12} + s_{18} j_{13} - s_{17} j_{12} + s_{16}(j_{16} - 2j_{14}) + 2[(s_4 - 2s_2)s_{16} + s_1 s_{18}]j^{17}$$
$$+ 2s_1 s_{16} j_{19} + 3s_1^2 s_{16} j_{20},$$
$$u_{19} = s_{24} j_{11} + s_{20} j_{12} + s_{19} j_{13} - s_{18} j_{12} - s_{17} j_{13} - 2s_{16} j_{15} - 2[2s_3 s_{16} + s_1 s_{17} - s_1 s_{19}]j^{17}$$
$$- 4s_1 s_{16} j_{18},$$
$$u_{20} = s_{25} j_{11} - s_{19} j_{12} + s_{20} j_{13} - s_{18} j_{13} + s_{16}(j_{11} - 2j_{16}) + 2[(-2s_4 + s_2)s_{16} - s_1(s_{18} - s_{20})]j_{17}$$
$$- 4s_1 s_{16} j_{19} - 6s_1^2 s_{16} j_{20},$$

$u_{21} = s_{26}\, j_{11} - s_{20}\, j_{12} - s_{19}\, j_{13} + s_{16}\, j_{16} + 2\,(s_3\, s_{16} - s_1\, s_{19})\, j_{17} + 2\, s_1\, s_{16}\, j_{18}$,

$u_{22} = s_{27}\, j_{11} - s_{20}\, j_{13} + s_{16}\, j_{16} + 2\,(s_4\, s_{16} - s_1\, s_{20})\, j_{17} + 2\, s_1\, s_{16}\, j_{19} + 3\, s_1^2\, s_{16}\, j_{20}$, $\qquad u_{23} = s_{38}\, j_{11}$,

$u_{24} = 2\, s_1\, s_{16}\, j_{17}$, $\qquad u_{25} = s_{39}\, j_{11} + s_{38}\, j_{12} + 2\, s_2\, s_{16}\, j_{17}$,

$u_{26} = s_{40}\, j_{11} + s_{38}\, j_{13} + 2\,(s_3\, s_{16} + s_1\, s_{17})\, j_{17} + 2\, s_1\, s_{16}\, j_{18} + 3\, s_1\, s_{16}^2\, j_{20}$,

$u_{27} = s_{41}\, j_{11} - s_{38}\, j_{12} + 2\,[(s_4 - s_2)\, s_{16} + s_1\, s_{18}]\, j_{17} + 2\, s_1\, s_{16}\, j_{19}$,

$u_{28} = s_{42}\, j_{11} - s_{38}\, j_{13} + 2\,[- s_3\, s_{16} + s_1\, s_{19}]\, j_{17} - 2\, s_1\, s_{16}\, j_{18} - 3\, s_1\, s_{16}^2\, j_{20}$,

$u_{29} = 2\,[- s_4\, s_{16} + s_1\, s_{20}]\, j_{17} - 2\, s_1\, s_{16}\, j_{19}$, $\qquad u_{30} = s_{50}\, j_{11} + 2\, s_{16}\, s_{38}\, j_{17} + s_{16}^3\, j_{20}$.

The corresponding portion of $\dfrac{df'}{d\phi}$ is

$$\begin{aligned}\frac{df'}{d\phi} = {} & u'_1 + (u'_2 + u'_3\, C)\, K + (u'_4 + u'_5\, C + u'_6\, C^2)\, K^2 + (u'_7 + u'_8\, C + u'_9\, C^2 + u'_{10}\, C^3)\, K^3 \\ & + \left\{ \overline{u}'_{11} \sqrt{1 - C^2}\, K + \frac{u'_{12} + u'_{13}\, C + u'_{14}\, C^2 + u'_{15}\, C^3}{(1 - C^2)^{\frac{1}{2}}}\, K^2 + \frac{u'_{16} + u'_{17}\, C + \ldots + u'_{22}\, C^6}{(1 - C^2)^{\frac{3}{2}}}\, K^3 \right\} z \\ & + \left\{ (u'_{23} + u'_{24}\, C)\,(1 - C^2)\, K^2 + (u'_{25} + u'_{26}\, C + \ldots + u'_{29}\, C^4)\, K^3 \right\} z^2 \\ & + u'_{30}\,(1 - C^2)^{\frac{3}{2}}\, K^3\, z^3,\end{aligned}$$

where the coefficients are given by the preceding group of equations, but, in which, the u and j are accented.

By uniting to the preceding portion of $\dfrac{df}{d\phi}$ the similar portion which arises from the fraction $\dfrac{S}{1 - x}$ we have

$$\begin{aligned}\frac{df}{d\phi} = {} & v_1 + (v_2 + v_3\, C)\, K + (v_4 + v_5\, C + \ldots + v_{11}\, C^7)\, \frac{K^2}{(1 - C^2)^2} + (v_{12} + v_{13}\, C + \ldots + v_{22}\, C^{10}\, \frac{K^3}{(1 - C^2)^3} \\ & + \Big\{ v_{23} \sqrt{1 - C^2}\, K + [v_{24} + v_{25}\, C + v_{26}\, C^2 + v_{27}\, C^3]\, \frac{K^2}{(1 - C^2)^{\frac{1}{2}}} \\ & \qquad + [v_{28} + v_{29}\, C + \ldots + v_{35}\, C^7]\, \frac{K^3}{(1 - C^2)^{\frac{3}{2}}} \Big\} \cos\theta' \\ & + 2\, \left\{ [v_{36} + v_{37}\, C]\,(1 - C^2)\, K^2 + [v_{38} + v_{39}\, C + \ldots + v_{42}\, C^4]\, K^3 \right\} \cos 2\,\theta' \\ & + 3\, v_{43}\,(1 - C^2)^{\frac{3}{2}}\, K^3 \cos 3\,\theta'.\end{aligned}$$

By the integration of this we have

$$\begin{aligned} f = (f) + {} & \Big\{ v_1 + (v_2 + v_3\, C)\, K + (v_4 + v_5\, C + \ldots + v_{11}\, C^7)\, \frac{K^2}{(1 - C^2)^2} \\ & \qquad + (v_{12} + v_{13}\, C + \ldots + v_{22}\, C^{10})\, \frac{K^3}{(1 - C^2)^3} \Big\}\, (\phi + c) \\ & + \Big\{ w_{23} \sqrt{1 - C^2}\, K + [w_{24} + w_{25}\, C + w_{26}\, C^2 + w_{27}\, C^3]\, \frac{K^2}{(1 - C^2)^{\frac{1}{2}}} \\ & \qquad + [w_{28} + w_{29}\, C + \ldots + w_{35}\, C^7]\, \frac{K^3}{(1 - C^2)^{\frac{3}{2}}} \Big\} \sin\theta' \\ & + \left\{ (w_{36} + w_{37}\, C)\,(1 - C^2)\, K^2 + [w_{38} + w_{39}\, C + \ldots + w_{42}\, C^4]\, K^3 \right\} \sin 2\,\theta' \\ & + w_{43}\,(1 - C^2)^{\frac{3}{2}}\, K^3 \sin 3\,\theta'.\end{aligned}$$

The coefficients v are given by the formulae

$$v_1 = u_1, \qquad v_2 = u_2 + j_{21}\frac{t_1}{s_{16}}, \qquad v_3 = u_3 + j_{22}\frac{t_1}{s_{16}}, \qquad v_4 = u_4 + \tfrac{1}{2}u_{23} + t_{130},$$

$$v_5 = u_5 + \tfrac{1}{2}u_{24} + t_{131}, \qquad v_6 = u_6 - 2u_4 - \tfrac{3}{2}u_{23} + t_{132}, \qquad v_7 = -2u_5 - \tfrac{3}{2}u_{24} + t_{133},$$

$$v_8 = -2u_6 + u_4 + \tfrac{3}{2}u_{23} + t_{134}, \qquad v_9 = u_5 + \tfrac{3}{2}u_{24} + t_{135}, \qquad v_{10} = u_6 - \tfrac{1}{2}u_{23} + t_{136},$$

$$v_{11} = -\tfrac{1}{2}u_{24}, \qquad v_{12} = u_7 + \tfrac{1}{2}u_{25} - \tfrac{1}{2}j_{21}\frac{t_{26}}{s_{16}} + t_{137}, \qquad v_{13} = u_8 + \tfrac{1}{2}u_{26} - \tfrac{1}{2}j_{22}\frac{t_{26}}{s_{16}} + t_{138},$$

$$v_{14} = u_9 - 3u_7 + \tfrac{1}{2}u_{27} - \tfrac{3}{2}u_{25} + \tfrac{1}{4}j_{21}\frac{t_{26}}{s_{16}} + t_{139},$$

$$v_{15} = u_{10} - 3u_8 + \tfrac{1}{2}u_{28} - \tfrac{3}{2}u_{26} + 2j_{22}\frac{t_{26}}{s_{16}} + t_{140},$$

$$v_{16} = -3u_9 + 3u_7 + \tfrac{1}{2}u_{29} - \tfrac{3}{2}u_{27} + \tfrac{3}{2}u_{25} - 3j_{21}\frac{t_{26}}{s_{16}} + t_{141},$$

$$v_{17} = -3u_{10} + 3u_8 - \tfrac{3}{2}u_{28} + \tfrac{3}{2}u_{26} - 3j_{22}\frac{t_{26}}{s_{16}} + t_{142},$$

$$v_{18} = 3u_9 - u_7 - \tfrac{3}{2}u_{29} + \tfrac{3}{2}u_{27} - \tfrac{1}{2}u_{25} + 2j_{21}\frac{t_{26}}{s_{16}} + t_{143},$$

$$v_{19} = 3u_{10} - u_8 + \tfrac{3}{2}u_{28} - \tfrac{1}{2}u_{26} + 2j_{22}\frac{t_{26}}{s_{16}} + t_{144},$$

$$v_{20} = -u_9 + \tfrac{3}{2}u_{29} - \tfrac{1}{2}u_{27} - \tfrac{1}{2}j_{21}\frac{t_{26}}{s_{16}} + t_{145}, \qquad v_{21} = -u_{10} - \tfrac{1}{2}u_{28} - \tfrac{1}{2}j_{22}\frac{t_{26}}{s_{16}} + t_{146},$$

$$v_{22} = -\tfrac{1}{2}u_{29} + t_{147}, \qquad v_{23} = u_{11}, \qquad v_{24} = u_{12} - j_{21}\frac{t_{27}}{s_{16}}, \qquad v_{25} = u_{13} - j_{22}\frac{t_{27}}{s_{16}},$$

$$v_{26} = u_{14} + j_{21}\frac{t_{27}}{s_{16}}, \qquad v_{27} = u_{15} + j_{22}\frac{t_{27}}{s_{16}}, \qquad v_{28} = u_{16} + \tfrac{3}{4}u_{30} + t_{114}, \qquad v_{29} = u_{17} + t_{115},$$

$$v_{30} = u_{18} - \tfrac{9}{4}u_{30} + t_{116}, \qquad v_{31} = u_{19} + t_{117}, \qquad v_{32} = u_{20} + \tfrac{9}{4}u_{30} + t_{118}, \qquad v_{33} = u_{21} + t_{119},$$

$$v_{34} = u_{22} - \tfrac{3}{4}u_{30} + t_{120}, \qquad v_{35} = t_{121}, \qquad v_{36} = \tfrac{1}{4}u_{23}, \qquad v_{37} = \tfrac{1}{4}u_{24}, \qquad v_{38} = \tfrac{1}{4}u_{25} - \tfrac{1}{4}j_{21}\frac{t_{26}}{s_{16}},$$

$$v_{39} = \tfrac{1}{4}u_{26} - \tfrac{1}{4}j_{22}\frac{t_{26}}{s_{16}}, \qquad v_{40} = \tfrac{1}{4}u_{27} + \tfrac{1}{4}j_{21}\frac{t_{26}}{s_{16}}, \qquad v_{41} = \tfrac{1}{4}u_{28} + \tfrac{1}{4}j_{22}\frac{t_{26}}{s_{16}},$$

$$v_{42} = \tfrac{1}{4}u_{29}, \qquad v_{43} = \tfrac{1}{12}u_{30}.$$

The coefficients w are given by the formulae

$$w_{23} = p_1 v_{23}, \qquad w_{24} = p_1 v_{24} + p_2 v_{23}, \qquad w_{25} = p_1 v_{25} + p_3 v_{23}, \qquad w_{26} = p_1 v_{26} - p_2 v_{23},$$
$$w_{27} = p_1 v_{27} - p_3 v_{23}, \quad w_{28} = p_1 v_{28} + p_2 v_{24} + p_4 v_{23}, \quad w_{29} = p_1 v_{29} + p_2 v_{25} + p_3 v_{24} + p_5 v_{23},$$
$$w_{30} = p_1 v_{30} + p_2(v_{26} - v_{24}) + p_3 v_{25} + (p_6 - 2p_4)v_{23},$$
$$w_{31} = p_1 v_{31} + p_2(v_{27} - v_{25}) + p_3(v_{26} - v_{24}) - 2p_5 v_{23},$$
$$w_{32} = p_1 v_{32} - p_2 v_{26} + p_3(v_{27} - v_{25}) + (p_4 - 2p_6)v_{23},$$
$$w_{33} = p_1 v_{33} - p_2 v_{27} - p_3 v_{26} + p_5 v_{23}, \qquad w_{34} = p_1 v_{34} - p_3 v_{27} + p_6 v_{23}, \qquad w_{35} = p_1 v_{35},$$
$$w_{36} = p_1 v_{36}, \qquad w_{37} = p_1 v_{37}, \qquad w_{38} = p_1 v_{38} + p_2 v_{36}, \qquad w_{39} = p_1 v_{39} + p_2 v_{37} + p_3 v_{36},$$
$$w_{40} = p_1 v_{40} - p_2 v_{36} + p_3 v_{37}, \qquad w_{41} = p_1 v_{41} - p_2 v_{37} - p_3 v_{36}, \qquad w_{42} = p_1 v_{42} - p_3 v_{37},$$
$$w_{43} = p_1 v_{43}.$$

The corresponding equation for f' is

$$f' = (f') + \Big\{ v_1' + (v_2' + v_3' C) K + (v_4' + v_5' C + \ldots + v_{11}' C^7) \frac{K^2}{(1-C^2)^2} + (v_{12}' + v_{13}' C + \ldots + v_{22}' C^{10}) \frac{K^3}{(1-C^2)^3} \Big\} (\phi + c)$$

$$+ \Big\{ w_{23}' \sqrt{1-C^2}\, K + (w_{24}' + w_{25}' C + \ldots + w_{27}' C^3) \frac{K^2}{(1-C^2)^{\frac{1}{2}}} + (w_{28}' + w_{29}' C + \ldots + w_{35}' C^7) \frac{K^3}{(1-C^2)^{\frac{3}{2}}} \Big\} \sin \theta'$$

$$+ \Big\{ (w_{36}' + w_{37}' C)(1-C^2) K^2 + (w_{38}' + w_{39}' C + \ldots + w_{42}' C^4) K^3 \Big\} \sin 2\theta'$$

$$+ w_{43}' (1-C^2)^{\frac{3}{2}} K^3 \sin 3\theta',$$

where the coefficients v' are given by equations precisely similar to those for v; it is necessary only to accent the u, accent the j and reverse their signs and augment the subscripts of the t from t_{114} to t_{121} by 8 and those of t_{130} to t_{147} by 18; for the expressions of w_{23}' to w_{43}', simply accent the v in the equations for w_{23} to w_{43}.

The portions of the motions of f and f', which are of fractional form, are

$$\frac{df}{d\phi} = \frac{H_3}{1 - B_0 - D \cos\theta'}, \qquad \frac{df'}{d\phi} = \frac{H_3'}{1 + B_0 + D' \cos\theta'};$$

and, on integration, they give

$$f = \frac{2 H_3}{\theta_0 \sqrt{(1-B_0)^2 - D^2}} \text{ arc tan} \left[\frac{1 - B_0 + D}{\sqrt{(1-B_0)^2 - D^2}} \tan \tfrac{1}{2}\theta' \right],$$

$$f' = \frac{2 H_3'}{\theta_0 \sqrt{(1+B_0)^2 - D'^2}} \text{ arc tan} \left[\frac{1 + B_0 - D'}{\sqrt{(1+B_0)^2 - D'^2}} \tan \tfrac{1}{2}\theta' \right].$$

The two radicals in the denominators can be computed from the equivalents

$$\sqrt{[1 - B_0 - B_2]^2 - B_1^2 - B_3^2 + (1-B_0)^2 (2 D_2 - D_1^2)},$$

$$\sqrt{[1 + B_0 + B_2]^2 - B_1^2 - B_3^2 + (1+B_0)^2 (2 D_2' - D_1'^2)}.$$

The last terms under the radical signs have the values

$$- B_3^2 + (1-B_0)^2 (2 D_2 - D_1^2) = [a_1 (1 - s_1 C)^2 - s_{38}^2 (1 - C^2)] (1 - C^2) K^2 + [a_{22} + a_{23} C + \ldots + a_{29} C^7 + a_{30} C^8] \frac{K^3}{1 - C^2},$$

$$- B_3^2 + (1+B_0)^2 (2 D_2' - D_1'^2) = [a_1 (1 + s_1 C)^2 - s_{38}^2 (1 - C^2)] (1 - C^2) K^2 + [a_{31} + a_{32} C + \ldots + a_{38} C^7 + a_{39} C^8] \frac{K^3}{1 - C^2}.$$

To facilitate the derivation of the constants α we employ the subsidiary constants $\alpha_2, \ldots \alpha_{21}$; then

$$\alpha_1 = 2t_{27} - t_1^2, \quad \alpha_2 = 2(t_{28} - t_1 t_{38}), \quad \alpha_3 = 2(t_{29} - t_1 t_{39}), \quad \alpha_4 = 2(t_{30} - t_1 t_{40}),$$
$$\alpha_5 = 2(t_{31} - t_1 t_{41}), \quad \alpha_6 = 2(t_{32} - t_1 t_{42}), \quad \alpha_7 = 2(t_{33} - t_1 t_{43}), \quad \alpha_8 = 2t_{34},$$
$$\alpha_9 = -2s_2, \quad \alpha_{10} = 2(s_1 s_2 - s_3), \quad \alpha_{11} = 2(s_1 s_3 - s_4), \quad \alpha_{12} = 2 s_1 s_4, \quad \alpha_{13} = 2(t_{35} - t_1 t_{54}),$$
$$\alpha_{14} = 2(t_{29} - t_1 t_{55}), \quad \alpha_{15} = 2(t_{36} - t_1 t_{56}), \quad \alpha_{16} = 2(t_{31} - t_1 t_{57}), \quad \alpha_{17} = 2(t_{32} - t_1 t_{58}),$$
$$\alpha_{18} = 2(t_{33} - t_1 t_{59}), \quad \alpha_{19} = 2 s_2, \quad \alpha_{20} = 2(s_3 + s_1 s_2), \quad \alpha_{21} = 2(s_4 + s_1 s_3),$$
$$\alpha_{22} = \alpha_2 + \alpha_1 \alpha_9 - 2 s_{38} s_{39}, \quad \alpha_{23} = \alpha_3 - 2 s_1 \alpha_2 + \alpha_1 \alpha_{10} - 2 s_{38} s_{40},$$
$$\alpha_{24} = \alpha_4 - 2 s_1 \alpha_3 + s_1^2 \alpha_2 + \alpha_1(\alpha_{11} - 2\alpha_9) - 2 s_{38}(s_{41} - 2 s_{39}),$$
$$\alpha_{25} = \alpha_5 - 2 s_1 \alpha_4 + s_1^2 \alpha_3 + \alpha_1(\alpha_{12} - 2\alpha_{10}) - 2 s_{38}(s_{42} - 2 s_{40}),$$
$$\alpha_{26} = \alpha_6 - 2 s_1 \alpha_5 + s_1^2 \alpha_4 + \alpha_1(-2\alpha_{11} + \alpha_9) - 2 s_{38}(-2 s_{41} + s_{39}),$$
$$\alpha_{27} = \alpha_7 - 2 s_1 \alpha_6 + s_1^2 \alpha_5 + \alpha_1(-2\alpha_{12} + \alpha_{10}) - 2 s_{38}(-2 s_{42} + s_{40}),$$
$$\alpha_{28} = \alpha_8 - 2 s_1 \alpha_7 + s_1^2 \alpha_6 + \alpha_1 \alpha_{11} - 2 s_{38} s_{41}, \quad \alpha_{29} = -2 s_1 \alpha_8 + s_1^2 \alpha_7 + \alpha_1 \alpha_{12} - 2 s_{38} s_{42},$$
$$\alpha_{30} = s_1^2 \alpha_8, \quad \alpha_{31} = \alpha_{13} + \alpha_1 \alpha_{19} - 2 s_{38} s_{39}, \quad \alpha_{32} = \alpha_{14} + 2 s_1 \alpha_{13} + \alpha_1 \alpha_{20} - 2 s_{38} s_{40},$$
$$\alpha_{33} = \alpha_{15} + 2 s_1 \alpha_{14} + s_1^2 \alpha_{13} + \alpha_1(\alpha_{21} - 2\alpha_{19}) - 2 s_{38}(s_{41} - 2 s_{39}),$$
$$\alpha_{34} = \alpha_{16} + 2 s_1 \alpha_{15} + s_1^2 \alpha_{14} + \alpha_1(\alpha_{12} - 2\alpha_{20}) - 2 s_{38}(s_{42} - 2 s_{40}),$$
$$\alpha_{53} = \alpha_{17} + 2 s_1 \alpha_{16} + s_1^2 \alpha_{15} + \alpha_1(-2\alpha_{21} + \alpha_{19}) - 2 s_{38}(-2 s_{41} + s_{39}),$$
$$\alpha_{36} = \alpha_{18} + 2 s_1 \alpha_{17} + s_1^2 \alpha_{16} + \alpha_1(-2\alpha_{12} + \alpha_{20}) - 2 s_{38}(-2 s_{42} + s_{40}),$$
$$\alpha_{37} = \alpha_8 + 2 s_1 \alpha_{18} + s_1^2 \alpha_{17} + \alpha_1 \alpha_{21} - 2 s_{38} s_{41}, \quad \alpha_{38} = 2 s_1 \alpha_8 + s_1^2 \alpha_{18} + \alpha_1 \alpha_{12} - 2 s_{38} s_{42},$$
$$\alpha_{39} = s_1^2 \alpha_8.$$

We can put

$$B_0 + B_2 = \beta_1 C + [\beta_2 + \beta_3 C + \beta_4 C^2] K + [\beta_5 + \beta_6 C + \ldots + \beta_8 C^3] K^2 + [\beta_9 + \beta_{10} C + \ldots + \beta_{15} C^6] \frac{K^3}{1 - C^2},$$

where

$$\beta_1 = s_1, \quad \beta_2 = s_2 + s_{38}, \quad \beta_3 = s_3, \quad \beta_4 = s_4 - s_{38}, \quad \beta_5 = s_5 + s_{39}, \quad \beta_6 = s_6 + s_{46},$$
$$\beta_7 = s_7 + s_{41}, \quad \beta_8 = s_8 + s_{42}, \quad \beta_9 = s_9 + s_{43}, \quad \beta_{10} = s_{10} + s_{44} \quad \beta_{11} = s_{11} + s_{45},$$
$$\beta_{12} = s_{12} + s_{46}, \quad \beta_{13} = s_{13} + s_{47}, \quad \beta_{14} = s_{14} + s_{48}, \quad \beta_{15} = s_{15} + s_{49}.$$

Let us call the quantity under the radical sign in the expression for f, R^2, then

$$R^2 = [-s_1 + C]^2 + 2[\beta_{16} + \beta_{17} C + \beta_{18} C^2 + \beta_{19} C^3] K + [\beta_{20} + \beta_{21} C + \ldots + \beta_{26} C_6] \frac{K^2}{1 - C^2}$$
$$+ [\beta_{27} + \beta_{28} C + \ldots + \beta_{37} C^{10}] \frac{K^3}{(1 - C^2)^2},$$

where

$$\beta_{16} = -\beta_2 - s_{16} s_{17}, \quad \beta_{17} = -\beta_3 + \beta_1 \beta_2 - s_{16} s_{18}, \quad \beta_{18} = -\beta_4 + \beta_1 \beta_3 - s_{16} s_{19}, \quad \beta_{19} = \beta_1 \beta_4 - s_{16} s_{20},$$
$$\beta_{20} = -2\beta_5 + \beta_2^2 - 2 s_{16} s_{21} - s_{17}^2 + \alpha_1 - s_{38}^2, \quad \beta_{21} = -2\beta_6 + 2\beta_1\beta_5 + 2\beta_2\beta_3 - 2 s_{16} s_{22} - 2 s_{17} s_{18} - 2 s_1 \alpha_1,$$
$$\beta_{22} = -2\beta_7 + 2\beta_5 + 2\beta_1\beta_6 + 2\beta_2\beta_4 + \beta_3^2 - \beta_2^2 - 2 s_{16} s_{23} - 2 s_{17} s_{19} + (s_1^2 - 2)\alpha_1 + 3 s_{38}^2,$$
$$\beta_{23} = -2\beta_8 + 2\beta_6 + 2\beta_1\beta_7 - 2\beta_1\beta_5 + 2\beta_3\beta_4 - 2\beta_2\beta_3 - 2 s_{16} s_{24} - 2 s_{18} s_{19} + 4 s_1 \alpha_1,$$
$$\beta_{24} = 2\beta_7 + 2\beta_1\beta_8 - 2\beta_1\beta_6 - 2\beta_2\beta_4 + \beta_4^2 - \beta_3^2 - 2 s_{16} s_{25} - 2 s_{18} s_{20} + (1 - 2 s_1^2)\alpha_1 - 3 s_{38}^2,$$
$$\beta_{25} = 2\beta_8 - 2\beta_1\beta_7 - 2\beta_3\beta_4 - 2 s_{16} s_{26} - 2 s_{19} s_{20} - 2 s_1 \alpha_1,$$
$$\beta_{26} = -2\beta_1\beta_8 - \beta_4^2 - 2 s_{16} s_{27} - s_{20}^2 + s_1^2 \alpha_1 + s_{38}^2,$$
$$\beta_{27} = 2(-\beta_9 + \beta_2\beta_5 - s_{16} s_{28} - s_{17} s_{21}) + \alpha_{22},$$
$$\beta_{28} = 2(-\beta_{10} + \beta_1\beta_9 + \beta_2\beta_6 + \beta_3\beta_5 - s_{16} s_{29} - s_{17} s_{22} - s_{18} s_{21}) + \alpha_{23},$$

$$\beta_{29} = 2\,(-\beta_{11} + \beta_9 + \beta_1\,\beta_{10} + \beta_2\,(\beta_7 - 2\,\beta_6) + \beta_3\,\beta_6 + \beta_4\,\beta_5 - s_{16}\,s_{30} - s_{17}\,s_{23} - s_{18}\,s_{22} - s_{19}\,s_{21})$$
$$+ \alpha_{24} - \alpha_{22},$$
$$\beta_{30} = 2\,(-\beta_{12} + \beta_{10} + \beta_1\,(\beta_{11} - \beta_9) + \beta_2\,(\beta_8 - 2\,\beta_6) + \beta_3\,(\beta_7 - 2\,\beta_5) + \beta_4\,\beta_6 - s_{16}\,s_{31} - s_{17}\,s_{24}$$
$$- s_{18}\,s_{23} - s_{19}\,s_{22} - s_{20}\,s_{21}) + \alpha_{25} - \alpha_{23},$$
$$\beta_{31} = 2\,(-\beta_{13} + \beta_{11} + \beta_1\,(\beta_{12} - \beta_{10}) - \beta_2\,(2\,\beta_7 - \beta_5) + \beta_3\,(\beta_8 - 2\,\beta_6) + \beta_4\,(\beta_7 - 2\,\beta_5) - s_{16}\,s_{32}$$
$$- s_{17}\,s_{25} - s_{18}\,s_{24} - s_{19}\,s_{23} - s_{20}\,s_{22}) + \alpha_{26} - \alpha_{24},$$
$$\beta_{32} = 2\,(-\beta_{14} + \beta_{12} + \beta_1\,(\beta_{13} - \beta_{11}) + \beta_2\,(-2\,\beta_8 + \beta_6) + \beta_3\,(-2\,\beta_7 + \beta_5) + \beta_4\,(\beta_8 - 2\,\beta_6)$$
$$- s_{16}\,s_{33} - s_{17}\,s_{26} - s_{18}\,s_{25} - s_{19}\,s_{24} - s_{20}\,s_{23}) + \alpha_{27} - \alpha_{25},$$
$$\beta_{33} = 2\,(-\beta_{15} + \beta_{13} + \beta_1\,(\beta_{14} - \beta_{12}) + \beta_2\,\beta_7 + \beta_3\,(-2\,\beta_8 + \beta_6) + \beta_4\,(-2\,\beta_7 + \beta_5) - s_{18}\,s_{34}$$
$$- s_{17}\,s_{27} - s_{18}\,s_{26} - s_{19}\,s_{25} - s_{20}\,s_{24}) + \alpha_{28} - \alpha_{26},$$
$$\beta_{34} = 2\,(\beta_{14} + \beta_1\,(\beta_{15} - \beta_{13}) + \beta_2\,\beta_8 + \beta_3\,\beta_7 + \beta_4\,(-2\,\beta_8 + \beta_6) - s_{16}\,s_{35} - s_{18}\,s_{27} - s_{19}\,s_{26} - s_{20}\,s_{25})$$
$$+ \alpha_{29} - \alpha_{27},$$
$$\beta_{35} = 2\,(\beta_{15} - \beta_1\,\beta_{14} + \beta_3\,\beta_8 + \beta_4\,\beta_7 - s_{16}\,s_{36} - s_{19}\,s_{27} - s_{20}\,s_{26}) + \alpha_{30} - \alpha_{28},$$
$$\beta_{36} = 2\,(-\beta_1\,\beta_{15} + \beta_4\,\beta_8 - s_{16}\,s_{37} - s_{20}\,s_{27}) - \alpha_{29}, \qquad \beta_{37} = -\alpha_{30}.$$

In like manner the quantity under the radical sign in the expression for f',

$$R'^2 = [-s_1 - C]^2 + 2\,[\beta'_{16} + \beta'_{17}\,C + \beta'_{18}\,C^2 + \beta'_{19}\,C^3]\,K + [\beta'_{20} + \beta'_{21}\,C + \ldots + \beta'_{26}\,C^6]\,\frac{K^2}{1 - C^2}$$
$$+ [\beta'_{27} + \beta'_{28}\,C + \ldots + \beta'_{37}\,C^{10}]\,\frac{K^3}{(1 - C^2)^2},$$

where the coefficients are determined by the equations

$$\beta'_{16} = \beta_{16} + 2\,\beta_2, \quad \beta'_{17} = \beta_{17} + 2\,\beta_3, \quad \beta'_{18} = \beta_{18} + 2\,\beta_4, \quad \beta'_{19} = \beta_{19}, \quad \beta'_{20} = \beta_{20} + 4\,\beta_5,$$
$$\beta'_{21} = \beta_{21} + 4\,\beta_6 + 4\,s_1\,\alpha_1, \quad \beta'_{22} = \beta_{22} + 4\,\beta_7 - 4\,\beta_5, \quad \beta'_{23} = \beta_{23} + 4\,\beta_8 - 4\,\beta_6 - 8\,s_1\,\alpha_1,$$
$$\beta'_{24} = \beta_{24} - 4\,\beta_7, \quad \beta'_{25} = \beta_{25} - 4\,\beta_8 + 4\,s_1\,\alpha_1, \quad \beta'_{26} = \beta_{26}, \quad \beta'_{27} = \beta_{27} + 4\,\beta_9 - \alpha_{22} + \alpha_{31},$$
$$\beta'_{28} = \beta_{28} + 4\,\beta_{10} + \alpha_{32} - \alpha_{23}, \quad \beta'_{29} = \beta_{29} + 4\,(\beta_{11} - \beta_9) + \alpha_{33} - \alpha_{31} - \alpha_{24} + \alpha_{22},$$
$$\beta'_{30} = \beta_{30} + 4\,(\beta_{12} - \beta_{10}) + \alpha_{34} - \alpha_{32} - \alpha_{25} + \alpha_{23},$$
$$\beta'_{31} = \beta_{31} + 4\,(\beta_{13} - \beta_{11}) + \alpha_{35} - \alpha_{33} - \alpha_{26} + \alpha_{24},$$
$$\beta'_{32} = \beta_{32} + 4\,(\beta_{14} - \beta_{12}) + \alpha_{36} - \alpha_{34} - \alpha_{27} + \alpha_{25},$$
$$\beta'_{33} = \beta_{33} + 4\,(\beta_{15} - \beta_{13}) + \alpha_{37} - \alpha_{35} - \alpha_{28} + \alpha_{26},$$
$$\beta'_{34} = \beta_{34} - 4\,\beta_{14} + \alpha_{38} - \alpha_{36} - \alpha_{29} + \alpha_{27}, \qquad \beta'_{35} = \beta_{35} - 4\,\beta_{15} + \alpha_{39} - \alpha_{37} - \alpha_{30} + \alpha_{28},$$
$$\beta'_{36} = \beta_{36} - \alpha_{38} + \alpha_{29}, \qquad \beta'_{37} = \beta_{37} - \alpha_{39} + \alpha_{30}.$$

From the value of R^2 we derive

$$\frac{1}{R} = \frac{1}{-s_1 + C} - \frac{\beta_{16} + \beta_{17}\,C + \beta_{18}\,C^2 + \beta_{19}\,C^3}{(-s_1 + C)^3}\,K + \frac{\beta_{38} + \beta_{39}\,C + \ldots + \beta_{46}\,C^8}{(-s_1 + C)^5}\,\frac{K^2}{1 - C^2}$$
$$+ \frac{\beta_{47} + \beta_{48}\,C + \ldots + \beta_{61}\,C^{14}}{(-s_1 + C)^7}\,\frac{K^3}{(1 - C^2)^2}.$$

In order to facilitate the expression of the coefficients we adopt the following subsidiary quantities

$$\delta_1 = \beta_{16}\,\beta_{18} + \beta_{17}\,\beta_{17}, \qquad \delta_2 = \beta_{16}\,\beta_{21} + \beta_{17}\,\beta_{20}, \qquad \delta_3 = \beta_{16}\,\beta_{22} + \beta_{17}\,\beta_{21} + \beta_{18}\,\beta_{20},$$
$$\delta_4 = \beta_{16}\,\beta_{23} + \beta_{17}\,\beta_{22} + \beta_{18}\,\beta_{21} + \beta_{19}\,\beta_{20}, \qquad \delta_5 = \beta_{16}\,\beta_{24} + \beta_{17}\,\beta_{23} + \beta_{18}\,\beta_{22} + \beta_{19}\,\beta_{21},$$
$$\delta_6 = \beta_{16}\,\beta_{25} + \beta_{17}\,\beta_{24} + \beta_{18}\,\beta_{23} + \beta_{19}\,\beta_{22}, \qquad \delta_7 = \beta_{16}\,\beta_{26} + \beta_{17}\,\beta_{25} + \beta_{18}\,\beta_{24} + \beta_{19}\,\beta\ \ ,$$
$$\delta_8 = \beta_{17}\,\beta_{26} + \beta_{18}\,\beta_{25} + \beta_{19}\,\beta_{24}, \qquad \delta_9 = \beta_{18}\,\beta_{26} + \beta_{19}\,\beta_{25}, \qquad \delta_{10} = \beta_{19}\,\beta_{26},$$

then the coefficients β satisfy the following equations

$$
\begin{aligned}
2\beta_{38} &= 3\beta_{16}^2 - s_1^2\beta_{20}, \qquad 2\beta_{39} = 6\beta_{16}\beta_{17} + 2s_1\beta_{20} - s_1^2\beta_{21}, \\
2\beta_{40} &= 3(2\delta_1 - \beta_{16}^2 - \beta_{17}^2) - \beta_{20} + 2s_1\beta_{21} - s_1^2\beta_{22}, \\
2\beta_{41} &= 6(\beta_{16}\beta_{19} + \beta_{17}\beta_{18} - \beta_{16}\beta_{17}) - \beta_{21} + 2s_1\beta_{22} - s_1^2\beta_{23}, \\
2\beta_{42} &= 3(\beta_{18}^2 + 2\beta_{17}\beta_{19} - 2\beta_{16}\beta_{18} - \beta_{17}^2) - \beta_{22} + 2s_1\beta_{23} - s_1^2\beta_{24}, \\
2\beta_{43} &= 6(\beta_{18}\beta_{19} - \beta_{16}\beta_{19} - \beta_{17}\beta_{18}) - \beta_{23} + 2s_1\beta_{24} - s_1^2\beta_{25}, \\
2\beta_{44} &= 3(\beta_{19}^2 - \beta_{18}^2 - 2\beta_{17}\beta_{19}) - \beta_{24} + 2s_1\beta_{25} - s_1^2\beta_{26}, \qquad 2\beta_{45} = 6\beta_{18}\beta_{19} - \beta_{25} + 2s_1\beta_{26}, \\
2\beta_{46} &= -3\beta_{19}^2 - \beta_{26}, \qquad 2\beta_{47} = -s_1^4\beta_{27} - 2\beta_{16}\beta_{38} - 2\beta_{16}^3 + 2s_1^2\beta_{16}\beta_{20}, \\
2\beta_{48} &= -s_1^4\beta_{28} + 4s_1^3\beta_{27} - 2\beta_{16}\beta_{39} - 2\beta_{17}\beta_{38} - 6\beta_{16}^2\beta_{17} + 2s_1^2\delta_2 - 4s_1\beta_{16}\beta_{20}, \\
2\beta_{49} &= -s_1^4\beta_{29} + 4s_1^3\beta_{28} - 6s_1^2\beta_{27} - 2\beta_{16}\beta_{40} - 2\beta_{17}\beta_{39} - 2(\beta_{18} - \beta_{16})\beta_{38} - 6\beta_{16}\delta_1 + 4\beta_{16}^3 \\
&\quad + 2s_1^2\delta_3 - 2s_1\delta_2 + 2(1 - s_1^2)\beta_{16}\beta_{20}, \\
2\beta_{50} &= -s_1^4\beta_{30} + 4s_1^3\beta_{29} - 6s_1^2\beta_{28} + 4s_1\beta_{27} - 2\beta_{16}\beta_{41} - 2\beta_{17}\beta_{40} - 2(\beta_{18} - \beta_{16})\beta_{39} - 2(\beta_{19} - \beta_{17})\beta_{38} \\
&\quad - 2(3\beta_{16}^2\beta_{19} + 6\beta_{16}\beta_{17}\beta_{18} + \beta_{17}^3) + 12\beta_{16}^2\beta_{17} + 2s_1^2\delta_4 - 4s_1\delta_3 + 2(1 - s_1^2)\delta_2 + 4s_1\beta_{16}\beta_{20}, \\
2\beta_{51} &= -s_1^4\beta_{31} + 4s_1^3\beta_{30} - 6s_1^2\beta_{29} + 4s_1\beta_{28} - \beta_{27} - 2\beta_{16}\beta_{42} - 2\beta_{17}\beta_{41} - 2(\beta_{18} - \beta_{16})\beta_{40} \\
&\quad - 2(\beta_{19} - \beta_{17})\beta_{39} + 2\beta_{18}\beta_{38} - 6\beta_{16}(2\beta_{17}\beta_{19} + \beta_{18}^2) - 6\beta_{17}^2\beta_{18} - 12\beta_{16}\delta_1 - 2\beta_{16}^3 \\
&\quad + 2s_1^2\delta_5 - 4s_1\delta_4 + 2(1 - s_1^2)\delta_3 + 4s_1\delta_2 - 2\beta_{16}\beta_{20}, \\
2\beta_{52} &= -s_1^4\beta_{32} + 4s_1^3\beta_{31} - 6s_1^2\beta_{30} + 4s_1\beta_{29} - \beta_{28} - 2\beta_{16}\beta_{43} - 2\beta_{17}\beta_{42} - 2(\beta_{18} - \beta_{16})\beta_{41} \\
&\quad - 2(\beta_{19} - \beta_{17})\beta_{40} + 2\beta_{18}\beta_{39} + 2\beta_{19}\beta_{38} - 6(2\beta_{16}\beta_{18}\beta_{19} + \beta_{17}^2\beta_{19} + \beta_{17}\beta_{18}^2) + 4(3\beta_{16}^2\beta_{19} \\
&\quad + 6\beta_{16}\beta_{17}\beta_{18} + \beta_{17}^3) - 6\beta_{16}^2\beta_{17} + 2s_1^2\delta_6 - 4s_1\delta_5 + 2(1 - s_1^2)\delta_4 + 4s_1\delta_3 - 2\delta_2, \\
2\beta_{53} &= -s_1^4\beta_{33} + 4s_1^3\beta_{32} - 6s_1^2\beta_{31} + 4s_1\beta_{30} - \beta_{29} - 2\beta_{16}\beta_{44} - 2\beta_{17}\beta_{43} - 2(\beta_{18} - \beta_{16})\beta_{42} \\
&\quad - 2(\beta_{19} - \beta_{17})\beta_{41} + 2\beta_{18}\beta_{40} + 2\beta_{19}\beta_{39} - 2(3\beta_{16}\beta_{19}^2 + 6\beta_{17}\beta_{18}\beta_{19} + \beta_{18}^3) + 12\beta_{16}(2\beta_{17}\beta_{19} \\
&\quad + \beta_{18}^2) + 12\beta_{17}^2\beta_{18} - 6\beta_{16}\delta_1 + 2s_1^2\delta_7 - 4s_1\delta_6 + 2(1 - s_1^2)\delta_5 + 4s_1\delta_4 - 2\delta_3, \\
2\beta_{54} &= -s_1^4\beta_{34} + 4s_1^3\beta_{33} - 6s_1^2\beta_{32} + 4s_1\beta_{31} - \beta_{30} - 2\beta_{16}\beta_{45} - 2\beta_{17}\beta_{44} - 2(\beta_{18} - \beta_{16})\beta_{43} \\
&\quad - 2(\beta_{19} - \beta_{17})\beta_{42} + 2\beta_{18}\beta_{41} + 2\beta_{19}\beta_{40} - 6\beta_{19}(\beta_{19}\beta_{17} + \beta_{18}^2) + 12(2\beta_{16}\beta_{18}\beta_{19} + \beta_{17}^2\beta_{19} \\
&\quad + \beta_{17}\beta_{18}^2) - 2(3\beta_{16}^2\beta_{19} + 6\beta_{16}\beta_{17}\beta_{18} + \beta_{17}^3) + 2s_1\delta_8 - 4s_1\delta_7 + 2(1 - s_1^2)\delta_6 + 4s_1\delta_5 - 2\delta_4, \\
2\beta_{55} &= -s_1^4\beta_{35} + 4s_1^3\beta_{34} - 6s_1^2\beta_{33} + 4s_1\beta_{32} - \beta_{31} - 2\beta_{16}\beta_{46} - 2\beta_{17}\beta_{45} - 2(\beta_{18} - \beta_{16})\beta_{44} \\
&\quad - 2(\beta_{19} - \beta_{17})\beta_{43} + 2\beta_{18}\beta_{42} + 2\beta_{19}\beta_{41} - 6\beta_{18}\beta_{19}^2 + 4(3\beta_{16}\beta_{19}^2 + 6\beta_{17}\beta_{18}\beta_{19} + \beta_{18}^3) \\
&\quad - 6\beta_{16}(2\beta_{17}\beta_{19} + \beta_{18}^2) - 6\beta_{17}^2\beta_{18} + 2s_1^2\delta_9 - 4s_1\delta_8 + 2(1 - s_1^2)\delta_7 + 4s_1\delta_6 - 2\delta_5, \\
2\beta_{56} &= -s_1^4\beta_{36} + 4s_1^3\beta_{35} - 6s_1^2\beta_{34} + 4s_1\beta_{33} - \beta_{32} - 2\beta_{17}\beta_{46} - 2(\beta_{18} - \beta_{16})\beta_{45} + 2(\beta_{19} - \beta_{17})\beta_{44} \\
&\quad + 2\beta_{18}\beta_{43} + 2\beta_{19}\beta_{42} - 2\beta_{19}^3 - 6\beta_{19}(\beta_{19}\beta_{17} + \beta_{18}^2) - 6(2\beta_{16}\beta_{18}\beta_{19} + \beta_{17}^2\beta_{19} + \beta_{17}\beta_{18}^2) \\
&\quad + 2s_1^2\delta_{10} - 4s_1\delta_9 + 2(1 - s_1^2)\delta_8 + 4s_1\delta_7 - 2\delta_6, \\
2\beta_{57} &= -s_1^4\beta_{37} + 4s_1^3\beta_{36} - 6s_1^2\beta_{35} + 4s_1\beta_{34} - \beta_{33} - 2(\beta_{18} - \beta_{16})\beta_{46} - 2(\beta_{19} - \beta_{17})\beta_{45} + 2\beta_{18}\beta_{44} \\
&\quad + 2\beta_{19}\beta_{43} + 12\beta_{18}\beta_{19}^2 - 2(3\beta_{16}\beta_{19}^2 + 6\beta_{17}\beta_{18}\beta_{19} + \beta_{18}^3) - 4s_1\delta_{10} + 2(1 - s_1^2)\delta_9 + 4s_1\delta_8 - 2\delta_7, \\
2\beta_{58} &= -4s_1^3\beta_{37} - 6s_1^4\beta_{36} + 4s_1\beta_{35} - \beta_{34} - 2(\beta_{19} - \beta_{17})\beta_{46} + 2\beta_{18}\beta_{45} + 2\beta_{19}\beta_{44} + 4\beta_{19}^3 \\
&\quad - 6\beta_{19}(\beta_{17}\beta_{19} + \beta_{18}^2) + 2(1 - s_1^2)\delta_{10} + 4s_1\delta_9 - 2\delta_8, \\
2\beta_{59} &= -6s_1^2\beta_{37} + 4s_1\beta_{36} - \beta_{35} + 2\beta_{18}\beta_{46} + 2\beta_{19}\beta_{45} - 6\beta_{18}\beta_{19}^2 + 4s_1\delta_{10} - 2\delta_9, \\
2\beta_{60} &= 4s_1\beta_{37} - \beta_{36} + 2\beta_{19}\beta_{46} - 2\beta_{19}^3 - 2\delta_{10}, \qquad 2\beta_{61} = -\beta_{37}.
\end{aligned}
$$

In like manner we have

$$
\begin{aligned}
\frac{1}{R'} = \frac{1}{-s_1 - C} &+ \frac{\beta'_{16} + \beta'_{17}C + \beta'_{18}C^2 + \beta'_{19}C^3}{(-s_1 - C)^3}K + \frac{\beta'_{38} + \beta'_{39}C + \ldots + \beta'_{46}C^8}{(-s_1 - C)^5}\,\frac{K^2}{1 - C^2} \\
&+ \frac{\beta'_{47} + \beta'_{48}C + \ldots + \beta'_{61}C^{14}}{(-s_1 - C)^7}\,\frac{K^3}{(1 - C^2)^2},
\end{aligned}
$$

where the coefficients from β'_{38} to β'_{61} satisfy equations which are obtained from the preceding group by accenting the β from β_{16} to β_{37}, reversing the sign of s_1 and finally the sign of the whole expression.

We may write the terms of f and f' we are considering

$$f = \frac{M}{R} \operatorname{arc\,tan} \left(\frac{N}{R} \tan \tfrac{1}{2} \theta' \right), \qquad f' = \frac{M'}{R'} \operatorname{arc\,tan} \left(\frac{N'}{R'} \tan \tfrac{1}{2} \theta' \right).$$

We have

$$M = \gamma_1 + \gamma_2 C + [\gamma_3 + \gamma_4 C + \gamma_5 C^2] K + [\gamma_6 + \gamma_7 C + \cdots + \gamma_{13} C^7] \frac{K^2}{(1 - C^2)^2}$$
$$+ [\gamma_{14} + \cdots + \gamma_{25} C^{11}] \frac{K^3}{(1 - C^2)^3},$$

where the coefficients γ are given by the equations

$$\gamma_1 = p_1 j_{21}, \quad \gamma_2 = p_1 j_{22}, \quad \gamma_3 = p_1 t_{166} + p_2 j_{21}, \quad \gamma_4 = p_1 t_{167} + p_2 j_{22} + p_3 j_{21},$$
$$\gamma_5 = p_1 t_{168} + p_3 j_{22}, \quad \gamma_6 = p_1 t_{169} + p_2 t_{166} + p_4 j_{21}, \quad \gamma_7 = p_1 t_{170} + p_2 t_{167} + p_3 t_{166} + p_4 j_{22} + p_5 j_{21},$$
$$\gamma_8 = p_1 t_{171} + p_2 (t_{168} - 2 t_{166}) + p_3 t_{167} + p_5 j_{22} + (p_6 - 2 p_4) j_{21},$$
$$\gamma_9 = p_1 t_{172} - 2 p_2 t_{167} + p_3 (t_{168} - 2 t_{166}) - 2 p_5 j_{21} + (p_6 - 2 p_4) j_{22},$$
$$\gamma_{10} = p_1 t_{173} + p_2 (- 2 t_{168} + t_{166}) - 2 p_3 t_{167} - 2 p_5 j_{22} + (- 2 p_6 + p_4) j_{21},$$
$$\gamma_{11} = p_1 t_{174} + p_2 t_{167} + p_3 (- 2 t_{168} + t_{166}) + p_5 j_{21} + (- 2 p_6 + p_4) j_{22},$$
$$\gamma_{12} = p_1 t_{175} + p_2 t_{168} + p_3 t_{167} + p_5 j_{22} + p_6 j_{21}, \qquad \gamma_{13} = p_1 t_{176} + p_3 t_{168} + p_6 j_{22},$$
$$\gamma_{14} = p_1 t_{177} + p_2 t_{169} + p_4 t_{166} + p_7 j_{21},$$
$$\gamma_{15} = p_1 t_{178} + p_2 t_{170} + p_3 t_{169} + p_4 t_{167} + p_5 t_{166} + p_7 j_{22} + p_8 j_{21},$$
$$\gamma_{16} = p_1 t_{179} + p_2 (t_{171} - t_{169}) + p_3 t_{170} + p_4 (t_{168} - 3 t_{166}) + p_5 t_{167} + p_6 t_{166} + p_8 j_{22} + (p_9 - 3 p_7) j_{21},$$
$$\gamma_{17} = p_1 t_{180} + p_2 (t_{172} - t_{170}) + p_3 (t_{171} - t_{169}) - 3 p_4 t_{167} + p_5 (t_{168} - 3 t_{166}) + p_6 t_{167} + (p_9 - 3 p_7) j_{22}$$
$$+ (p_{10} - 3 p_8) j_{21},$$
$$\gamma_{18} = p_1 t_{181} + p_2 (t_{173} - t_{171}) + p_3 (t_{172} - t_{170}) + p_4 (- 3 t_{168} + 3 t_{166}) - 3 p_5 t_{167} + p_6 (t_{168} - 3 t_{166})$$
$$+ (p_{10} - 3 p_8) j_{22} + (- 3 p_9 + p_7) j_{21},$$
$$\gamma_{19} = p_1 t_{182} + p_2 (t_{174} - t_{172}) + p_3 (t_{173} - t_{171}) + 3 p_4 t_{167} + p_5 (- 3 t_{168} + 3 t_{166}) - 3 p_6 t_{167}$$
$$+ (- 3 p_9 + 3 p_7) j_{22} + (- 3 p_{10} + 3 p_8) j_{21},$$
$$\gamma_{20} = p_1 t_{183} + p_2 (t_{175} - t_{173}) + p_3 (t_{174} - t_{172}) + p_4 (3 t_{168} - t_{166}) + 3 p_5 t_{167} + p_6 (- 3 t_{168} + 3 t_{166})$$
$$+ (- 3 p_{10} + 3 p_8) j_{22} + (3 p_9 - p_7) j_{21},$$
$$\gamma_{21} = p_1 t_{184} + p_2 (t_{176} - t_{174}) + p_3 (t_{175} - t_{173}) - p_4 t_{167} + p_5 (3 t_{168} - t_{166}) + 3 p_6 t_{167} + (3 p_9 - p_7) j_{22}$$
$$+ (3 p_{10} - p_8) j_{21},$$
$$\gamma_{22} = p_1 t_{185} - p_2 t_{175} + p_3 (t_{176} - t_{174}) - p_4 t_{168} - p_5 t_{167} + p_6 (3 t_{168} - t_{166}) + (3 p_{10} - p_8) j_{22} - p_9 j_{21},$$
$$\gamma_{23} = p_1 t_{186} - p_2 t_{176} - p_3 t_{175} - p_5 t_{168} - p_6 t_{167} - p_9 j_{22} - p_{10} j_{21},$$
$$\gamma_{24} = p_1 t_{187} - p_3 t_{176} - p_6 t_{168} - p_{10} j_{22}, \qquad \gamma_{25} = p_1 t_{188}.$$

Also we have

$$M' = \gamma'_1 + \gamma'_2 C + [\gamma'_3 + \gamma'_4 C + \gamma'_5 C^2] K + [\gamma'_6 + \gamma'_7 C + \cdots + \gamma'_{13} C^7] \frac{K^2}{(1 - C^2)^2}$$
$$+ [\gamma'_{14} + \cdots + \gamma'_{25} C^{11}] \frac{K^3}{(1 - C^2)^3},$$

where the equations determining the γ' are derivable from the preceding group by simply accenting j_{21} and j_{22} and augmenting the subscripts of the t by 23.

The quantities N and N' are given by the equations

$$
\begin{aligned}
N = 1 - s_1 C + s_{16}\sqrt{1 - C^2} &+ [(t_{70} + t_{71} C + \ldots + t_{73} C^3)(1 - C^2)^{-\frac{1}{2}} - (s_2 + s_3 C + s_4 C^2)] K \\
&+ [(t_{74} + t_{75} C + \ldots + t_{80} C^6)(1 - C^2)^{-\frac{3}{2}} - (s_5 + s_6 C + \ldots + s_8 C^3)] K^2 \\
&+ [(t_{81} + t_{82} C + \ldots + t_{91} C^{10})(1 - C^2)^{-\frac{5}{2}} - (s_9 + s_{10} C + \ldots + s_{15} C^6)] K^3, \\
N' = 1 + s_1 C - s_{16}\sqrt{1 - C^2} &- [(t_{92} + t_{93} C + \ldots + t_{95} C^3)(1 - C^2)^{-\frac{1}{2}} - (s_2 + s_3 C + s_4 C^2)] K \\
&- [(t_{96} + t_{97} C + \ldots + t_{102} C^6)(1 - C^2)^{-\frac{3}{2}} - (s_5 + s_6 C + \ldots + s_8 C^3)] K^2 \\
&- [(t_{103} + t_{104} C + \ldots + t_{113} C^{10})(1 - C^2)^{-\frac{5}{2}} - (s_9 + s_{10} C + \ldots + s_{15} C^6)] K^3.
\end{aligned}
$$

By multiplication

$$
\begin{aligned}
\frac{M}{R} = \frac{\gamma_1 + \gamma_2 C}{-s_1 + C} + \frac{\gamma_{26} + \gamma_{27} C + \ldots + \gamma_{30} C^4}{(-s_1 + C)^3} K + \frac{\gamma_{31} + \gamma_{32} C + \ldots + \gamma_{42} C^{11}}{(-s_1 + C)^5 (1 - C^2)^2} K^2 \\
+ \frac{\gamma_{43} + \ldots + \gamma_{60} C^{17}}{(-s_1 + C)^7 (1 - C^2)^3} K^3.
\end{aligned}
$$

To facilitate the expression of the coefficients γ we adopt the following

SUBSIDIARY QUANTITIES.

$$
\begin{aligned}
&\varepsilon_1 = s_1^2 \gamma_3, \quad \varepsilon_2 = s_1^2 \gamma_4 - 2 s_1 \gamma_3, \quad \varepsilon_3 = s_1^2 \gamma_5 - 2 s_1 \gamma_4 + \gamma_3, \quad \varepsilon_4 = -2 s_1 \gamma_5 + \gamma_4, \quad \varepsilon_5 = \gamma_5, \\
&\varepsilon_6 = s_1^2 \beta_{38}, \quad \varepsilon_7 = s_1^2 \beta_{39} - 2 s_1 \beta_{38}, \quad \varepsilon_8 = s_1^2 \beta_{40} - 2 s_1 \beta_{39} + \beta_{33}, \quad \varepsilon_9 = s_1^2 \beta_{41} - 2 s_1 \beta_{40} + \beta_{39}, \\
&\varepsilon_{10} = s_1^2 \beta_{42} - 2 s_1 \beta_{41} + \beta_{40}, \quad \varepsilon_{11} = s_1^2 \beta_{43} - 2 s_1 \beta_{42} + \beta_{41}, \quad \varepsilon_{12} = s_1^2 \beta_{44} - 2 s_1 \beta_{43} + \beta_{42}, \\
&\varepsilon_{13} = s_1^2 \beta_{45} - 2 s_1 \beta_{44} + \beta_{43}, \quad \varepsilon_{14} = s_1^2 \beta_{46} - 2 s_1 \beta_{45} + \beta_{44}, \quad \varepsilon_{15} = -2 s_1 \beta_{46} + \beta_{45}, \quad \varepsilon_{16} = \beta_{46}, \\
&\zeta_1 = s_1^4 \beta_{16}, \quad \zeta_2 = s_1^4 \beta_{17} - 4 s_1^3 \beta_{16}, \quad \zeta_3 = s_1^4 \beta_{18} - 4 s_1^3 \beta_{17} + 6 s_1^2 \beta_{16}, \\
&\zeta_4 = s_1^4 \beta_{19} - 4 s_1^3 \beta_{18} + 6 s_1^2 \beta_{17} - 4 s_1 \beta_{16}, \quad \zeta_5 = -4 s_1^3 \beta_{19} + 6 s_1^2 \beta_{18} - 4 s_1 \beta_{17} + \beta_{16}, \\
&\zeta_6 = 6 s_1^2 \beta_{19} - 4 s_1 \beta_{18} + \beta_{17}, \quad \zeta_7 = -4 s_1 \beta_{19} + \beta_{18}, \quad \zeta_8 = \beta_{19}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\gamma_{26} &= -\gamma_1 \beta_{16} + \varepsilon_1, \quad \gamma_{27} = -\gamma_1 \beta_{17} - \gamma_2 \beta_{16} + \varepsilon_2, \quad \gamma_{28} = -\gamma_1 \beta_{18} - \gamma_2 \beta_{17} + \varepsilon_3, \\
\gamma_{29} &= -\gamma_1 \beta_{19} - \gamma_2 \beta_{18} + \varepsilon_4, \quad \gamma_{30} = -\gamma_2 \beta_{19} + \varepsilon_5, \quad \gamma_{31} = \gamma_1 \beta_{38} - \varepsilon_1 \beta_{16} + s_1^4 \gamma_6, \\
\gamma_{32} &= \gamma_1 \beta_{39} + \gamma_2 \beta_{38} - \varepsilon_2 \beta_{16} - s_1^2 \gamma_4 \beta_{17} + s_1^4 \gamma_7 - 4 s_1^3 \gamma_6, \\
\gamma_{33} &= \gamma_1 (\beta_{40} - \beta_{38}) + \gamma_2 \beta_{39} - \varepsilon_1 (\beta_{18} - 2 \beta_{16}) - \varepsilon_2 \beta_{17} - \varepsilon_3 \beta_{16} + s_1^4 \gamma_8 - 4 s_1^3 \gamma_7 + 6 s_1^2 \gamma_6, \\
\gamma_{34} &= \gamma_1 (\beta_{41} - \beta_{39}) + \gamma_2 (\beta_{40} - \beta_{38}) - \varepsilon_1 (\beta_{19} - 2 \beta_{17}) - \varepsilon_2 (\beta_{18} - 2 \beta_{16}) - \varepsilon_3 \beta_{17} - \varepsilon_4 \beta_{16} \\
&\quad + s_1^4 \gamma_9 - 4 s_1^3 \gamma_8 + 6 s_1^2 \gamma_7 - 4 s_1 \gamma_6, \\
\gamma_{35} &= \gamma_1 (\beta_{42} - \beta_{40}) + \gamma_2 (\beta_{41} - \beta_{39}) - \varepsilon_1 (-2 \beta_{18} + \beta_{16}) - \varepsilon_2 (\beta_{19} - 2 \beta_{17}) - \varepsilon_3 (\beta_{18} - 2 \beta_{16}) \\
&\quad - \varepsilon_4 \beta_{17} + s_1^4 \gamma_{10} - 4 s_1^3 \gamma_9 + 6 s_1^2 \gamma_8 - 4 s_1 \gamma_7 + \gamma_6, \\
\gamma_{36} &= \gamma_1 (\beta_{43} - \beta_{41}) + \gamma_2 (\beta_{42} - \beta_{40}) - \varepsilon_1 (-2 \beta_{19} + \beta_{17}) - \varepsilon_2 (-2 \beta_{18} + \beta_{16}) - \varepsilon_3 (\beta_{19} - 2 \beta_{17}) \\
&\quad - \varepsilon_4 (\beta_{18} - 2 \beta_{16}) - \varepsilon_5 \beta_{17} + s_1^4 \gamma_{11} - 4 s_1^3 \gamma_{10} + 6 s_1^2 \gamma_9 - 4 s_1 \gamma_8 + \gamma_7, \\
\gamma_{37} &= \gamma_1 (\beta_{44} - \beta_{42}) + \gamma_2 (\beta_{43} - \beta_{41}) - \varepsilon_1 \beta_{18} - \varepsilon_2 (-2 \beta_{19} + \beta_{17}) - \varepsilon_3 (-2 \beta_{18} + \beta_{16}) \\
&\quad - \varepsilon_4 (\beta_{19} - 2 \beta_{17}) - \varepsilon_5 (\beta_{18} - 2 \beta_{16}) + s_1^4 \gamma_{12} - 4 s_1^3 \gamma_{11} + 6 s_1^2 \gamma_{10} - 4 s_1 \gamma_9 + \gamma_8, \\
\gamma_{38} &= \gamma_1 (\beta_{45} - \beta_{43}) + \gamma_2 (\beta_{44} - \beta_{42}) - \varepsilon_1 \beta_{19} - \varepsilon_2 \beta_{18} - \varepsilon_3 (-2 \beta_{19} + \beta_{17}) - \varepsilon_4 (-2 \beta_{18} + \beta_{16}) \\
&\quad - \varepsilon_5 (\beta_{19} - 2 \beta_{17}) + s_1^4 \gamma_{13} - 4 s_1^3 \gamma_{12} + 6 s_1^2 \gamma_{11} - 4 s_1 \gamma_{10} + \gamma_9, \\
\gamma_{39} &= \gamma_1 (\beta_{46} - \beta_{44}) + \gamma_2 (\beta_{45} - \beta_{43}) - \varepsilon_2 \beta_{19} - \varepsilon_3 \beta_{18} - \varepsilon_4 (-2 \beta_{19} + \beta_{17}) - \varepsilon_5 (-2 \beta_{18} + \beta_{16}) \\
&\quad - 4 s_1^3 \gamma_{13} + 6 s_1^2 \gamma_{12} - 4 s_1 \gamma_{11} + \gamma_{10},
\end{aligned}
$$

$$\gamma_{40}=-\gamma_1\beta_{45}+\gamma_2(\beta_{46}-\beta_{44})-\varepsilon_3\beta_{19}-\varepsilon_4\beta_{18}-\varepsilon_5(-2\beta_{19}+\beta_{17})+6s_1^2\gamma_{13}-4s_1\gamma_{12}+\gamma_{11},$$

$$\gamma_{41}=-\gamma_1\beta_{46}-\gamma_2\beta_{45}-\varepsilon_4\beta_{19}-\varepsilon_5\beta_{18}-4s_1\gamma_{13}+\gamma_{12},\qquad \gamma_{42}=-\gamma_2\beta_{46}-\varepsilon_5\beta_{19}+\gamma_{13},$$

$$\gamma_{43}=\gamma_1\beta_{47}+\varepsilon_6\gamma_3-\gamma_6\zeta_1+s_1^6\gamma_{14},$$

$$\gamma_{44}=\gamma_1\beta_{48}+\gamma_2\beta_{47}+\gamma_3\varepsilon_7+\gamma_4\varepsilon_6-\gamma_6\zeta_2-\gamma_7\zeta_1+s_1^6\gamma_{15}-6s_1^5\gamma_{14},$$

$$\begin{aligned}\gamma_{45}=\;&\gamma_1(\beta_{49}-\beta_{47})+\gamma_2\beta_{48}+\gamma_3\varepsilon_8+\gamma_4\varepsilon_7+(\gamma_5-2\gamma_3)\varepsilon_6-\gamma_6\zeta_3-\gamma_7\zeta_2-(\gamma_8-\gamma_6)\zeta_1\\&+s_1^6\gamma_{16}-6s_1^5\gamma_{15}+15s_1^4\gamma_{14},\end{aligned}$$

$$\begin{aligned}\gamma_{46}=\;&\gamma_1(\beta_{50}-\beta_{48})+\gamma_2(\beta_{49}-\beta_{47})+\gamma_3\varepsilon_9+\gamma_4\varepsilon_8+(\gamma_5-2\gamma_3)\varepsilon_7-2\gamma_4\varepsilon_6-\gamma_6\zeta_4-\gamma_7\zeta_3\\&-(\gamma_8-\gamma_6)\zeta_2+(\gamma_9-\gamma_7)\zeta_1+s_1^6\gamma_{17}-6s_1^5\gamma_{16}+15s_1^4\gamma_{15}-20s_1^3\gamma_{14},\end{aligned}$$

$$\begin{aligned}\gamma_{47}=\;&\gamma_1(\beta_{51}-\beta_{49})+\gamma_2(\beta_{50}-\beta_{48})+\gamma_3\varepsilon_{10}+\gamma_4\varepsilon_9+(\gamma_5-2\gamma_3)\varepsilon_8-2\gamma_4\varepsilon_7-(2\gamma_5-\gamma_3)\varepsilon_6\\&-\gamma_6\zeta_5-\gamma_7\zeta_4-(\gamma_8-\gamma_6)\zeta_3-(\gamma_9-\gamma_7)\zeta_2-(\gamma_{10}-\gamma_8)\zeta_1+s_1^6\gamma_{18}-6s_1^5\gamma_{17}+15s_1^4\gamma_{16}\\&-20s_1^3\gamma_{15}+15s_1^2\gamma_{14},\end{aligned}$$

$$\begin{aligned}\gamma_{48}=\;&\gamma_1(\beta_{52}-\beta_{50})+\gamma_2(\beta_{51}-\beta_{49})+\gamma_3\varepsilon_{11}+\gamma_4\varepsilon_{10}+(\gamma_5-2\gamma_3)\varepsilon_9-2\gamma_4\varepsilon_8-(2\gamma_5-\gamma_3)\varepsilon_7\\&+\gamma_4\varepsilon_6-\gamma_6\zeta_6-\gamma_7\zeta_5-(\gamma_8-\gamma_6)\zeta_4-(\gamma_9-\gamma_7)\zeta_3-(\gamma_{10}-\gamma_8)\zeta_2-(\gamma_{11}-\gamma_9)\zeta_1\\&+s_1^6\gamma_{19}-6s_1^5\gamma_{18}+15s_1^4\gamma_{17}-20s_1^3\gamma_{16}+15s_1^2\gamma_{15}-6s_1\gamma_{14},\end{aligned}$$

$$\begin{aligned}\gamma_{49}=\;&\gamma_1(\beta_{53}-\beta_{51})+\gamma_2(\beta_{52}-\beta_{50})+\gamma_3\varepsilon_{12}+\gamma_4\varepsilon_{11}+(\gamma_5-2\gamma_3)\varepsilon_{10}-2\gamma_4\varepsilon_9-(2\gamma_5-\gamma_3)\varepsilon_8\\&+\gamma_4\varepsilon_7+\gamma_5\varepsilon_6-\gamma_6\zeta_7-\gamma_7\zeta_6-(\gamma_8-\gamma_6)\zeta_5-(\gamma_9-\gamma_7)\zeta_4-(\gamma_{10}-\gamma_8)\zeta_3-(\gamma_{11}-\gamma_9)\zeta_2\\&-(\gamma_{12}-\gamma_{10})\zeta_1+s_1^6\gamma_{20}-6s_1^5\gamma_{19}+15s_1^4\gamma_{18}-20s_1^3\gamma_{17}+15s_1^2\gamma_{16}-6s_1\gamma_{15}+\gamma_{14},\end{aligned}$$

$$\begin{aligned}\gamma_{50}=\;&\gamma_1(\beta_{54}-\beta_{52})+\gamma_2(\beta_{53}-\beta_{51})+\gamma_3\varepsilon_{13}+\gamma_4\varepsilon_{12}+(\gamma_5-2\gamma_3)\varepsilon_{11}-2\gamma_4\varepsilon_{10}-(2\gamma_5-\gamma_3)\varepsilon_9\\&+\gamma_4\varepsilon_8+\gamma_5\varepsilon_7-\gamma_6\zeta_8-\gamma_7\zeta_7-(\gamma_8-\gamma_6)\zeta_6-(\gamma_9-\gamma_7)\zeta_5-(\gamma_{10}-\gamma_8)\zeta_4-(\gamma_{11}-\gamma_9)\zeta_3\\&-(\gamma_{12}-\gamma_{10})\zeta_2-(\gamma_{13}-\gamma_{11})\zeta_1+s_1^6\gamma_{21}-6s_1^5\gamma_{20}+15s_1^4\gamma_{19}-20s_1^3\gamma_{18}+15s_1^2\gamma_{17}\\&-6s_1\gamma_{16}+\gamma_{15},\end{aligned}$$

$$\begin{aligned}\gamma_{51}=\;&\gamma_1(\beta_{55}-\beta_{53})+\gamma_2(\beta_{54}-\beta_{52})+\gamma_3\varepsilon_{14}+\gamma_4\varepsilon_{13}+(\gamma_5-2\gamma_3)\varepsilon_{12}-2\gamma_4\varepsilon_{11}-(2\gamma_5-\gamma_3)\varepsilon_{10}\\&+\gamma_4\varepsilon_9+\gamma_5\varepsilon_8-\gamma_7\zeta_8-(\gamma_8-\gamma_6)\zeta_7-(\gamma_9-\gamma_7)\zeta_6-(\gamma_{10}-\gamma_8)\zeta_5-(\gamma_{11}-\gamma_9)\zeta_4-(\gamma_{12}-\gamma_{10})\zeta_3\\&-(\gamma_{13}-\gamma_{11})\zeta_2+\gamma_{12}\zeta_1+s_1^6\gamma_{22}-6s_1^5\gamma_{21}+15s_1^4\gamma_{20}-20s_1^3\gamma_{19}+15s_1^2\gamma_{18}-6s_1\gamma_{17}+\gamma_{16},\end{aligned}$$

$$\begin{aligned}\gamma_{52}=\;&\gamma_1(\beta_{56}-\beta_{54})+\gamma_2(\beta_{55}-\beta_{53})+\gamma_3\varepsilon_{15}+\gamma_4\varepsilon_{14}+(\gamma_5-2\gamma_3)\varepsilon_{13}-2\gamma_4\varepsilon_{12}-(2\gamma_5-\gamma_3)\varepsilon_{11}\\&+\gamma_4\varepsilon_{10}+\gamma_5\varepsilon_9-(\gamma_8-\gamma_6)\zeta_8-(\gamma_9-\gamma_7)\zeta_7-(\gamma_{10}-\gamma_8)\zeta_6-(\gamma_{11}-\gamma_9)\zeta_5-(\gamma_{12}-\gamma_{10})\zeta_4\\&-(\gamma_{13}-\gamma_{11})\zeta_3+\gamma_{12}\zeta_2+\gamma_{13}\zeta_1+s_1^6\gamma_{23}-6s_1^5\gamma_{22}+15s_1^4\gamma_{21}-20s_1^3\gamma_{20}+15s_1^2\gamma_{19}\\&-6s_1\gamma_{18}+\gamma_{17},\end{aligned}$$

$$\begin{aligned}\gamma_{53}=\;&\gamma_1(\beta_{57}-\beta_{55})+\gamma_2(\beta_{56}-\beta_{54})+\gamma_3\varepsilon_{16}+\gamma_4\varepsilon_{15}+(\gamma_5-2\gamma_3)\varepsilon_{14}-2\gamma_4\varepsilon_{13}-(2\gamma_5-\gamma_3)\varepsilon_{12}\\&+\gamma_4\varepsilon_{11}+\gamma_6\varepsilon_{10}-(\gamma_9-\gamma_7)\zeta_8-(\gamma_{10}-\gamma_8)\zeta_7-(\gamma_{11}-\gamma_9)\zeta_6-(\gamma_{12}-\gamma_{10})\zeta_5-(\gamma_{13}-\gamma_{11})\zeta^4\\&+\gamma_{12}\zeta_3+\gamma_{13}\zeta_2+s_1^6\gamma_{24}-6s_1^5\gamma_{23}+15s_1^4\gamma_{22}-20s_1^3\gamma_{21}+15s_1^2\gamma_{20}-6s_1\gamma_{19}+\gamma_{18},\end{aligned}$$

$$\begin{aligned}\gamma_{54}=\;&\gamma_1(\beta_{58}-\beta_{56})+\gamma_2(\beta_{57}-\beta_{55})+\gamma_4\varepsilon_{16}+(\gamma_5-2\gamma_3)\varepsilon_{15}-2\gamma_4\varepsilon_{14}-(2\gamma_5-\gamma_3)\varepsilon_{13}\\&+\gamma_4\varepsilon_{12}+\gamma_5\varepsilon_{11}-(\gamma_{10}-\gamma_8)\zeta_8-(\gamma_{11}-\gamma_9)\zeta_7-(\gamma_{12}-\gamma_{10})\zeta_6-(\gamma_{13}-\gamma_{11})\zeta_5+\gamma_{12}\zeta_4\\&+\gamma_{13}\zeta_3-s_1^6\gamma_{25}-6s_1^5\gamma_{24}+15s_1^4\gamma_{23}-20s_1^3\gamma_{22}+15s_1^2\gamma_{21}-6s_1\gamma_{20}+\gamma_{19},\end{aligned}$$

$$\begin{aligned}\gamma_{55}=\;&\gamma_1(\beta_{59}-\beta_{57})+\gamma_2(\beta_{58}-\beta_{56})+(\gamma_5-2\gamma_3)\varepsilon_{16}-2\gamma_4\varepsilon_{15}-(2\gamma_5-\gamma_3)\varepsilon_{14}+\gamma_4\varepsilon_{13}+\gamma_5\varepsilon_{12}\\&-(\gamma_{11}-\gamma_9)\zeta_8-(\gamma_{12}-\gamma_{10})\zeta_7-(\gamma_{13}-\gamma_{11})\zeta_6+\gamma_{12}\zeta_5+\gamma_{13}\zeta_4-6s_1^5\gamma_{25}+15s_1^4\gamma_{24}-20s_1^3\gamma_{23}\\&+15s_1^2\gamma_{22}-6s_1\gamma_{21}+\gamma_{20},\end{aligned}$$

$$\begin{aligned}\gamma_{56}=\;&\gamma_1(\beta_{60}-\beta_{58})+\gamma_2(\beta_{59}-\beta_{57})-2\gamma_4\varepsilon_{16}-(2\gamma_5-\gamma_3)\varepsilon_{15}+\gamma_4\varepsilon_{14}+\gamma_5\varepsilon_{13}-(\gamma_{12}-\gamma_{10})\zeta_8\\&-(\gamma_{13}-\gamma_{11})\zeta_7+\gamma_{12}\zeta_6+\gamma_{13}\zeta_5+15s_1^4\gamma_{25}-20s_1^3\gamma_{24}+15s_1^2\gamma_{23}-6s_1\gamma_{22}+\gamma_{21},\end{aligned}$$

$$\begin{aligned}\gamma_{57}=\;&\gamma_1(\beta_{61}-\beta_{59})+\gamma_2(\beta_{60}-\beta_{58})-(2\gamma_5-\gamma_3)\varepsilon_{16}+\gamma_4\varepsilon_{15}+\gamma_5\varepsilon_{14}-(\gamma_{13}-\gamma_{11})\zeta_8+\gamma_{12}\zeta_7\\&+\gamma_{13}\zeta_6-20s_1^3\gamma_{25}+15s_1^2\gamma_{24}-6s_1\gamma_{23}+\gamma_{22},\end{aligned}$$

$$\gamma_{58}=-\gamma_1\beta_{60}+\gamma_2(\beta_{61}-\beta_{59})+\gamma_4\varepsilon_{16}+\gamma_5\varepsilon_{15}+\gamma_{12}\zeta_8+\gamma_{13}\zeta_7+15s_1^2\gamma_{25}-6s_1\gamma_{24}+\gamma_{23},$$

$$\gamma_{59}=-\gamma_1\beta_{61}-\gamma_2\beta_{60}+\gamma_5\varepsilon_{16}+\gamma_{13}\zeta_8-6s_1\gamma_{25}+\gamma_{24},\qquad \gamma_{60}=-\gamma_2\beta_{61}+\gamma_{25}.$$

In like manner

$$\frac{M'}{R'} = \frac{\gamma'_1 + \gamma'_2 C}{-s_1 - C} + \frac{\gamma'_{26} + \gamma'_{27} C + \ldots + \gamma'_{30} C^4}{(-s_1 - C)^3} K + \frac{\gamma'_{31} + \gamma'_{32} C + \ldots + \gamma'_{42} C^{11}}{(-s_1 - C)^5 (1 - C^2)^2} K^2 + \frac{\gamma'_{43} + \ldots + \gamma'_{60} C^{17}}{(-s_1 - C)^7 (1 - C^2)^3} K^3,$$

where the coefficients γ' are given by equations similar to those of the preceding group; it is necessary only to accent the γ and the β, change the signs of s_1, $\beta_{16}, \ldots \beta_{19}$ and the sign of the whole expression.

By multiplication of the factors we find

$$\begin{aligned}\frac{N}{R} = {} & \frac{1 - s_1 C + s_{16} \sqrt{1 - C^2}}{-s_1 + C} + \frac{\gamma_{61} + \ldots + \gamma_{66} C^5 + (\gamma_{67} + \ldots + \gamma_{71} C^4) \sqrt{1 - C^2}}{(-s_1 + C)^3 \sqrt{1 - C^2}} K \\ & + \frac{\gamma_{72} + \ldots + \gamma_{82} C^{10} + (\gamma_{83} + \ldots + \gamma_{92} C^9) \sqrt{1 - C^2}}{(-s_1 + C)^5 (1 - C^2)^{\frac{3}{2}}} K^2 \\ & + \frac{\gamma_{93} + \ldots + \gamma_{110} C^{17} + (\gamma_{111} + \ldots + \gamma_{127} C^{16}) \sqrt{1 - C^2}}{(-s_1 + C)^7 (1 - C^2)^{\frac{5}{2}}} K^3.\end{aligned}$$

To facilitate the expression of the coefficients we adopt the following

Subsidiary Quantities.

$\eta_1 = s_9$, $\eta_2 = s_{10}$, $\eta_3 = s_{11} - 2 s_9$, $\eta_4 = s_{12} - 2 s_{10}$, $\eta_5 = s_{13} - 2 s_{11} + s_9$, $\eta_6 = s_{14} - 2 s_{12} + s_{10}$,
$\eta_7 = s_{15} - 2 s_{13} + s_{11}$, $\eta_8 = -2 s_{14} + s_{12}$, $\eta_9 = -2 s_{15} + s_{13}$, $\eta_{10} = s_{14}$, $\eta_{11} = s_{15}$,
$\lambda_1 = t_{74}$, $\lambda_2 = t_{75}$, $\lambda_3 = t_{76} - 2 t_{74}$, $\lambda_4 = t_{77} - 2 t_{75}$, $\lambda_5 = t_{78} - 2 t_{76} + t_{74}$,
$\lambda_6 = t_{79} - 2 t_{77} + t_{75}$, $\lambda_7 = t_{80} - 2 t_{78} + t_{76}$, $\lambda_8 = -2 t_{79} + t_{77}$, $\lambda_9 = -2 t_{80} + t_{78}$,
$\lambda_{10} = t_{79}$, $\lambda_{11} = t_{80}$.

The values of the coefficients are then

$\gamma_{61} = -s_{16} \beta_{16} + s_1^2 t_{70}$, $\gamma_{62} = -s_{16} \beta_{17} + s_1^2 t_{71} - 2 s_1 t_{70}$,
$\gamma_{63} = -s_{16} (\beta_{18} - \beta_{16}) + s_1^2 t_{72} - 2 s_1 t_{71} + t_{70}$, $\gamma_{64} = -s_{16} (\beta_{19} - \beta_{17}) + s_1^2 t_{73} - 2 s_1 t_{72} + t_{71}$,
$\gamma_{65} = s_{16} \beta_{18} - 2 s_1 t_{73} + t_{72}$, $\gamma_{66} = s_{16} \beta_{19} + t_{73}$, $\gamma_{67} = -\beta_{16} - s_1^2 s_2$,
$\gamma_{68} = -\beta_{17} + s_1 \beta_{16} - s_1^2 s_3 + 2 s_1 s_2$, $\gamma_{69} = -\beta_{18} + s_1 \beta_{17} - s_1^2 s_4 + 2 s_1 s_3 - s_2$,
$\gamma_{70} = -\beta_{19} + s_1 \beta_{18} + 2 s_1 s_4 - s_3$, $\gamma_{71} = s_1 \beta_{19} - s_4$, $\gamma_{72} = s_{16} \beta_{38} - \beta_{16} t_{70} + s_1^4 t_{74}$,
$\gamma_{73} = s_{16} \beta_{39} - \beta_{16} t_{71} - \beta_{17} t_{70} + s_1^4 t_{75} - 4 s_1^3 t_{74}$,
$\gamma_{74} = s_{16} (\beta_{40} - \beta_{38}) - \beta_{16} t_{72} - \beta_{17} t_{71} - (\beta \;\; - \beta_{16}) t_{70} + s_1^4 t_{76} - 4 s_1^3 t_{75} + 6 s_1^2 t_{74}$,
$\gamma_{75} = s_{16} (\beta_{41} - \beta_{39}) - \beta_{16} t_{73} - \beta_{17} t_{72} - (\beta_{18} - \beta_{16}) t_{71} - (\beta_{19} - \beta_{17}) t_{70} + s_1^4 t_{77} - 4 s_1^3 t_{76}$
$\quad + 6 s_1^2 t_{75} - 4 s_1 t_{74}$,
$\gamma_{76} = s_{16} (\beta_{42} - \beta_{40}) - \beta_{17} t_{73} - (\beta_{18} - \beta_{16}) t_{72} - (\beta_{19} - \beta_{17}) t_{71} + \beta_{18} t_{70} + s_1^4 t_{78} - 4 s_1^3 t_{77}$
$\quad + 6 s_1^2 t_{76} - 4 s_1 t_{75} + t_{74}$,
$\gamma_{77} = s_{16} (\beta_{43} - \beta_{41}) - (\beta_{18} - \beta_{16}) t_{73} - (\beta_{19} - \beta_{17}) t_{72} + \beta_{18} t_{71} + \beta_{19} t_{70} + s_1^4 t_{79} - 4 s_1^3 t_{78}$
$\quad + 6 s_1^2 t_{77} - 4 s_1 t_{76} + t_{75}$,
$\gamma_{78} = s_{16} (\beta_{44} - \beta_{42}) - (\beta_{19} - \beta_{17}) t_{73} + \beta_{18} t_{72} + \beta_{19} t_{71} + s_1^4 t_{80} - 4 s_1^3 t_{79} + 6 s_1^2 t_{78} - 4 s_1 t_{77} + t_{76}$,
$\gamma_{79} = s_{16} (\beta_{45} - \beta_{43}) + \beta_{18} t_{73} + \beta_{19} t_{72} - 4 s_1^3 t_{80} + 6 s_1^2 t_{79} - 4 s_1 t_{78} + t_{77}$,
$\gamma_{80} = s_{16} (\beta_{46} - \beta_{44}) + \beta_{19} t_{73} + 6 s_1^2 t_{80} - 4 s_1 t_{79} + t_{78}$, $\gamma_{81} = -s_{16} \beta_{45} - 4 s_1 t_{80} + t_{79}$,
$\gamma_{82} = -s_{16} \beta_{46} + t_{80}$, $\gamma_{83} = \beta_{38} + s_1^2 s_2 \beta_{16} - s_1^4 s_5$,
$\gamma_{84} = \beta_{39} - s_1 \beta_{38} + s_1^2 s_2 \beta_{17} + (s_1^2 s_3 - 2 s_1 s_2) \beta_{16} - s_1^4 s_6 + 4 s_1^3 s_5$,

$$\gamma_{85} = \beta_{40} - s_1\beta_{39} + s_1^2 s_2(\beta_{18} - \beta_{16}) + (s_1^2 s_3 - 2s_1 s_2)\beta_{17} + (s_1^2 s_4 - 2s_1 s_3 + s_2)\beta_{16} - s_1^4(s_7 - s_5) + 4s_1^3 s_6 - 6s_1^2 s_5\,,$$

$$\gamma_{86} = \beta_{41} - s_1\beta_{40} + s_1^2 s_2(\beta_{19} - \beta_{17}) + (s_1^2 s_3 - 2s_1 s_2)(\beta_{18} - \beta_{16}) + (s_1^2 s_4 - 2s_1 s_3 + s_2)\beta_{17} + (-2s_1 s_4 + s_3)\beta_{16} - s_1^4(s_8 - s_6) + 4s_1^3(s_7 - s_5) - 6s_1^2 s_6 + 4s_1 s_5\,,$$

$$\gamma_{87} = \beta_{42} - s_1\beta_{41} - s_1^2 s_2\beta_{18} + (s_1^2 s_3 - 2s_1 s_2)(\beta_{19} - \beta_{17}) + (s_1^2 s_4 - 2s_1 s_3 + s_2)(\beta_{18} - \beta_{16}) + (-2s_1 s_4 + s_3)\beta_{17} + s_4\beta_{16} + s_1^4 s_7 + 4s_1^3(s_8 - s_6) - 6s_1^2(s_7 - s_5) + 4s_1 s_6 - s_5\,,$$

$$\gamma_{88} = \beta_{43} - s_1\beta_{42} - s_1^2 s_2\beta_{19} - (s_1^2 s_3 - 2s_1 s_2)\beta_{18} + (s_1^2 s_4 - 2s_1 s_3 + s_2)(\beta_{19} - \beta_{17}) + (-2s_1 s_4 + s_3)(\beta_{18} - \beta_{16}) + s_4\beta_{17} + s_1^4 s_8 - 4s_1^3 s_7 - 6s_1^2(s_8 - s_6) + 4s_1(s_7 - s_5) - s_6\,,$$

$$\gamma_{89} = \beta_{44} - s_1\beta_{43} - (s_1^2 s_3 - 2s_1 s_2)\beta_{19} - (s_1^2 s_4 - 2s_1 s_3 + s_2)\beta_{18} + (-2s_1 s_4 + s_3)(\beta_{19} - \beta_{17}) + s_4(\beta_{18} - \beta_{16}) - 4s_1^3 s_8 + 6s_1^2 s_7 + 4s_1(s_8 - s_6) - (s_7 - s_5)\,,$$

$$\gamma_{90} = \beta_{45} - s_1\beta_{44} - (s_1^2 s_4 - 2s_1 s_3 + s_2)\beta_{19} - (-2s_1 s_4 + s_3)\beta_{18} + s_4(\beta_{19} - \beta_{17}) + 6s_1^2 s_8 - 4s_1 s_7 - (s_8 - s_6)\,,$$

$$\gamma_{91} = \beta_{46} - s_1\beta_{45} - (-2s_1 s_4 + s_3)\beta_{19} - s_4\beta_{18} - 4s_1 s_8 + s_7\,, \qquad \gamma_{92} = -s_1\beta_{46} - s_4\beta_{19} + s_8\,,$$

$$\gamma_{93} = s_{16}\beta_{47} + \varepsilon_6 t_{70} - \zeta_1\lambda_1 + s_1^6 t_{81}\,,$$

$$\gamma_{94} = s_{16}\beta_{48} + \varepsilon_6 t_{71} + \varepsilon_7 t_{70} - \zeta_1\lambda_2 - \zeta_2 t_{74} + s_1^6 t_{82} - 6s_1^5 t_{81}\,,$$

$$\gamma_{95} = s_{16}(\beta_{49} - \beta_{47}) + \varepsilon_6(t_{72} - t_{70}) + \varepsilon_7 t_{71} + \varepsilon_8 t_{70} - \zeta_1\lambda_3 - \zeta_2\lambda_2 - \zeta_3\lambda_1 + s_1^6 t_{83} - 6s_1^5 t_{82} + 15s_1^4 t_{81}\,,$$

$$\gamma_{96} = s_{16}(\beta_{50} - \beta_{48}) + \varepsilon_6(t_{73} - t_{71}) + \varepsilon_7(t_{72} - t_{70}) + \varepsilon_8 t_{71} + \varepsilon_9 t_{70} - \zeta_1\lambda_4 - \zeta_2\lambda_3 - \zeta_3\lambda_2 - \zeta_4\lambda_1 + s_1^6 t_{84} - 6s_1^5 t_{83} + 15s_1^4 t_{82} - 20s_1^3 t_{81}\,,$$

$$\gamma_{97} = s_{16}(\beta_{51} - \beta_{49}) - \varepsilon_6 t_{72} + \varepsilon_7(t_{73} - t_{71}) + \varepsilon_8(t_{72} - t_{70}) + \varepsilon_9 t_{71} + \varepsilon_{10} t_{70} - \zeta_1\lambda_5 - \zeta_2\lambda_4 - \zeta_3\lambda_3 - \zeta_4\lambda_2 - \zeta_5\lambda_1 + s_1^6 t_{85} - 6s_1^5 t_{84} + 15s_1^4 t_{83} - 20s_1^3 t_{82} + 15s_1^2 t_{81}\,,$$

$$\gamma_{98} = s_{16}(\beta_{52} - \beta_{50}) - \varepsilon_6 t_{73} - \varepsilon_7 t_{72} + \varepsilon_8(t_{73} - t_{71}) + \varepsilon_9(t_{72} - t_{70}) + \varepsilon_{10} t_{71} + \varepsilon_{11} t_{70} - \zeta_1\lambda_6 - \zeta_2\lambda_5 - \zeta_3\lambda_4 - \zeta_4\lambda_3 - \zeta_5\lambda_2 - \zeta_6\lambda_1 + s_1^6 t_{86} - 6s_1^5 t_{85} + 15s_1^4 t_{84} - 20s_1^3 t_{83} + 15s_1^2 t_{82} - 6s_1 t_{86}\,,$$

$$\gamma_{99} = s_{16}(\beta_{53} - \beta_{51}) - \varepsilon_7 t_{73} - \varepsilon_8 t_{72} + \varepsilon_9(t_{73} - t_{71}) + \varepsilon_{10}(t_{72} - t_{70}) + \varepsilon_{11} t_{71} + \varepsilon_{12} t_{70} - \zeta_1\lambda_7\; \zeta_2\lambda_6 - \zeta_3\lambda_5 - \zeta_4\lambda_4 - \zeta_5\lambda_3 - \zeta_6\lambda_2 - \zeta_7\lambda_1 + s_1^6 t_{87} - 6s_1^5 t_{86} + 15s_1^4 t_{85} - 20s_1^3 t_{84} + 15s_1^2 t_{83} - 6s_1 t_{82} + t_{81}\,,$$

$$\gamma_{100} = s_{16}(\beta_{54} - \beta_{52}) - \varepsilon_8 t_{73} - \varepsilon_9 t_{72} + \varepsilon_{10}(t_{73} - t_{71}) + \varepsilon_{11}(t_{72} - t_{70}) + \varepsilon_{12} t_{71} + \varepsilon_{13} t_{70} - \zeta_1\lambda_8 - \zeta_2\lambda_7 - \zeta_3\lambda_6 - \zeta_4\lambda_5 - \zeta_5\lambda_4 - \zeta_6\lambda_3 - \zeta_7\lambda_2 - \zeta_8\lambda_1 + s_1^6 t_{88} - 6s_1^5 t_{87} + 15s_1^4 t_{86} - 20s_1^3 t_{85} + 15s_1^2 t_{84} - 6s_1 t_{83} + t_{82}\,,$$

$$\gamma_{101} = s_{16}(\beta_{55} - \beta_{53}) - \varepsilon_9 t_{73} - \varepsilon_{10} t_{72} + \varepsilon_{11}(t_{73} - t_{71}) + \varepsilon_{12}(t_{72} - t_{70}) + \varepsilon_{13} t_{71} + \varepsilon_{14} t_{70} - \zeta_1\lambda_9 - \zeta_2\lambda_8 - \zeta_3\lambda_7 - \zeta_4\lambda_6 - \zeta_5\lambda_5 - \zeta_6\lambda_4 - \zeta_7\lambda_3 - \zeta_8\lambda_2 + s_1^6 t_{89} - 6s_1^5 t_{88} + 15s_1^4 t_{87} - 20s_1^3 t_{86} + 15s_1^2 t_{85} - 6s_1 t_{84} + t_{83}\,,$$

$$\gamma_{102} = s_{16}(\beta_{56} - \beta_{54}) - \varepsilon_{10} t_{73} - \varepsilon_{11} t_{72} + \varepsilon_{12}(t_{73} - t_{71}) + \varepsilon_{13}(t_{72} - t_{70}) + \varepsilon_{14} t_{71} + \varepsilon_{15} t_{70} - \zeta_1\lambda_{10} - \zeta_2\lambda_9 - \zeta_3\lambda_8 - \zeta_4\lambda_7 - \zeta_5\lambda_6 - \zeta_6\lambda_5 - \zeta_7\lambda_4 - \zeta_8\lambda_3 + s_1^6 t_{90} - 6s_1^5 t_{89} + 15s_1^4 t_{88} - 20s_1^3 t_{87} + 15s_1^2 t_{86} - 6s_1 t_{85} + t_{84}\,,$$

$$\gamma_{103} = s_{16}(\beta_{57} - \beta_{55}) - \varepsilon_{11} t_{73} - \varepsilon_{12} t_{72} + \varepsilon_{13}(t_{73} - t_{71}) + \varepsilon_{14}(t_{72} - t_{70}) + \varepsilon_{15} t_{71} + \varepsilon_{16} t_{70} - \zeta_1\lambda_{11} - \zeta_2\lambda_{10} - \zeta_3\lambda_9 - \zeta_4\lambda_8 - \zeta_5\lambda_7 - \zeta_6\lambda_6 - \zeta_7\lambda_5 - \zeta_8\lambda_4 + s_1^6 t_{91} - 6s_1^5 t_{90} + 15s_1^4 t_{89} - 20s_1^3 t_{88} + 15s_1^2 t_{87} - 6s_1 t_{86} + t_{85}\,,$$

$$\gamma_{104} = s_{16}(\beta_{58} - \beta_{56}) - \varepsilon_{12} t_{73} - \varepsilon_{13} t_{72} - \varepsilon_{14}(t_{73} - t_{71}) - \varepsilon_{15}(t_{72} - t_{70}) - \varepsilon_{16} t_{71} - \zeta_2\lambda_{11} - \zeta_3\lambda_{10} - \zeta_4\lambda_9 - \zeta_5\lambda_8 - \zeta_6\lambda_7 - \zeta_7\lambda_6 - \zeta_8\lambda_5 - 6s_1^5 t_{91} + 15s_1^4 t_{90} - 20s_1^3 t_{89} + 15s_1^2 t_{88} - 6s_1 t_{87} + t_{86}\,,$$

$$\gamma_{105} = s_{16}(\beta_{59} - \beta_{57}) - \varepsilon_{13} t_{73} - \varepsilon_{14} t_{72} + \varepsilon_{15}(t_{73} - t_{71}) + \varepsilon_{16}(t_{72} - t_{70}) - \zeta_3\lambda_{11} - \zeta_4\lambda_{10} - \zeta_5\lambda_9 - \zeta_6\lambda_8 - \zeta_7\lambda_7 - \zeta_8\lambda_6 + 15s_1^4 t_{91} - 20s_1^3 t_{90} + 15s_1^2 t_{89} - 6s_1 t_{88} + t_{87}\,,$$

$$\gamma_{106} = s_{16}(\beta_{60} - \beta_{58}) - \varepsilon_{14} t_{73} - \varepsilon_{15} t_{72} + \varepsilon_{16}(t_{73} - t_{71}) - \zeta_4 \lambda_{11} - \zeta_5 \lambda_{10} - \zeta_6 \lambda_9 - \zeta_7 \lambda_8 - \zeta_8 \lambda_7$$
$$- 20 s_1^3 t_{91} + 15 s_1^2 t_{90} - 6 s_1 t_{89} + t_{88},$$

$$\gamma_{107} = s_{16}(\beta_{61} - \beta_{59}) - \varepsilon_{15} t_{73} - \varepsilon_{16} t_{72} - \zeta_5 \lambda_{11} - \zeta_6 \lambda_{10} - \zeta_7 \lambda_9 - \zeta_8 \lambda_8 + 15 s_1^2 t_{91} - 6 s_1 t_{90} + t_{89},$$

$$\gamma_{108} = - s_{16} \beta_{60} - \varepsilon_{16} t_{73} - \zeta_6 \lambda_{11} - \zeta_7 \lambda_{10} - \zeta_8 \lambda_9 - 6 s_1 t_{91} + t_{90},$$

$$\gamma_{109} = - s_{16} \beta_{61} - \zeta_7 \lambda_{11} - \zeta_8 \lambda_{10} + t_{91}, \qquad \gamma_{110} = - \zeta_8 \lambda_{11}, \qquad \gamma_{111} = \beta_{47} - \varepsilon_6 s_2 + \zeta_1 s_5 - s_1^6 \eta_1,$$

$$\gamma_{112} = \beta_{48} - s_1 \beta_{47} - \varepsilon_6 s_3 - \varepsilon_7 s_2 + \zeta_1 s_6 + \zeta_2 s_5 - s_1^6 \eta_2 - 6 s_1^5 \eta_1,$$

$$\gamma_{113} = \beta_{49} - s_1 \beta_{48} - \varepsilon_6 (s_4 - s_2) - \varepsilon_7 s_3 - \varepsilon_8 s_2 + \zeta_1 (s_7 - 2 s_5) + \zeta_2 s_6 + \zeta_3 s_5 - s_1^6 (s_{11} - s_9)$$
$$+ 6 s_1^5 s_{10} - 15 s_1^4 s_9,$$

$$\gamma_{114} = \beta_{50} - s_1 \beta_{49} + \varepsilon_6 s_3 - \varepsilon_7 (s_4 - s_2) - \varepsilon_8 s_3 - \varepsilon_9 s_2 + \zeta_1 (s_8 - 2 s_6) + \zeta_2 (s_7 - 2 s_5) + \zeta_3 s_6$$
$$+ \zeta_4 s_5 - s_1^6 \eta_4 + 6 s_1^5 \eta_3 - 15 s_1^4 \eta_2 + 20 s_1^3 \eta_1,$$

$$\gamma_{115} = \beta_{51} - s_1 \beta_{50} - \varepsilon_6 (-2 s_4 + s_2) + 2 \varepsilon_7 s_3 - \varepsilon_8 (s_4 - 2 s_2) - \varepsilon_9 s_3 - \varepsilon_{10} s_2 + \zeta_1 (-2 s_7 + s_5)$$
$$+ \zeta_2 (s_8 - 2 s_6) + \zeta_3 (s_7 - 2 s_5) + \zeta_4 s_6 + \zeta_5 s_5 - s_1^6 \eta_5 + 6 s_1^5 \eta_4 - 15 s_1^4 \eta_3 + 20 s_1^3 \eta_2 - 15 s_1^2 \eta_1,$$

$$\gamma_{116} = \beta_{52} - s_1 \beta_{51} - \varepsilon_6 s_3 - \varepsilon_7 (-2 s_4 + s_2) + 2 \varepsilon_8 s_3 - \varepsilon_9 (s_4 - 2 s_2) - \varepsilon_{10} s_3 - \varepsilon_{11} s_2$$
$$+ \zeta_1 (-2 s_8 + s_6) + \zeta_2 (-2 s_7 + s_5) + \zeta_3 (s_8 - 2 s_6) + \zeta_4 (s_7 - 2 s_5) + \zeta_5 s_6 + \zeta_6 s_5$$
$$- s_1^6 \eta_6 + 6 s_1^5 \eta_5 - 15 s_1^4 \eta_4 + 20 s_1^3 \eta_3 - 15 s_1^2 \eta_2 + 6 s_1 \eta_1,$$

$$\gamma_{117} = \beta_{53} - s_1 \beta_{52} - \varepsilon_6 s_4 - \varepsilon_7 s_3 - \varepsilon_8 (-2 s_4 + s_2) + 2 \varepsilon_9 s_3 - \varepsilon_{10} (s_4 - 2 s_2) - \varepsilon_{11} s_3 - \varepsilon_{12} s_2$$
$$+ \zeta_1 s_7 + \zeta_2 (-2 s_8 + s_6) + \zeta_3 (-2 s_7 + s_5) + \zeta_4 (s_8 - 2 s_6) + \zeta_5 (s_7 - 2 s_5) + \zeta_6 s_6 + \zeta_7 s_5$$
$$- s_1^6 \eta_7 + 6 s_1^5 \eta_6 - 15 s_1^4 \eta_5 + 20 s_1^3 \eta_4 - 15 s_1^2 \eta_3 + 6 s_1 \eta_2 - \eta_1,$$

$$\gamma_{118} = \beta_{54} - s_1 \beta_{53} - \varepsilon_7 s_4 - \varepsilon_8 s_3 - \varepsilon_9 (-2 s_4 + s_2) + 2 \varepsilon_{10} s_3 - \varepsilon_{11} (s_4 - 2 s_2) - \varepsilon_{12} s_3 - \varepsilon_{13} s_2$$
$$+ \zeta_1 s_8 + \zeta_2 s_7 + \zeta_3 (-2 s_8 + s_6) + \zeta_4 (-2 s_7 + s_5) + \zeta_5 (s_8 - 2 s_6) + \zeta_6 (s_7 - 2 s_5)$$
$$+ \zeta_7 s_6 + \zeta_8 s_5 - s_1^6 \eta_8 + 6 s_1^5 \eta_7 - 15 s_1^4 \eta_6 + 20 s_1^3 \eta_5 - 15 s_1^3 \eta_4 + 6 s_1 \eta_3 - \eta_2,$$

$$\gamma_{119} = \beta_{55} - s_1 \beta_{54} - \varepsilon_8 s_4 - \varepsilon_9 s_3 - \varepsilon_{10} (-2 s_4 + s_2) + 2 \varepsilon_{11} s_3 - \varepsilon_{12} (s_4 - 2 s_2) - \varepsilon_{13} s_3 - \varepsilon_{14} s_2$$
$$+ \zeta_2 s_8 + \zeta_3 s_7 + \zeta_4 (-2 s_8 + s_6) + \zeta_5 (-2 s_7 + s_5) + \zeta_6 (s_8 - 2 s_6) + \zeta_7 (s_7 - 2 s_5)$$
$$+ \zeta_8 s_6 - s_1^6 \eta_9 + 6 s_1^5 \eta_8 - 15 s_1^4 \eta_7 + 20 s_1^3 \eta_6 - 15 s_1^2 \eta_5 + 6 s_1 \eta_4 - \eta_3,$$

$$\gamma_{120} = \beta_{56} - s_1 \beta_{55} - \varepsilon_9 s_4 - \varepsilon_{10} s_3 - \varepsilon_{11} (-2 s_4 + s_2) + 2 \varepsilon_{12} s_3 - \varepsilon_{13} (s_4 - 2 s_2) - \varepsilon_{14} s_3 - \varepsilon_{15} s_2$$
$$+ \zeta_3 s_8 + \zeta_4 s_7 + \zeta_5 (-2 s_8 + s_6) + \zeta_6 (-2 s_7 + s_5) + \zeta_7 (s_8 - 2 s_6) + \zeta_8 (s_7 - 2 s_5)$$
$$- s_1^6 \eta_{10} + 6 s_1^5 \eta_9 - 15 s_1^4 \eta_8 + 20 s_1^3 \eta_7 - 15 s_1^2 \eta_6 + 6 s_1 \eta_5 - \eta_4,$$

$$\gamma_{121} = \beta_{57} - s_1 \beta_{56} - \varepsilon_{10} s_4 - \varepsilon_{11} s_3 - \varepsilon_{12} (-2 s_4 + s_2) + 2 \varepsilon_{13} s_3 - \varepsilon_{14} (s_4 - 2 s_2) - \varepsilon_{15} s_3 - \varepsilon_{16} s_2$$
$$+ \zeta_4 s_8 + \zeta_5 s_7 + \zeta_6 (-2 s_8 + s_6) + \zeta_7 (-2 s_7 + s_5) + \zeta_8 (s_8 - 2 s_6) - s_1^6 \eta_{11} + 6 s_1^5 \eta_{10}$$
$$- 15 s_1^4 \eta_9 + 20 s_1^3 \eta_8 - 15 s_1^2 \eta_7 + 6 s_1 \eta_6 - \eta_5,$$

$$\gamma_{122} = \beta_{58} - s_1 \beta_{57} - \varepsilon_{11} s_4 - \varepsilon_{12} s_3 - \varepsilon_{13} (-2 s_4 + s_2) + 2 \varepsilon_{14} s_3 - \varepsilon_{15} (s_4 - 2 s_2) - \varepsilon_{16} s_3 + \zeta_5 s_8$$
$$+ \zeta_6 s_7 + \zeta_7 (-2 s_8 + s_6) + \zeta_8 (-2 s_7 + s_5) + 6 s_1^5 \eta_{11} - 15 s_1^4 \eta_{10} + 20 s_1^3 \eta_9$$
$$- 15 s_1^2 \eta_8 + 6 s_1 \eta_7 + \eta_6,$$

$$\gamma_{123} = \beta_{59} - s_1 \beta_{58} - \varepsilon_{12} s_4 - \varepsilon_{13} s_3 - \varepsilon_{14} (-2 s_4 + s_2) + 2 \varepsilon_{15} s_3 - \varepsilon_{16} (s_4 - 2 s_2) + \zeta_6 s_8 + \zeta_7 s^7$$
$$+ \zeta_8 (-2 s_8 + s_6) - 15 s_1^4 \eta_{11} + 20 s_1^3 \eta_{10} - 15 s_1^2 \eta_9 + 6 s_1 \eta_8 - \eta_7,$$

$$\gamma_{124} = \beta_{60} - s_1 \beta_{59} - \varepsilon_{13} s_4 - \varepsilon_{14} s_3 - \varepsilon_{15} (-2 s_4 + s_2) + 2 \varepsilon_{16} s_3 + \zeta_7 s_8 + \zeta_8 s_7 + 20 s_1^3 \eta_{11}$$
$$- 15 s_1^2 \eta_{10} + 6 s_1 \eta_9 - \eta_8,$$

$$\gamma_{125} = \beta_{61} - s_1 \beta_{60} - \varepsilon_{14} s_4 - \varepsilon_{15} s_3 - \varepsilon_{16} (-2 s_4 + s_2) + \zeta_8 s_8 - 15 s_1^2 \eta_{11} + 6 s_1 \eta_{10} - \eta_9,$$

$$\gamma_{126} = - s_1 \beta_{61} - \varepsilon_{15} s_4 - \varepsilon_{16} s_3 + 6 s_1 \eta_{11} - \eta_{10}, \qquad \gamma_{127} = - \varepsilon_{16} s_4 - \eta_{11}.$$

Similarly we have

$$\frac{N'}{R'}=\frac{1+s_1 C-s_{16}\sqrt{1-C'^2}}{-s_1-C}+\frac{\gamma'_{61}+\ldots+\gamma'_{66}C^5+(\gamma'_{67}+\ldots+\gamma'_{71}C^4)\sqrt{1-C^2}}{(-s_1-C)^3\sqrt{1-C^2}}K$$
$$+\frac{\gamma'_{72}+\ldots+\gamma'_{82}C^{10}+(\gamma'_{83}+\ldots+\gamma'_{92}C^9)\sqrt{1-C^2}}{(-s_1+C)^5(1-C^2)^{\frac{3}{2}}}K^2$$
$$+\frac{\gamma'_{93}+\ldots+\gamma'_{110}C^{17}+(\gamma'_{111}+\ldots+\gamma'_{127}C^{16})\sqrt{1-C^2}}{(-s_1-C)^7(1-C^2)^{\frac{5}{2}}}K^3.$$

where the coefficients γ' are given by equations similar to those of the preceding group; it is only necessary to accent the β, change the signs of the t and augment their subscripts by 22, change the signs of s_1, s_{16}, $\beta_{16}, \ldots \beta_{19}$ and finally the sign of the whole expression.

We have now fulfilled the task of integrating the differential equations of Case III. It must be mentioned that the arbitrary constants C and K are not the conjugates of f and f'. If new terms are added to W the differential equations determining C, K, f, f' are (using the Poisson symbols)

$$\frac{dC}{d\phi}=[C,f]\frac{\partial W}{\partial f}+[C,f']\frac{\partial W}{\partial f'},$$
$$\frac{dK}{d\phi}=[K,f]\frac{\partial W}{\partial f}+[K,f']\frac{\partial W}{\partial f'},$$
$$\frac{df}{d\phi}=-[C,f]\frac{\partial W}{\partial C}-[K,f]\frac{\partial W}{\partial K},$$
$$\frac{df'}{d\phi}=-[C,f']\frac{\partial W}{\partial C}-[K,f']\frac{\partial W}{\partial K}.$$

MEMOIR No. 81.

Remarks Supplementary to Memoir No. 79.

(This memoir appears here for the first time.)

The Protometers of Jupiter and Saturn.

Further reflection has led me to the conclusion that the values of the protometers of Jupiter and Saturn given in Memoir No. 79 are not the best. The moduli e_0 and e'_0 of the departure of the orbits from circularity having evidently but little influence on the average values of the radii r and r', it is thence suggested that the well known periodic solution, where the planets always cross the line of syzygies at right angles, will afford us satisfactory values for the protometers.

Provided we agree to neglect quantities of the order of the squares and products of the planetary masses, the treatment of this question is exceedingly simple, as no integrations are required.

It is plain that the terms of the potential function Ω involving the variable ϕ have an influence on the mean values of r and r' only to the extent of quantities of two dimensions with respect to planetary masses. Hence they may be set aside. Thus we may put

$$\Omega = \frac{Mm}{r} + \frac{Mm'}{r'} + \frac{1}{2}\,\frac{mm'}{1-\varkappa}\,\frac{1}{r'}\,b_{\frac{1}{2}}^{(0)},$$

where $b_{\frac{1}{2}}^{(0)}$ denotes the same function of $\frac{r}{r'}$ that Laplace's $b_{\frac{1}{2}}^{(0)}$ is of α. With this value of the potential function it is plain the planets may move uniformly about the Sun in circles, and our task is only to discover the values of the constant radii. As, in this case, $s=0$, $s'=0$, and T involves the squares of these quantities, we may put $s=0$, $s'=0$ in T and, making q and q' the variables conjugate to v and v', thus have

$$T = \frac{1}{2\mathrm{m}}\,\frac{q^2}{r^2} + \frac{1}{2\mathrm{m}'}\,\frac{q'^2}{r'^2}.$$

Making $F=\Omega - T$, we have six equations for the problem, viz.,

$$0=\frac{\partial F}{\partial r},\quad 0=\frac{\partial F}{\partial r'},\quad \frac{dq}{dt}=0,\quad \frac{dq'}{dt}=0,\quad \frac{dv}{dt}=\frac{1}{\mathrm{m}}\,\frac{q}{r^2},\quad \frac{dv'}{dt}=\frac{1}{\mathrm{m}'}\,\frac{q'}{r'^2}.$$

With regard to the last two equations the observations furnish us with the values

$$n = \frac{1}{\mathrm{m}}\frac{q}{r^2}, \quad n' = \frac{1}{\mathrm{m}'}\frac{q'}{r'^2}.$$

The first and second equations are

$$\frac{\partial F}{\partial r} = -\frac{Mm}{r^2} + \frac{1}{2}\frac{mm'}{1-x}\frac{1}{r'}\frac{db_{\frac{1}{2}}^{(0)}}{dr} + \frac{1}{\mathrm{m}}\frac{q^2}{r^3} = 0,$$

$$\frac{\partial F}{\partial r'} = -\frac{Mm'}{r'^2} + \frac{1}{2}\frac{mm'}{1-x}\left(-\frac{1}{r'^2}b_{\frac{1}{2}}^{(0)} + \frac{1}{r'}\frac{db_{\frac{1}{2}}^{(0)}}{dr'}\right) + \frac{1}{\mathrm{m}'}\frac{q'^2}{r'^3} = 0.$$

Eliminating q and q' from these by means of the preceding equations we have

$$-\frac{Mm}{r^2} + \frac{1}{2}\frac{mm'}{1-x}\frac{1}{r'}\frac{db_{\frac{1}{2}}^{(0)}}{dr} + \mathrm{m}n^2 r = 0,$$

$$-\frac{Mm'}{r'^2} + \frac{1}{2}\frac{mm'}{1-x}\left(-\frac{1}{r'^2}b_{\frac{1}{2}}^{(0)} + \frac{1}{r'}\frac{db_{\frac{1}{2}}^{(0)}}{dr'}\right) + \mathrm{m}'n'^2 r' = 0.$$

Thus the values of the protometers satisfy the equations

$$\mathrm{m}\, n^2 a^3 = Mm - \frac{1}{2}\frac{mm'}{1-x}\alpha^2\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha},$$

$$\mathrm{m}' n'^2 a'^3 = Mm' + \frac{1}{2}\frac{mm'}{1-x}\left(b_{\frac{1}{2}}^{(0)} + \alpha\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right).$$

We assume $\log\alpha = 9.73655498$, which employed in the preceding equations leads to

$$\log a = 0.7162325165, \quad \log a' = 0.9796775356.$$

Modifications produced in the values of the protometers by the inclusion of terms of two dimensions with respect to planetary masses.

It seems so important to have the most serviceable values for the protometers that we do not hesitate to determine the corrections arising from considering the terms of the second order with respect to planetary masses.

Here it is more convenient to employ the four differential equations of the second order. For brevity we will put

$$\frac{dv}{dt} = \gamma, \quad \frac{dv'}{dt} = \gamma'.$$

Then these equations may be written

$$\frac{d\left[(\gamma-\gamma')\frac{dr}{d\phi}\right]}{d\phi} + \frac{\frac{\mu}{r^2} - \gamma^2 r}{\gamma-\gamma'} = \frac{1}{\mathrm{m}(\gamma-\gamma')}\frac{\partial R}{\partial r},$$

$$\frac{d\left[(\gamma-\gamma')\frac{dr'}{d\phi}\right]}{d\phi} + \frac{\frac{\mu'}{r'^2} - \gamma'^2 r'}{\gamma-\gamma'} = -\frac{1}{\mathrm{m}'(\gamma-\gamma')}\frac{1}{r'}\left[r\frac{\partial R}{\partial r} + R\right],$$

$$\frac{d(\gamma r^2)}{d\phi} = \frac{1}{\mathrm{m}(\gamma-\gamma')}\frac{\partial R}{\partial\phi},$$

$$\frac{d(\gamma' r'^2)}{d\phi} = -\frac{1}{\mathrm{m}'(\gamma-\gamma')}\frac{\partial R}{\partial\phi}.$$

Since the first terms of the first members of the first and second equations can have no non-periodic terms, it follows that the two functions

$$\frac{\dfrac{\mu}{r^2}-\gamma^2 r-\dfrac{1}{\mathrm{m}}\dfrac{\partial R}{\partial r}}{\gamma-\gamma'} \quad \text{and} \quad \frac{\dfrac{\mu'}{r'^2}-\gamma'^2 r'+\dfrac{1}{\mathrm{m}' r'}\left[r\dfrac{\partial R}{\partial r}+R\right]}{\gamma-\gamma'}$$

have no non-periodic terms. These conditions determine the protometers. Let us put

$$r=a(1+u),\ r'=a'(1+u'),\ \gamma=n(1+s),\ \gamma'=n'(1+s'),$$

it being understood that u, u', s, s', are to have no non-periodic terms. Before we can reduce the two foregoing expressions to a manageable form, it is necessary to know the values of u, u', s, s' correct to quantities of the first order. In this case, the third and fourth of the differential equations reduce to

$$\frac{d(s+2u)}{d\phi}=\frac{n}{n-n'}\frac{a}{\mu \mathrm{m}}\frac{\partial R}{\partial \phi},$$

$$\frac{d(s'+2u')}{d\phi}=-\frac{n'}{n-n'}\frac{a'}{\mu' \mathrm{m}'}\frac{\partial R}{\partial \phi}.$$

They are immediately integrable and give

$$s+2u=\frac{n}{n-n'}\frac{a}{\mu \mathrm{m}}R_0,$$

$$s'+2u'=-\frac{n'}{n-n'}\frac{a'}{\mu' \mathrm{m}'}R_0,$$

where R_0 denotes R divested of its non-periodic term. Eliminating s and s' by means of these equations, the first and second of the differential equations become, to a like degree of approximation,

$$\frac{d^2 u}{d\phi^2}+\frac{n^2}{(n-n')^2}u=\frac{n^2}{(n-n')^2}\frac{1}{\mu \mathrm{m}}\left[a^2\frac{\partial R_0}{\partial a}+\frac{2n}{n-n'}aR_0\right]$$

$$\frac{d^2 u'}{d\phi^2}+\frac{n'^2}{(n-n')^2}u'=-\frac{n'^2}{(n-n')^2}\frac{1}{\mu' \mathrm{m}'}\frac{a'}{a}\left[a^2\frac{\partial R_0}{\partial a}+\frac{n+n'}{n-n'}aR_0\right].$$

These linear differential equations with known final terms are immediately integrable. Put

$$\xi=-\frac{n}{n-n'}s+\frac{n'}{n-n'}s',$$

then the functions

$$\left[\frac{\mu}{r^2}-\gamma^2 r-\frac{1}{\mathrm{m}}\frac{\partial R}{\partial r}\right](1+\xi) \text{ and } \left[\frac{\mu'}{r'^2}-\gamma'^2 r'+\frac{1}{\mathrm{m}' r'}\left(r\frac{\partial R}{\partial r}+R\right)\right](1+\xi)$$

should have no non-periodic terms. Developing the indicated multiplications

and throwing out the terms which have no non-periodic parts, the first function becomes

$$\frac{\mu}{a^2} - an^2 - \frac{an^2}{\mu \mathrm{m}} a^2 \frac{\partial R}{\partial a} + an^2 (3u^2 - 2us - s^2) - an^2 \left[3u + 2s + \frac{a^2}{\mu \mathrm{m}} \frac{\partial R_0}{\partial a} \right] \xi$$
$$- \frac{an^2}{\mu \mathrm{m}} a^3 \frac{\partial^2 R_0}{\partial a^2} + u \frac{an^2}{\mu \mathrm{m}} \left[a^3 \frac{\partial^2 R_0}{\partial a^2} + a^2 \frac{\partial R_0}{\partial a} \right] u',$$

and the second

$$\frac{\mu'}{a'^2} - a'n'^2 + \frac{a'^2 n'^2}{\mu' \mathrm{m}'} \frac{a'}{a} \left[a^2 \frac{\partial R}{\partial a} + aR \right] + a'n'^2 (3u'^2 - 2u's' - s'^2)$$
$$- a'n'^2 \left[3u' + 2s' - \frac{1}{\mu' \mathrm{m}'} \frac{a'}{a} \left(a^2 \frac{\partial R_0}{\partial a} + aR_0 \right) \right] \xi$$
$$+ \frac{a'n'^2}{\mu' \mathrm{m}'} \frac{a'}{a} \left[a^3 \frac{\partial^2 R_0}{\partial a^2} + 2a^2 \frac{\partial R_0}{\partial a} \right] u - \frac{a'n'^2}{\mu' \mathrm{m}'} \frac{a'}{a} \left[a^3 \frac{\partial^2 R_0}{\partial a^2} + 4a^2 \frac{\partial R_0}{\partial a} + 2aR_0 \right] u'.$$

The form of the function R is

$$R = \tfrac{1}{2} \frac{mm'}{1-x} \frac{1}{a'} b_{\frac{1}{2}}^{(0)} + \tfrac{1}{4} x^2 M \left(m \frac{a'^2}{a^3} + m' \frac{a^2}{a'^3} \right)$$
$$+ \left[\frac{mm'}{1-x} \frac{1}{a'} b_{\frac{1}{2}}^{(1)} - xM \left(m \frac{a'}{a^2} + m' \frac{a}{a'^2} \right) \right] \cos \phi$$
$$+ \frac{mm'}{1-x} \frac{1}{a'} \sum_{i=2}^{i=\infty} b_{\frac{1}{2}}^{(i)} \cos i\phi.$$

In the numerical elaboration the data of Memoir No. 79 are used. We begin with the argument $\log \alpha = 9.73655$. In the following table the common logarithms of the coefficients are given:

$i.$	$b_{\frac{1}{2}}^{(i)}.$	$\alpha \dfrac{db_{\frac{1}{2}}^{(i)}}{d\alpha}.$	$\alpha^2 \dfrac{d^2 b_{\frac{1}{2}}^{(i)}}{d\alpha^2}.$	$\dfrac{aR_0}{\mu \mathrm{m}}.$	$\dfrac{a^2 \frac{\partial R_0}{\partial a}}{\mu \mathrm{m}}.$	$\dfrac{a^3 \frac{\partial^2 R_0}{\partial a^2}}{\mu \mathrm{m}}.$
0	0.33848	9.64420	9.93156			
1	9.79272	9.90764	9.87985	6.51314n	6.96019	7.36175n
2	9.41079	9.77981	0.01959	5.60317	5.79219	6.21197
3	9.07149	9.59756	0.02114	5.26387	5.78994	6.21352
4	8.75208	9.39254	9.94935	4.94446	5.58492	6.14173
5	8.4442	9.1749	9.8351	4.63663	5.3672	6.0275
6	8.1440	8.9491	9.6935	4.3364	5.1415	5.8859
7	7.8491	8.7177	9.5329	4.0415	4.9101	5.7253
8	7.5582	8.4821	9.3585	3.7506	4.6745	5.5509

$i.$	$u.$	$u'.$	$s.$	$s'.$	$\xi.$
1	6.44438n	6.87524	5.03128	6.18861n	6.08662n
2	6.72751n	6.16915	7.05502	6.66359n	7.34459n
3	5.74575n	5.53323	6.15263	6.15868n	6.52513n
4	5.15910n	5.02485	5.63930	5.75984n	6.04822n
5	4.6787 n	4.5794	5.2252	5.4062 n	5.6561 n
6	4.2511 n	4.1715	4.8572	5.0766 n	5.3031 n
7	3.8554 n	3.7885	4.5153	4.7614 n	4.9720 n
8	3.4812 n	3.4233	4.1899	4.4568 n	4.6554 n

The parts of the non-periodic terms of the two functions mentioned above may be set down as follows:

Non-periodic term of	$3u^2 - 2us - s^2$	$= + 0.00000\ 05110$
" "	$-\left[3u + 2s + \frac{a^2}{\mu \mathrm{m}}\frac{\partial R_0}{\partial a}\right]\xi$	$= + 0.00000\ 08837$
" "	$-\frac{a^3}{\mu \mathrm{m}}\frac{\partial^2 R_0}{\partial a^2}\,u$	$= - 0.00000\ 02705$
" "	$\frac{1}{\mu \mathrm{m}}\left[a^3\frac{\partial^2 R_0}{\partial a^2} + a^2\frac{\partial R_0}{\partial a}\right]u'$	$= - 0.00000\ 04966$
	Sum	$= + 0.00000\ 06276$
" "	$3u'^2 - 2u's' - s'^2$	$= + 0.00000\ 09378$
" "	$-\left[3u' + 2s' - \frac{1}{\mu' \mathrm{m}'}\frac{a'}{a}\left(a^2\frac{\partial R_0}{\partial a} + aR_0\right)\right]\xi$	$= - 0.00000\ 16774$
" "	$\frac{1}{\mu' \mathrm{m}'}\frac{a'}{a}\left[a^3\frac{\partial^2 R_0}{\partial a^2} + 2a^2\frac{\partial R_0}{\partial a}\right]u$	$= - 0.00000\ 02296$
" "	$-\frac{1}{\mu' \mathrm{m}'}\frac{a'}{a}\left[a^3\frac{\partial^2 R_0}{\partial a^2} + 4a^2\frac{\partial R_0}{\partial a} + 2aR_0\right]u'$	$= - 0.00000\ 19428$
	Sum	$= - 0.00000\ 29120$

The small terms arising from the non-periodic portion of R_0, viz.,

$$\tfrac{1}{4}\,x^2 M\left(m\,\frac{a'^2}{a^3} + m'\,\frac{a^2}{a'^3}\right)$$

are still to be applied to the quantities $\mu\mathrm{m}$ and $\mu'\mathrm{m}'$, in consequence of which the logarithm of the first is augmented by 0.00000 00524, and the logarithm of the second is diminished by 0.00000 02114. Hence we have the equations

$$[8.02387\ 99252]a^3 = [0.17259\ 27054] - [6.6284215]\,\tfrac{1}{2}\alpha^2\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha},$$

$$[6.70993\ 18580]a'^3 = [9.64841\ 98186] + [6.6284215]\,\tfrac{1}{2}\left(b_{\frac{1}{2}}^{(0)} + \alpha\frac{db_{\frac{1}{2}}^{(0)}}{d\alpha}\right).$$

We assume that $\log\alpha = 9.73655558$, and thus

$$\log a = 0.71623\ 26249, \qquad \log a' = 0.97967\ 70438;$$

which regive the value of $\log\alpha$ employed.

On Certain Properties of the Function W.

When the masses of the planets vanish the variables η and η' become constant; thus we should, in that case, have

$$\frac{\partial W}{\partial f} = 0, \qquad \frac{\partial W}{\partial f'} = 0,$$

and, in any other case, they should be of the order of the planetary masses. This is not immediately apparent from the form of W. Terms of the form

$$A \cos (if + i'f')$$

are still present in W after we have made $R = 0$, where A is seemingly of the zero order in reference to planetary masses.

In the case where $R = 0$, putting

$$\frac{1}{\sqrt{\mathrm{m}\, r}} = u, \quad \frac{1}{\sqrt{\mathrm{m}'\, r'}} = u', \quad \mathrm{a} = \frac{\mathrm{m}\sqrt{\mu a}}{h} - \frac{\eta}{h}, \quad \mathrm{a}' = \frac{\mathrm{m}'\sqrt{\mu' a'}}{h} - \frac{\eta'}{h},$$

the quadratic determining V becomes

$$u^2 (1 + V)^2 + u'^2 (1 - V)^2 = \mathrm{a}^2 u^2 + \mathrm{a}'^2 u'^2,$$

whose solution gives

$$V = \frac{-(u^2 - u'^2) + \sqrt{(\mathrm{a}u^2 - \mathrm{a}'u'^2)^2 + [(\mathrm{a} + \mathrm{a}')^2 - 4]\, u^2 u'^2}}{u^2 + u'^2}.$$

When the planetary masses vanish not only $R = 0$ but

$$\mathrm{a} + \mathrm{a}' = 2,$$

and the value of V, reducing to $\mathrm{a} - 1$ or $1 - \mathrm{a}'$, becomes a constant. But it would be a mistake to suppose that, on account of this, $\frac{\partial W}{\partial \eta}$ and $\frac{\partial W}{\partial \eta'}$ vanish. We have

$$\mathrm{a} + \mathrm{a}' - 2 = \frac{\mathrm{m}\sqrt{\mu a}}{h} + \frac{\mathrm{m}'\sqrt{\mu' a'}}{h} - 2 - \frac{\eta + \eta'}{h},$$

and this expression must be considered as of the order of the planetary masses. It follows that all the terms of V which arise from the term $[(\mathrm{a} + \mathrm{a}')^2 - 4]\, u^2 u'^2$ under the radical sign are at least of the same order. All the coefficients of these periodic terms of V then have the factor

$$\mathrm{m} \sqrt{\mu a} + \mathrm{m} \sqrt{\mu' a'} - 2h - \eta - \eta'.$$

Thus the derivatives $\frac{\partial W}{\partial f}$ and $\frac{\partial W}{\partial f'}$ are of the order of the planetary masses. But when $\frac{\partial W}{\partial \eta}$ and $\frac{\partial W}{\partial \eta'}$ are formed, the derivatives of this factor are -1 in each case, and the factor is lowered one dimension in respect to planetary masses. With the values of the constants involved, given in Memoir No. 79, the mentioned factor is

$$0.00094\ 15897 - \eta - \eta'.$$

MEMOIR No. 82.

Development, in Terms of the True Anomaly, of odd Negative Powers of the Distance between two Planets Moving in the Same Plane.

(This memoir appears here for the first time.)

Developments of these quantities in terms of the mean anomalies are familiar enough, but Gyldén's method of absolute orbits requires developments in terms of the true anomalies. The latter are far simpler and easier to obtain than the former. Gyldén has treated this subject himself,* but his formulae are not so ready in the application as the tabular method here proposed.

Employing the notation universally in use and ϕ denoting the true elongation of the planets, s one of the numbers $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c., we have

$$\left(\frac{a'}{\Delta}\right)^{2s} = \left(\frac{a'}{r'}\right)^{2s} \left[1 - 2\,\frac{r}{r'}\cos\phi + \frac{r^2}{r'^2}\right]^{-s}$$
$$= \tfrac{1}{2}\left(\frac{a'}{r'}\right)^{2s} \sum_{i=-\infty}^{i=+\infty} B_s^{(i)} \cos i\phi\,,$$

where $B_s^{(i)}$ is the same function of $\frac{r}{r'}$ that Laplace's $b_s^{(i)}$ is of $\frac{a}{a'} = \alpha$. The approximate value of $\frac{r}{r'}$ being α, any function of $\frac{r}{r'}$ can be expanded in a series of ascending powers of $\frac{r}{r'} - \alpha$ by Taylor's Theorem. And as we have

$$\frac{r}{r'} - \alpha = \alpha\left(\frac{r}{a}\,\frac{a'}{r'} - 1\right),$$

consequently

$$B_s^{(i)} = \sum_{n=0}^{n=\infty} \frac{1}{n\,!}\,\alpha^n\,\frac{d^n b_s^{(i)}}{d\alpha^n}\left(\frac{a'}{r'}\,\frac{r}{a} - 1\right)^n.$$

For brevity we write $\mathrm{b}_n^{(i)}$ for $\frac{1}{n\,!}\alpha^n\,\frac{d^n b_s^{(i)}}{d\alpha^n}$. Then

$$\left(\frac{a'}{\Delta}\right)^{2s} = \frac{1}{2}\sum_{i=-\infty}^{i=+\infty}\left[\mathrm{b}_0^{(i)}\left(\frac{a'}{r'}\right)^{2s} + \mathrm{b}_1^{(i)}\left(\frac{a'}{r'}\right)^{2s}\left(\frac{a'}{r'}\,\frac{r}{a} - 1\right) + \mathrm{b}_2^{(i)}\left(\frac{a'}{r'}\right)^{2s}\right.$$
$$\left.\times\left(\frac{a'}{r'}\,\frac{r}{a} - 1\right)^2 + \,..\,\right]\cos i\phi.$$

---

* *Orbites Absolues*, Tom. I, pp. 357–385.

Suppose now that we decide to cut short the infinite series at quantities of the 8^{th} order with respect to the eccentricities. This limitation enables us to use a remarkable abbreviation in the notation. We put

$$
\begin{aligned}
K_0^{(i)} &= b_0^{(i)} - b_1^{(i)} + b_2^{(i)} - b_3^{(i)} + b_4^{(i)} - b_5^{(i)} + b_6^{(i)} - b_7^{(i)} + b_8^{(i)},\\
K_1^{(i)} &= b_1^{(i)} - 2b_2^{(i)} + 3b_3^{(i)} - 4b_4^{(i)} + 5b_5^{(i)} - 6b_6^{(i)} + 7b_7^{(i)} - 8b_8^{(i)},\\
K_2^{(i)} &= b_2^{(i)} - 3b_3^{(i)} + 6b_4^{(i)} - 10b_5^{(i)} + 15b_6^{(i)} - 21b_7^{(i)} + 28b_8^{(i)},\\
K_3^{(i)} &= b_3^{(i)} - 4b_4^{(i)} + 10b_5^{(i)} - 20b_6^{(i)} + 35b_7^{(i)} - 56b_8^{(i)},\\
K_4^{(i)} &= b_4^{(i)} - 5b_5^{(i)} + 15b_6^{(i)} - 35b_7^{(i)} + 70b_8^{(i)},\\
K_5^{(i)} &= b_5^{(i)} - 6b_6^{(i)} + 21b_7^{(i)} - 56b_8^{(i)},\\
K_6^{(i)} &= b_6^{(i)} - 7b_7^{(i)} + 28b_8^{(i)},\\
K_7^{(i)} &= b_7^{(i)} - 8b_8^{(i)},\\
K_8^{(i)} &= b_8^{(i)}.
\end{aligned}
$$

Then

$$
\left(\frac{a'}{\Delta}\right)^{2s} = \frac{1}{2}\sum_{i=-\infty}^{i=+\infty}\Bigg[K_0^{(i)}\left(\frac{a'}{r'}\right)^{2s} + K_1^{(i)}\left(\frac{a'}{r'}\right)^{2s+1}\frac{r}{a} + K_2^{(i)}\left(\frac{a'}{r'}\right)^{2s+2}\left(\frac{r}{a}\right)^2 + \ldots\ldots + K_8^{(i)}\left(\frac{a'}{r'}\right)^{2s+8}\left(\frac{r}{a}\right)^8\Bigg]\cos i\phi.
$$

When we compute the values of the $b_j^{(i)}$ we will naturally be led to adopt fewer decimals as i and j increase; no attention need be paid to this circumstance in computing the K, beyond taking care to make the multiplications and additions without throwing away any decimals.

It is thus seen that the development we seek depends on the expansion of integral powers positive and negative of the radius in terms of cosines of integral multiples of the true anomaly. It is well known that the coefficients of these terms are expressible rationally in terms of the two quantities e and $\sqrt{1-e^2}$. The briefest and most elegant exposition of this subject is given by Laplace.* But the expression of these coefficients in powers of e is more suitable to our purposes; and the easiest way to arrive at the latter form is by applying the binomial theorem to both factors of

$$\left(\frac{r}{a}\right)^n = (1-e^2)^n\,(1+e\cos f)^{-n},$$

and then substituting for the powers of $\cos f$ their expressions in terms of cosines of multiples of f. By this method the following table of the expansion of $\left(\frac{r}{a}\right)^n$ has been derived; it reaches from $n = -9$ to $n = 8$, and the coefficients being generally fractions, all those falling in the same column have been reduced to a common denominator, always a power of 2. The

**Mécanique Céleste*, Liv. II, Chap. III, Art. 16.

latter is placed in the third line of the table. No signs are written unless negative, and vacancy must be understood as 0.

Table of the Expansion of $\left(\frac{r}{a}\right)^n$.

		cos 0f.				cos f.				cos 2f.			
n.	e^0.	e^2.	e^4.	e^6.	e^8.	e.	e^3.	e^5.	e^7.	e^2.	e^4.	e^6.	e^8.
		2	8	16	128	1	4	8	64	2	2	16	16
−9	1	54	2034	22834	746235	9	576	8406	323100	36	450	22284	95211
−8	1	44	1394	13484	387235	8	424	5272	176408	28	294	12628	47719
−7	1	35	917	7553	188776	7	301	3143	90587	21	182	6685	22099
−6	1	27	573	3961	85248	6	204	1758	43104	15	105	3243	9258
−5	1	20	335	1910	34960	5	130	905	18600	10	55	1400	3400
−4	1	14	179	824	12640	4	76	416	7040	6	25	512	1040
−3	1	9	84	304	3840	3	39	162	2208	3	9	144	240
−2	1	5	32	88	896	2	16	48	512	1	2	24	32
−1	1	2	8	16	128	1	4	8	64				
0	1												
1	1	−1	−1	−1	−5	−1	1	1	5	1		−5	4
2	1	−1	−1	−1	−5	−2	4	2	8	3	−1	−35	61
3	1		−3	−2	−9	−3	6	3	12	6	−3	−132	375
4	1	2	−7	−4	−17	−4	4	8	24	10	−5	−388	1507
5	1	5	−10	−10	−35	−5	−5	20	50	15	5	−970	4715
6	1	9	−6	−26	−75	−6	−24	36	120	21		−2142	12453
7	1	14	14	−56	−175	−7	−56	42	308	28	14	−4284	29057
8	1	20	62	−92	−439	−8	−104	8	712	36	42	−7908	61653

	cos 3f.			cos 4f.			cos 5f.		cos 6f.		cos 7f.	cos 8f.
n.	e^3.	e^5.	e^7.	e^4.	e^6.	e^8.	e^5.	e^7.	e^6.	e^8.	e^7.	e^8.
	4	16	64	8	16	32	16	64	32	32	64	128
−9	84	3654	83916	126	2524	27279	126	4788	84	774	36	9
−8	56	2072	41384	70	1204	11431	56	1848	28	226	8	1
−7	35	1085	18641	35	511	4214	21	595	7	49	1	
−6	20	510	7440	15	183	1296	6	144	1	6		
−5	10	205	2500	5	50	300	1	20				
−4	4	64	640	1	8	40						
−3	1	12	96									
−2												
−1												
0												
1	−1	−1	−1	1	1	1	−1	−3	1	1	−1	1
2	−4	2	8	5	1	−1	−6	−8	7	4	−8	9
3	−10	15	24	15	−6	−9	−21		28	6	−36	45
4	−20	40	40	35	−28	−21	−56	56	84	−6	−120	165
5	−35	70	70	70	−70	−35	−126	210	210	−60	−330	495
6	−56	84	168	126	−126	−63	−252	504	462	−198	−792	1287
7	−84	42	420	210	−168	−147	−462	924	924	−462	−1716	3003
8	−120	−120	888	330	−132	−363	−792	1320	1716	−858	−3432	6435

The columns of integers in this table can be tested by differences. The order of differences indicated by the exponent of the power of e the integers

multiply should be constant. This property may be used to prolong the table to powers of $\frac{r}{a}$ beyond the limits -9 and 8.

As a sufficient illustration I give the development of $\frac{a'}{\Delta}$ derived from the preceding table. The length of the expansion is reduced to nearly half by the device of an ambiguous sign in some of the arguments; this is to be taken in each way in succession.

It may be desirable to get rid of ϕ the true elongation of the planets; this we may do by substituting for it the expression $f-f'+\pi-\pi'=f-f'+\gamma$ where γ would be a slow-moving argument.

$$
\begin{aligned}
\frac{a'}{\Delta} = \tfrac{1}{2} \sum_{i=-\infty}^{i=+\infty} \Big\{ & \Big\{ K_0^{(i)} [1 + e'^2 + e'^4 + e'^6 + e'^8] \\
& + K_1^{(i)} [1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4 - \tfrac{1}{16} e^6 - \tfrac{5}{128} e^8] \\
& \qquad \times [1 + \tfrac{5}{2} e'^2 + 4e'^4 + \tfrac{11}{2} e'^6 + 7e'^8] \\
& + K_2^{(i)} [1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4 - \tfrac{1}{16} e^6 - \tfrac{5}{128} e^8] \\
& \qquad \times [1 + \tfrac{9}{2} e'^2 + \tfrac{21}{2} e'^4 + 19e'^6 + 30e'^8] \\
& + K_3^{(i)} [1 + 0e^2 - \tfrac{3}{8} e^4 - \tfrac{1}{8} e^6 - \tfrac{9}{128} e^8] \\
& \qquad \times [1 + 7e'^2 + \tfrac{179}{8} e'^4 + \tfrac{103}{2} e'^6 + \tfrac{395}{4} e'^8] \\
& + K_4^{(i)} [1 + e^2 - \tfrac{7}{8} e^4 - \tfrac{1}{4} e^6 - \tfrac{17}{128} e^8] \\
& \qquad \times [1 + 10e'^2 + \tfrac{335}{8} e'^4 + \tfrac{955}{8} e'^6 + \tfrac{2185}{8} e'^8] \\
& + K_5^{(i)} [1 + \tfrac{5}{2} e^2 - \tfrac{5}{4} e^4 - \tfrac{5}{8} e^6 - \tfrac{35}{128} e^8] \\
& \qquad \times [1 + \tfrac{27}{2} e'^2 + \tfrac{573}{8} e'^4 + \tfrac{3961}{16} e'^6 + 666e'^8] \\
& + K_6^{(i)} [1 + \tfrac{9}{2} e^2 - \tfrac{3}{4} e^4 - \tfrac{13}{8} e^6 - \tfrac{75}{128} e^8] \\
& \qquad \times [1 + \tfrac{35}{2} e'^2 + \tfrac{917}{8} e'^4 + \tfrac{7553}{16} e'^6 + \tfrac{23577}{16} e'^6] \\
& + K_7^{(i)} [1 + 7e^2 + \tfrac{7}{4} e^4 - \tfrac{7}{2} e^6 - \tfrac{175}{128} e^8] \\
& \qquad \times [1 + 22e'^2 + \tfrac{697}{4} e'^4 + \tfrac{3371}{4} e'^6 + \tfrac{387235}{128} e'^8] \\
& + K_8^{(i)} [1 + 10e^2 + \tfrac{31}{4} e^4 - \tfrac{23}{4} e^6 - \tfrac{431}{128} e^8] \\
& \qquad \times [1 + 27e'^2 + \tfrac{1017}{4} e'^4 + \tfrac{2853}{2} e'^6 + \tfrac{746235}{128} e'^8] \Big\} \cos i\phi \\
+ \Big\{ & K_1^{(i)} [-1 + \tfrac{1}{4} e^2 + \tfrac{1}{8} e^4 + \tfrac{5}{64} e^6] [1 + \tfrac{5}{2} e'^2 + 4e'^4 + \tfrac{11}{2} e'^6] \\
& + K_2^{(i)} [-2 + e^2 + \tfrac{1}{4} e^4 + \tfrac{1}{8} e^6] [1 + \tfrac{9}{2} e'^2 + \tfrac{21}{2} e'^4 + 19e'^6] \\
& + K_3^{(i)} [-3 + \tfrac{3}{2} e^2 + \tfrac{3}{8} e^4 + \tfrac{3}{16} e^6] [1 + 7e'^2 + \tfrac{179}{8} e'^4 + \tfrac{103}{2} e'^6] \\
& + K_4^{(i)} [-4 + e^2 + e^4 + \tfrac{3}{8} e^6] [1 + 10e'^2 + \tfrac{335}{8} e'^4 + \tfrac{955}{8} e'^8] \\
& + K_5^{(i)} [-5 - \tfrac{5}{4} e^2 + \tfrac{5}{2} e^4 + \tfrac{25}{32} e^6] [1 + \tfrac{27}{2} e'^2 + \tfrac{573}{8} e'^4 + \tfrac{3961}{16} e'^6] \\
& + K_6^{(i)} [-6 - 6e^2 + \tfrac{9}{2} e^4 + \tfrac{15}{8} e^6] [1 + \tfrac{35}{2} e'^2 + \tfrac{917}{8} e'^4 + \tfrac{7553}{16} e'^6] \\
& + K_7^{(i)} [-7 - 14e^2 + \tfrac{21}{4} e^4 + \tfrac{77}{16} e^6] [1 + 22e'^2 + \tfrac{697}{4} e'^4 + \tfrac{3371}{4} e'^6] \\
& + K_8^{(i)} [-8 - 26e^2 + e^4 + \tfrac{89}{8} e^6] [1 + 27e'^2 + \tfrac{1017}{4} e'^4 \\
& \qquad + \tfrac{2853}{2} e'^6] \Big\} e \cos (f + i\phi)
\end{aligned}
$$

$$
\begin{aligned}
+ \Big\{ & K_0^{(i)} [1 + e'^2 + e'^4 + e'^6] \\
& + K_1^{(i)} [2 + 4e'^2 + 6e'^4 + 8e'^6] [1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4 - \tfrac{1}{16} e^6] \\
& + K_2^{(i)} [3 + \tfrac{39}{4} e'^2 + \tfrac{31}{4} e'^4 + \tfrac{69}{2} e'^6] [1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4 - \tfrac{1}{16} e^6] \\
& + K_3^{(i)} [4 + 19e'^2 + 52e'^4 + 110e'^6] [1 + 0e^2 - \tfrac{3}{6} e^4 - \tfrac{1}{8} e^6] \\
& + K_4^{(i)} [5 + \tfrac{65}{2} e'^2 + \tfrac{905}{8} e'^4 + \tfrac{2325}{8} e'^6] [1 + e^2 - \tfrac{7}{8} e^4 - \tfrac{1}{4} e^6] \\
& + K_5^{(i)} [6 + 51e'^2 + \tfrac{879}{4} e'^4 + \tfrac{1347}{2} e'^6] [1 + \tfrac{5}{2} e^2 - \tfrac{5}{4} e^4 - \tfrac{5}{8} e^6] \\
& + K_6^{(i)} [7 + \tfrac{301}{3} e'^2 + \tfrac{3143}{6} e'^4 + \tfrac{90587}{64} e'^6] [1 + \tfrac{9}{2} e^2 - \tfrac{3}{4} e^4 - \tfrac{13}{8} e^6] \\
& + K_7^{(i)} [8 + 106e'^2 + 659e'^4 + \tfrac{22051}{6} e'^6] [1 + 7e^2 + \tfrac{7}{4} e^4 - \tfrac{7}{2} e^6] \\
& + K_8^{(i)} [9 + 144e'^2 + \tfrac{4203}{4} e'^4 + \tfrac{80775}{16} e'^6] \\
& \qquad \times [1 + 10e^2 + \tfrac{31}{4} e^4 - \tfrac{23}{4} e^6] \Big\} e' \cos (f' + i\phi)
\end{aligned}
$$

$$
\begin{aligned}
+ \Big\{ & K_1^{(i)} [\tfrac{1}{2} + 0e^2 - \tfrac{\ }{16} e^4 + \tfrac{1}{4} e^6] [1 + \tfrac{5}{2} e'^2 + 4e'^4 + \tfrac{11}{2} e'^6] \\
& + K_2^{(i)} [\tfrac{3}{2} - \tfrac{1}{2} e^2 - \tfrac{35}{16} e^4 + \tfrac{61}{16} e^6] [1 + \tfrac{9}{2} e'^2 + \tfrac{21}{2} e'^4 + 19e'^6] \\
& + K_3^{(i)} [3 - \tfrac{3}{2} e^2 - \tfrac{33}{4} e^4 + \tfrac{375}{16} e^6] [1 + 7e'^2 + \tfrac{179}{8} e'^4 + \tfrac{103}{2} e'^6] \\
& + K_4^{(i)} [5 - \tfrac{5}{2} e^2 - \tfrac{97}{4} e^4 + \tfrac{1507}{16} e^6] [1 + 10e'^2 + \tfrac{335}{8} e'^4 + \tfrac{955}{8} e'^6] \\
& + K_5^{(i)} [\tfrac{15}{2} + \tfrac{5}{2} e^2 - \tfrac{485}{8} e^4 + \tfrac{4715}{16} e^6] [1 + \tfrac{27}{2} e'^2 + \tfrac{573}{8} e'^4 + \tfrac{3961}{16} e'^6] \\
& + K_6^{(i)} [\tfrac{21}{2} + 0e^2 - \tfrac{1071}{8} e^4 + \tfrac{12453}{16} e^6] [1 + \tfrac{35}{2} e'^2 + \tfrac{917}{8} e'^4 + \tfrac{7553}{16} e'^6] \\
& + K_7^{(i)} [14 + 7e^2 - \tfrac{1071}{4} e^4 + \tfrac{29057}{16} e^6] [1 + 22e'^2 + \tfrac{697}{4} e'^4 + \tfrac{3371}{4} e'^6] \\
& + K_8^{(i)} [18 + 21e^2 - \tfrac{1977}{4} e^4 + \tfrac{61653}{16} e^6] \\
& \qquad \times [1 + 27e'^2 + \tfrac{1017}{4} e'^4 + \tfrac{2853}{2} e'^6] \Big\} e^2 \cos (2f + i\phi)
\end{aligned}
$$

$$
\begin{aligned}
+ \tfrac{1}{2} \Big\{ & K_1^{(i)} [-1 + \tfrac{1}{4} e^2 + \tfrac{1}{8} e^4 + \tfrac{5}{64} e^6] [2 + 4e'^2 + 6e'^4 + 8e'^6] \\
& + K_2^{(i)} [-2 + e^2 + \tfrac{1}{4} e^4 + \tfrac{1}{8} e^6] [3 + \tfrac{39}{4} e'^2 + \tfrac{61}{4} e'^4 + \tfrac{69}{2} e'^6] \\
& + K_3^{(i)} [-3 + \tfrac{3}{2} e^2 + \tfrac{3}{8} e^4 + \tfrac{3}{16} e^6] [4 + 19e'^2 + 52e'^4 + 110e'^6] \\
& + K_4^{(i)} [-4 + e^2 + e^4 + \tfrac{3}{8} e^6] [5 + \tfrac{65}{2} e'^2 + \tfrac{905}{8} e'^4 + \tfrac{2325}{8} e'^6] \\
& + K_5^{(i)} [-5 - \tfrac{5}{4} e^2 + \tfrac{5}{2} e^4 + \tfrac{25}{32} e^6] [6 + 51e'^2 + \tfrac{679}{4} e'^4 + \tfrac{1347}{2} e'^6] \\
& + K_6^{(i)} [-6 - 6e^2 + \tfrac{9}{2} e^4 + \tfrac{15}{8} e^6] [7 + \tfrac{301}{4} e'^2 + \tfrac{3143}{8} e'^4 + \tfrac{90587}{64} e'^6] \\
& + K_7^{(i)} [-7 - 14e^2 + \tfrac{21}{4} e^4 + \tfrac{77}{16} e^6] [8 + 106e'^2 + 659e'^4 + \tfrac{22051}{8} e'^6] \\
& + K_8^{(i)} [-8 - 26e^2 + e^4 + \tfrac{89}{8} e^6] \\
& \qquad \times [9 + 144e'^2 + \tfrac{4203}{4} e'^4 + \tfrac{80775}{16} e'^6] \Big\} ee' \cos (f \pm f' + i\phi)
\end{aligned}
$$

$$
\begin{aligned}
+ \Big\{ & K_1^{(i)} [\tfrac{1}{2} + e'^2 + \tfrac{3}{2} e'^4 + 2e'^6] [1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4 - \tfrac{1}{16} e^6] \\
& + K_2^{(i)} [\tfrac{3}{2} + \tfrac{9}{2} e'^2 + 9e'^4 + 15e'^6] [1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4 - \tfrac{1}{16} e^6] \\
& + K_3^{(i)} [3 + \tfrac{25}{2} e'^2 + 32e'^4 + 65e'^6] [1 + 0e^2 - \tfrac{3}{8} e^4 - \tfrac{1}{8} e^6] \\
& + K_4^{(i)} [5 + \tfrac{55}{2} e'^2 + \tfrac{175}{2} e'^4 + \tfrac{425}{2} e'^6] [1 + e^2 - \tfrac{7}{8} e^4 - \tfrac{1}{4} e^6] \\
& + K_5^{(i)} [\tfrac{15}{2} + \tfrac{105}{2} e'^2 + \tfrac{3243}{16} e'^4 + \tfrac{4629}{8} e'^6] [1 + \tfrac{5}{2} e^2 - \tfrac{5}{4} e^4 - \tfrac{5}{8} e^6] \\
& + K_6^{(i)} [\tfrac{21}{2} + 91 e'^2 + \tfrac{6685}{16} e'^4 + \tfrac{22099}{16} e'^6] [1 + \tfrac{9}{2} e^2 - \tfrac{3}{4} e^4 - \tfrac{13}{8} e^6] \\
& + K_7^{(i)} [14 + 147e'^2 + \tfrac{3157}{4} e'^4 + \tfrac{47719}{16} e'^6] [1 + 7e^2 + \tfrac{7}{4} e^4 - \tfrac{7}{2} e^6] \\
& + K_8^{(i)} [18 + 225e'^2 + \tfrac{5571}{4} e'^4 + \tfrac{95211}{16} e'^6] \\
& \qquad \times [1 + 10e^2 + \tfrac{31}{4} e^4 - \tfrac{23}{4} e^6] \Big\} e'^2 \cos (2f' + i\phi)
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \{ K_1^{(i)} [-\tfrac{1}{4} - \tfrac{1}{16} e^2 - \tfrac{1}{64} e^4] [1 + \tfrac{5}{2} e'^2 + 4e'^4] + K_2^{(i)} [-1 + \tfrac{1}{8} e^2 + \tfrac{1}{8} e^4] \\
& \qquad \times [1 + \tfrac{9}{2} e'^2 + \tfrac{21}{2} e'^4] \\
& + K_3^{(i)} [-\tfrac{5}{2} + \tfrac{15}{16} e^2 + \tfrac{3}{8} e^4] [1 + 7e'^2 + \tfrac{179}{8} e'^4] + K_4^{(i)} [-5 + \tfrac{5}{2} e^2 + \tfrac{5}{8} e^4] \\
& \qquad \times [1 + 10e'^2 + \tfrac{335}{8} e'^4] \\
& + K_5^{(i)} [-\tfrac{35}{4} + \tfrac{35}{16} e^2 + \tfrac{35}{32} e^4] [1 + \tfrac{27}{2} e'^2 + \tfrac{573}{8} e'^4] \\
& \qquad + K_6^{(i)} [-14 + \tfrac{21}{4} e^2 + \tfrac{21}{8} e^4] [1 + \tfrac{35}{2} e'^2 + \tfrac{917}{6} e'^4] \\
& + K_7^{(i)} [-21 + \tfrac{21}{8} e^2 + \tfrac{105}{16} e^4] [1 + 22e'^2 + \tfrac{697}{4} e^4] \\
& \qquad + K_8^{(i)} [-30 - \tfrac{15}{2} e^2 + \tfrac{111}{8} e^4] [1 + 27e'^2 + \tfrac{1017}{4} e'^4] \} e^3 \cos (3f + i\phi) \\
+ \tfrac{1}{2} & \{ K_1^{(i)} [\tfrac{1}{2} + 0e^2 - \tfrac{5}{16} e^4] [2 + 4e'^2 + 6e'^4] + K_2^{(i)} [\tfrac{3}{2} - \tfrac{1}{2} e^2 - \tfrac{35}{16} e^4] \\
& \qquad \times [3 + \tfrac{39}{4} e'^2 + \tfrac{81}{4} e'^4] \\
& + K_3^{(i)} [3 - \tfrac{3}{2} e^2 - \tfrac{33}{4} e^4] [4 + 19e'^2 + 52e'^4] + K_4^{(i)} [5 - \tfrac{5}{2} e^2 - \tfrac{97}{4} e^4] \\
& \qquad \times [5 + \tfrac{65}{2} e'^2 + \tfrac{905}{8} e'^4] \\
& + K_5^{(i)} [\tfrac{15}{2} + \tfrac{5}{2} e^2 - \tfrac{485}{8} e^4] [6 + 51e'^2 + \tfrac{879}{4} e'^4] + K_6^{(i)} [\tfrac{21}{2} + 0e^2 - \tfrac{1071}{8} e^4] \\
& \qquad \times [7 + \tfrac{301}{4} e'^2 + \tfrac{3143}{8} e'^4] \\
& + K_7^{(i)} [14 + 7e^2 - \tfrac{1071}{4} e^4] [8 + 106e'^2 + 659e'^4] \\
& \qquad + K_8^{(i)} [18 + 21e^2 - \tfrac{1977}{4} e^4] [9 + 144e'^2 + \tfrac{4203}{4} e'^4] \} e^2 e' \cos (2f \pm f' + i\phi) \\
+ \tfrac{1}{2} & \{ K_1^{(i)} [-1 + \tfrac{1}{4} e^2 + \tfrac{1}{8} e^4] [\tfrac{1}{2} + e'^2 + \tfrac{3}{2} e'^4] + K_2^{(i)} [-2 + e^2 + \tfrac{1}{4} e^4] \\
& \qquad \times [\tfrac{3}{2} + \tfrac{9}{2} e'^2 + 9e'^4] \\
& + K_3^{(i)} [-3 + \tfrac{3}{2} e^2 + \tfrac{3}{8} e^4] [3 + \tfrac{35}{2} e'^2 + 32e'^4] + K_4^{(i)} [-4 + e^2 + e^4] \\
& \qquad \times [5 + \tfrac{55}{2} e'^2 + \tfrac{175}{2} e'^4] \\
& + K_5^{(i)} [-5 - \tfrac{5}{4} e^2 + \tfrac{5}{2} e^4] [\tfrac{15}{2} + \tfrac{105}{2} e'^2 + \tfrac{3243}{16} e'^4] + K_6^{(i)} [-6 - 6e^2 + \tfrac{9}{2} e'^4] \\
& \qquad \times [\tfrac{21}{2} + 91e'^2 + \tfrac{6685}{16} e'^4] \\
& + K_7^{(i)} [-7 - 14e^2 + \tfrac{21}{4} e^4] [14 + 147e'^2 + \tfrac{3157}{4} e'^4] \\
& \qquad + K_8^{(i)} [-8 - 26e^2 + e^4] [18 + 225e'^2 + \tfrac{5571}{4} e'^4] \} e e'^2 \cos (f \pm 2f' + i\phi) \\
+ \quad & \{ K_2^{(i)} [\tfrac{1}{4} + \tfrac{3}{4} e'^2 + \tfrac{3}{2} e'^4] [1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e'^4] + K_3^{(i)} [1 + 4e'^2 + 10e'^4] \\
& \qquad \times [1 + 0e^2 - \tfrac{3}{8} e^4] \\
& + K_4^{(i)} [\tfrac{5}{2} + \tfrac{205}{16} e'^2 + \tfrac{625}{16} e'^4] [1 + e^2 - \tfrac{7}{8} e^4] + K_5^{(i)} [5 + \tfrac{255}{8} e'^2 + \tfrac{465}{4} e'^4] \\
& \qquad \times [1 + \tfrac{5}{2} e^2 - \tfrac{5}{4} e^4] \\
& + K_6^{(i)} [\tfrac{35}{4} + \tfrac{1085}{16} e'^2 + \tfrac{18641}{64} e'^4] [1 + \tfrac{9}{2} e^2 - \tfrac{3}{4} e^4] \\
& \qquad + K_7^{(i)} [14 + \tfrac{259}{2} e'^2 + \tfrac{5173}{8} e'^4] [1 + \tfrac{7}{2} e^2 + \tfrac{7}{4} e^4] \\
& + K_8^{(i)} [21 + \tfrac{1827}{8} e'^2 + \tfrac{20979}{16} e'^4] [1 + 10e^2 + \tfrac{31}{4} e^4] \} e'^3 \cos (3f' + i\phi) \\
+ \quad & \{ K_1^{(i)} [\tfrac{1}{8} + \tfrac{1}{16} e^2 + \tfrac{1}{32} e^4] [1 + \tfrac{5}{2} e'^2 + 4e'^4] + K_2^{(i)} [\tfrac{5}{8} + \tfrac{1}{16} e^2 - \tfrac{1}{32} e^4] \\
& \qquad \times [1 + \tfrac{9}{2} e'^2 + \tfrac{21}{2} e'^4] \\
& + K_3^{(i)} [\tfrac{15}{8} - \tfrac{3}{8} e^2 - \tfrac{9}{32} e^4] [1 + 7e'^2 + \tfrac{179}{8} e'^4] + K_4^{(i)} [\tfrac{35}{8} - \tfrac{7}{4} e^2 - \tfrac{21}{32} e^4] \\
& \qquad \times [1 + 10e'^2 + \tfrac{335}{8} e'^4] \\
& + K_5^{(i)} [\tfrac{35}{4} - \tfrac{35}{6} e^2 - \tfrac{35}{32} e^4] [1 + \tfrac{27}{2} e'^2 + \tfrac{573}{8} e'^4] + K_6^{(i)} [\tfrac{63}{4} - \tfrac{63}{8} e^2 - \tfrac{63}{32} e^4] \\
& \qquad \times [1 + \tfrac{35}{2} e'^2 + \tfrac{917}{8} e'^4] \\
& + K_7^{(i)} [\tfrac{105}{4} - \tfrac{21}{2} e^2 - \tfrac{147}{32} e^4] [1 + 22e'^2 + \tfrac{697}{4} e'^4] + K_8^{(i)} [\tfrac{165}{4} - \tfrac{33}{4} e^2 - \tfrac{363}{32} e^4] \\
& \qquad \times [1 + 27e'^2 + \tfrac{1017}{4} e'^4] \} e^4 \cos (4f + i\phi)
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{1}{2} \Big\{ K_1^{(i)} [- \tfrac{1}{4} - \tfrac{1}{16} e^2 - \tfrac{1}{64} e^4] [2 + 4e'^2 + 6e'^4] + K_2^{(i)} [-1 + \tfrac{1}{8} e^2 + \tfrac{1}{8} e^4] \\
&\qquad \times [3 + \tfrac{39}{4} e'^2 + \tfrac{81}{4} e'^4] \\
&\quad + K_3^{(i)} [- \tfrac{5}{2} + \tfrac{15}{16} e^2 + \tfrac{3}{8} e^4] [4 + 19e'^2 + 52e'^4] \\
&\quad + K_4^{(i)} [- 5 + \tfrac{5}{2} e^2 + \tfrac{5}{8} e^4] [5 + \tfrac{65}{2} e'^2 + \tfrac{905}{8} e'^4] \\
&\quad + K_5^{(i)} [- \tfrac{35}{4} + \tfrac{35}{8} e^2 + \tfrac{35}{32} e^4] [6 + 51e'^2 + \tfrac{879}{4} e'^4] \\
&\quad + K_6^{(i)} [- 14 + \tfrac{21}{4} e^2 + \tfrac{21}{8} e^4] [7 + \tfrac{301}{4} e'^2 + \tfrac{3143}{8} e'^4] \\
&\quad + K_7^{(i)} [- 21 + \tfrac{21}{8} e^2 + \tfrac{105}{16} e^4] [8 + 106e'^2 + 659e'^4] \\
&\quad + K_8^{(i)} [- 30 - \tfrac{15}{2} e^2 + \tfrac{111}{8} e^4] [9 + 144e'^2 + \tfrac{4203}{4} e'^4] \Big\} e^3 e' \cos (3f \pm f' + i\phi) \\
&+ \tfrac{1}{2} \Big\{ K_1^{(i)} [\tfrac{1}{2} + 0e^2 - \tfrac{5}{16} e^4] [\tfrac{1}{2} + e'^2 + \tfrac{3}{2} e'^4] + K_2^{(i)} [\tfrac{3}{2} - \tfrac{1}{2} e^2 - \tfrac{35}{16} e^4] \\
&\qquad \times [\tfrac{3}{2} + \tfrac{9}{2} e'^2 + 9e'^4] \\
&\quad + K_3^{(i)} [3 - \tfrac{3}{2} e^2 - \tfrac{33}{4} e^4] [3 + \tfrac{25}{2} e'^2 + 32e'^4] + K_4^{(i)} [5 - \tfrac{5}{2} e^2 - \tfrac{97}{4} e^4] \\
&\qquad \times [5 + \tfrac{55}{2} e'^2 + \tfrac{175}{2} e'^4] \\
&\quad + K_5^{(i)} [\tfrac{15}{2} + \tfrac{5}{2} e^2 + \tfrac{485}{8} e^4] [\tfrac{15}{2} + \tfrac{105}{2} e'^2 + \tfrac{3243}{16} e'^4] \\
&\quad + K_6^{(i)} [\tfrac{21}{2} + 0e^2 - \tfrac{1071}{8} e^4] [\tfrac{21}{2} + 91e'^2 + \tfrac{6685}{16} e'^4] \\
&\quad + K_7^{(i)} [14 + 7e^2 - \tfrac{1071}{4} e^4] [14 + 147e'^2 + \tfrac{3157}{4} e'^4] \\
&\quad + K_8^{(i)} [18 + 21e^2 - \tfrac{1977}{4} e^2] [18 + 225e'^2 + \tfrac{5571}{4} e'^4] \Big\} e^2 e'^2 \cos (2f \pm 2f' + i\phi) \\
&+ \tfrac{1}{2} \Big\{ K_2^{(i)} [- 2 + e^2 + \tfrac{1}{4} e^4] [\tfrac{1}{4} + \tfrac{3}{4} e'^2 + \tfrac{3}{2} e'^4] \\
&\quad + K_3^{(i)} [- 3 + \tfrac{3}{2} e^2 + \tfrac{3}{8} e^4] [1 + 4e'^2 + 10e'^4] \\
&\quad + K_4^{(i)} [-4 + e^2 + e^4] [\tfrac{5}{2} + \tfrac{205}{16} e'^2 + \tfrac{625}{16} e'^4] \\
&\quad + K_5^{(i)} [- 5 - \tfrac{5}{4} e^2 + \tfrac{5}{2} e^4] [5 + \tfrac{255}{8} e'^2 + \tfrac{465}{4} e'^4] \\
&\quad + K_6^{(i)} [- 6 - 6e^2 + \tfrac{9}{2} e^4] [\tfrac{35}{4} + \tfrac{1085}{16} e'^2 + \tfrac{18641}{64} e'^4] \\
&\quad + K_7^{(i)} [- 7 - 14e^2 + \tfrac{21}{4} e^4] [14 + \tfrac{259}{2} e'^2 + \tfrac{5173}{8} e'^4] \\
&\quad + K_8^{(i)} [- 8 - 26e^2 + e^4] [21 + \tfrac{1827}{8} e'^2 + \tfrac{20979}{16} e'^4] \Big\} e e'^3 \cos (f \pm 3f' + i\phi) \\
&+ \Big\{ K_3^{(i)} [\tfrac{1}{8} + \tfrac{1}{2} e'^2 + \tfrac{5}{4} e'^4] [1 + 0e^2 - \tfrac{3}{8} e^4] \\
&\quad + K_4^{(i)} [\tfrac{5}{8} + \tfrac{25}{8} e'^2 + \tfrac{75}{8} e'^4] [1 + e^2 - \tfrac{7}{8} e^4] \\
&\quad + K_5^{(i)} [\tfrac{15}{8} + \tfrac{183}{16} e'^2 + \tfrac{81}{2} e'^4] [1 + \tfrac{5}{2} e^2 - \tfrac{5}{4} e^4] \\
&\quad + K_6^{(i)} [\tfrac{35}{8} + \tfrac{511}{16} e'^2 + \tfrac{2107}{16} e'^4] [1 + \tfrac{9}{2} e^2 - \tfrac{3}{4} e^4] \\
&\quad + K_7^{(i)} [\tfrac{35}{4} + \tfrac{301}{4} e'^2 + \tfrac{11431}{31} e'^4] [1 + 7e^2 + \tfrac{7}{4} e^4] \\
&\quad + K_8^{(i)} [\tfrac{63}{4} + \tfrac{631}{4} e'^2 + \tfrac{27279}{32} e'^4] [1 + 10e^2 + \tfrac{31}{4} e^4] \Big\} e'^4 \cos (4f' + i\phi) \\
&+ \Big\{ K_1^{(i)} [- \tfrac{1}{16} - \tfrac{3}{64} e^2] [1 + \tfrac{5}{2} e'^2] + K_2^{(i)} [- \tfrac{3}{8} - \tfrac{1}{8} e^2] [1 + \tfrac{9}{2} e'^2] \\
&\quad + K_3^{(i)} [- \tfrac{21}{16} + 0e^2] [1 + 7e'^2] \\
&\quad + K_4^{(i)} [- \tfrac{7}{2} + \tfrac{7}{8} e^2] [1 + 10e'^2] + K_5^{(i)} [- \tfrac{63}{8} + \tfrac{105}{32} e^2] [1 + \tfrac{27}{2} e'^2] \\
&\quad + K_6^{(i)} [- \tfrac{63}{4} + \tfrac{63}{16} e^2] [1 + \tfrac{35}{2} e'^2] \\
&\quad + K_7^{(i)} [- \tfrac{231}{8} + \tfrac{231}{16} e^2] [1 + 22e'^2] + K_8^{(i)} [- \tfrac{99}{2} + \tfrac{165}{8} e^2] \\
&\qquad \times [1 + 27e'^2] \Big\} e^5 \cos (5f + i\phi)
\end{aligned}
$$

$$+\tfrac{1}{2}\{K_1^{(i)}[\tfrac{1}{8}+\tfrac{1}{16}e^2][2+4e'^2]+K_2^{(i)}[\tfrac{5}{8}+\tfrac{1}{16}e^2][3+\tfrac{39}{4}e'^2]$$
$$+K_3^{(i)}[\tfrac{15}{8}-\tfrac{3}{8}e^2][4+19e'^2]$$
$$+K_4^{(i)}[\tfrac{35}{4}-\tfrac{7}{4}e^2][5+\tfrac{65}{2}e'^2]+K_5^{(i)}[\tfrac{35}{4}-\tfrac{35}{8}e^2][6+51e'^2]$$
$$+K_6^{(i)}[\tfrac{63}{4}-\tfrac{63}{8}e^2][7+\tfrac{301}{4}e'^2]$$
$$+K_7^{(i)}[\tfrac{105}{4}-\tfrac{21}{4}e^2][8+106e'^2]+K_8^{(i)}[\tfrac{165}{4}-\tfrac{33}{4}e^2]$$
$$\times[9+144e'^2]\}\,e^4e'\cos(4f\pm f'+i\phi)$$

$$+\tfrac{1}{2}\{K_1^{(i)}[-\tfrac{1}{4}-\tfrac{1}{16}e^2][\tfrac{1}{2}+e'^2]+K_2^{(i)}[-1+\tfrac{1}{8}e^2][\tfrac{3}{2}+\tfrac{9}{2}e'^2]$$
$$+K_3^{(i)}[-\tfrac{5}{2}+\tfrac{16}{16}e^2][3+\tfrac{25}{2}e'^2]$$
$$+K_4^{(i)}[-5+\tfrac{5}{2}e^2][5+\tfrac{55}{2}e'^2]+K_6^{(i)}[-\tfrac{35}{4}+\tfrac{35}{8}e^2][\tfrac{15}{2}+\tfrac{105}{2}e'^2]$$
$$+K_6^{(i)}[-14+\tfrac{21}{4}e^2][\tfrac{21}{2}+91e'^2]$$
$$+K_7^{(i)}[-21+\tfrac{21}{8}e^2][14+147e'^2]+K_8^{(i)}[-30-\tfrac{15}{2}e^2]$$
$$\times[18+225e'^2]\}\,e^3e'^2\cos(3f\pm 2f'+i\phi)$$

$$+\tfrac{1}{2}\{K_2^{(i)}[\tfrac{3}{2}-\tfrac{1}{2}e^2][\tfrac{1}{4}+\tfrac{3}{4}e'^2]+K_3^{(i)}[3-\tfrac{3}{2}e^2][1+4e'^2]$$
$$+K_4^{(i)}[5-\tfrac{5}{2}e^2][\tfrac{5}{2}+\tfrac{205}{16}e'^2]$$
$$+K_5^{(i)}[\tfrac{15}{2}+\tfrac{5}{2}e^2][5+\tfrac{255}{8}e'^2]+K_6^{(i)}[\tfrac{21}{2}+0e^2][\tfrac{35}{4}+\tfrac{1085}{16}e'^2]$$
$$+K_7^{(i)}[14+7e^2][14+\tfrac{259}{2}e'^2]$$
$$+K_8^{(i)}[18+21e^2][21+\tfrac{1827}{8}e'^2]\}\,e^2e'^3\cos(2f\pm 3f'+i\phi)$$

$$+\tfrac{1}{2}\{K_3^{(i)}[-3+\tfrac{3}{2}e^2][\tfrac{1}{8}+\tfrac{1}{2}e'^2]+K_4^{(i)}[-4+e^2][\tfrac{5}{8}+\tfrac{25}{8}e'^2]$$
$$+K_5^{(i)}[-5-\tfrac{6}{4}e^2][\tfrac{15}{8}+\tfrac{183}{16}e'^2]$$
$$+K_6^{(i)}[-6-6e^2][\tfrac{35}{8}+\tfrac{511}{6}e'^2]+K_7^{(i)}[-7-14e^2][\tfrac{35}{4}+\tfrac{301}{4}e'^2]$$
$$+K_8^{(i)}[-8-26e^2][\tfrac{63}{4}+\tfrac{631}{4}e'^2]\}\,ee'^4\cos(f\pm 4f'+i\phi)$$

$$+\{K_4^{(i)}[\tfrac{1}{16}+\tfrac{5}{16}e'^2][1+e^2]+K_5^{(i)}[\tfrac{3}{8}+\tfrac{9}{4}e'^2][1+\tfrac{5}{2}e^2)$$
$$+K_6^{(i)}[\tfrac{21}{16}+\tfrac{595}{64}e'^2][1+\tfrac{9}{2}e^2]$$
$$+K_7^{(i)}[\tfrac{7}{2}+\tfrac{231}{8}e'^2][1+7e^2]$$
$$+K_8^{(i)}[\tfrac{63}{8}+\tfrac{1197}{16}e'^2][1+10e^2]\}\,e'^5\cos(5f'+i\phi)$$

$$+\{K_1^{(i)}[\tfrac{1}{32}+\tfrac{1}{32}e^2][1+\tfrac{5}{2}e'^2]+K_2^{(i)}[\tfrac{7}{32}+\tfrac{1}{8}e^2][1+\tfrac{9}{2}e'^2]$$
$$+K_3^{(i)}[\tfrac{7}{4}+\tfrac{3}{16}e^2][1+7e'^2]$$
$$+K_4^{(i)}[\tfrac{21}{8}-\tfrac{3}{16}e^2][1+10e'^2]+K_5^{(i)}[\tfrac{105}{16}-\tfrac{15}{8}e^2][1+\tfrac{27}{2}e'^2]$$
$$+K_6^{(i)}[\tfrac{231}{16}-\tfrac{99}{16}e^2][1+\tfrac{35}{2}e'^2]$$
$$+K_7^{(i)}[\tfrac{231}{8}-\tfrac{231}{16}e^2][1+22e'^2]$$
$$+K_8^{(i)}[\tfrac{429}{8}-\tfrac{429}{16}e^2][1+27e'^2]\}\,e^6\cos(6f+i\phi)$$

$$+\tfrac{1}{2}\{K_1^{(i)}[-\tfrac{1}{16}-\tfrac{3}{64}e^2][2+4e'^2]+K_2^{(i)}[-\tfrac{3}{8}-\tfrac{1}{8}e^2][3+\tfrac{39}{4}e'^2]$$
$$+K_3^{(i)}[-\tfrac{21}{16}+0e^2][4+19e'^2]$$
$$+K_4^{(i)}[-\tfrac{7}{2}+\tfrac{7}{8}e^2][5+\tfrac{65}{2}e'^2]+K_5^{(i)}[-\tfrac{63}{8}+\tfrac{105}{32}e^2][6+51e'^2]$$
$$+K_6^{(i)}[-\tfrac{63}{4}+\tfrac{63}{8}e^2][7+\tfrac{301}{4}e'^2]$$
$$+K_7^{(i)}[-\tfrac{231}{8}+\tfrac{231}{16}e^2][8+106e'^2]$$
$$+K_8^{(i)}[-\tfrac{99}{2}+\tfrac{165}{8}e^2][9+144e'^2]\}\,e^5e'\cos(5f\pm f'+i\phi)$$

$$+ \tfrac{1}{2} \{ K_1^{(i)} [\tfrac{1}{8} + \tfrac{1}{18} e^2] [\tfrac{1}{2} + e'^2] + K_2^{(i)} [\tfrac{5}{8} + \tfrac{1}{16} e^2] [\tfrac{3}{2} + \tfrac{9}{2} e'^2]$$
$$+ K_3^{(i)} [\tfrac{15}{8} - \tfrac{3}{8} e^2] [3 + \tfrac{25}{2} e'^2]$$
$$+ K_4^{(i)} [\tfrac{35}{8} - \tfrac{7}{4} e^2] [5 + \tfrac{55}{2} e'^2] + K_5^{(i)} [\tfrac{35}{4} - \tfrac{35}{8} e^2] [\tfrac{15}{2} + \tfrac{105}{2} e'^2]$$
$$+ K_6^{(i)} [\tfrac{63}{4} - \tfrac{83}{8} e^2] [\tfrac{21}{2} + 91 e'^2] + K_7^{(i)} [\tfrac{105}{4} - \tfrac{21}{2} e^2] [14 + 147 e'^2]$$
$$+ K_8^{(i)} [\tfrac{165}{4} - \tfrac{33}{4} e^2] [18 + 225 e'^2] \} e^4 e'^2 \cos (4f \pm 2f' + i\phi)$$
$$+ \tfrac{1}{2} \{ K_2^{(i)} [-1 + \tfrac{1}{8} e^2] [\tfrac{1}{4} + \tfrac{3}{4} e'^2] + K_3^{(i)} [-\tfrac{5}{2} + \tfrac{5}{16} e^2] [1 + 4e'^2]$$
$$+ K_4^{(i)} [-5 + \tfrac{5}{2} e^2] [\tfrac{5}{2} + \tfrac{205}{16} e'^2]$$
$$+ K_5^{(i)} [-\tfrac{35}{4} + \tfrac{35}{8} e^2] [5 + \tfrac{255}{8} e'^2] + K_6^{(i)} [-14 + \tfrac{21}{4} e^2] [\tfrac{35}{4} + \tfrac{1085}{16} e'^2]$$
$$+ K_7^{(i)} [-21 + \tfrac{21}{8} e^2] [14 + 259 e'^2]$$
$$+ K_8^{(i)} [-30 - \tfrac{15}{2} e^2] [21 + \tfrac{1827}{8} e'^2] \} e^3 e'^3 \cos (3f \pm 3f' + i\phi)$$
$$+ \tfrac{1}{2} \{ K_3^{(i)} [3 - \tfrac{3}{2} e^2] [\tfrac{1}{8} + \tfrac{1}{2} e'^2] + K_4^{(i)} [5 - \tfrac{5}{2} e^2] [\tfrac{5}{8} + \tfrac{25}{8} e'^2]$$
$$+ K_5^{(i)} [\tfrac{15}{2} + \tfrac{5}{2} e^2] [\tfrac{15}{8} + \tfrac{183}{16} e'^2] + K_6^{(i)} [\tfrac{21}{2} + 0 e^2] [\tfrac{35}{8} + \tfrac{511}{18} e'^2]$$
$$+ K_7^{(i)} [14 + 7e^2] [\tfrac{35}{4} + \tfrac{301}{4} e'^2]$$
$$+ K_8^{(i)} [18 + 21e^2] [\tfrac{83}{4} + \tfrac{631}{4} e'^2] \} e^2 e'^4 \cos (2f \pm 4f' + i\phi)$$
$$+ \tfrac{1}{2} \{ K_4^{(i)} [-4 + e^2] [\tfrac{1}{16} + \tfrac{5}{16} e'^2] + K_5^{(i)} [-5 - \tfrac{5}{4} e^2] [\tfrac{3}{8} + \tfrac{9}{4} e'^2]$$
$$+ K_6^{(i)} [-6 - 6e^2] [\tfrac{21}{16} + \tfrac{595}{64} e'^2] + K_7^{(i)} [-7 - 14e^2] [\tfrac{7}{2} + \tfrac{231}{8} e'^2]$$
$$+ K_8^{(i)} [-8 - 26e^2] [\tfrac{63}{8} + \tfrac{1197}{16} e'^2] \} e e'^5 \cos (f \pm 5f' + i\phi)$$
$$+ \quad \{ K_5^{(i)} [\tfrac{1}{32} + \tfrac{3}{32} e'^2] [1 + 5e^2] + K_6^{(i)} [\tfrac{7}{32} + \tfrac{49}{32} e'^2] [1 + \tfrac{9}{2} e^2]$$
$$+ K_7^{(i)} [\tfrac{7}{8} + \tfrac{113}{16} e'^2] [1 + 7e^2] + K_8^{(i)} [\tfrac{21}{8} + \tfrac{387}{18} e'^2] [1 + 10e^2] \} e'^6 \cos (6f' + i\phi)$$
$$+ \quad \{ K_1^{(i)} [-\tfrac{1}{64}] [1] + K_2^{(i)} [-\tfrac{1}{8}] [1] + K_3^{(i)} [-\tfrac{9}{16}] [1] + K_4^{(i)} [-\tfrac{15}{8}] [1]$$
$$+ K_5^{(i)} [-\tfrac{165}{32}] [1] + K_6^{(i)} [-\tfrac{99}{8}] [1] + K_7^{(i)} [-\tfrac{429}{16}] [1]$$
$$+ K_8^{(i)} [-\tfrac{429}{8}] [1] \} e^7 \cos (7f + i\phi)$$
$$+ \tfrac{1}{2} \{ K_1^{(i)} [\tfrac{1}{32}] [2] + K_2^{(i)} [\tfrac{7}{32}] [3] + K_3^{(i)} [\tfrac{7}{8}] [4] + K_4^{(i)} [\tfrac{21}{8}] [5] + K_5^{(i)} [\tfrac{105}{16}] [6]$$
$$+ K_6^{(i)} [\tfrac{231}{16}] [7] + K_7^{(i)} [\tfrac{231}{8}] [8] + K_8^{(i)} [\tfrac{429}{8}] [9] \} e^6 e' \cos (6f \pm f' + i\phi)$$
$$+ \tfrac{1}{2} \{ K_1^{(i)} [-\tfrac{1}{16}] [\tfrac{1}{2}] + K_2^{(i)} [-\tfrac{3}{8}] [\tfrac{3}{2}] + K_3^{(i)} [-\tfrac{21}{16}] [3] + K_4^{(i)} [-\tfrac{7}{2}] [5]$$
$$+ K_5^{(i)} [-\tfrac{63}{8}] [\tfrac{15}{2}] + K_6^{(i)} [-\tfrac{63}{4}] [\tfrac{21}{2}] + K_7^{(i)} [-\tfrac{231}{8}] [14]$$
$$+ K_8^{(i)} [-\tfrac{99}{2}] [18] \} e^5 e'^2 \cos (5f \pm 2f' + i\phi)$$
$$+ \tfrac{1}{2} \{ K_2^{(i)} [\tfrac{5}{8}] [\tfrac{1}{4}] + K_3^{(i)} [\tfrac{15}{8}] [1] + K_4^{(i)} [\tfrac{35}{8}] [\tfrac{5}{2}] + K_6^{(i)} [\tfrac{63}{4}] [\tfrac{35}{4}]$$
$$+ K_7^{(i)} [\tfrac{105}{4}] [14] + K_8^{(i)} [\tfrac{185}{4}] [21] \} e^4 e'^3 \cos (4f \pm 3f' + i\phi)$$
$$+ \tfrac{1}{2} \{ K_3^{(i)} [-\tfrac{5}{2}] [\tfrac{1}{8}] + K_4^{(i)} [-5] [\tfrac{5}{8}] + K_5^{(i)} [-\tfrac{35}{4}] [\tfrac{15}{8}] + K_6^{(i)} [-14] [\tfrac{35}{8}]$$
$$+ K_7^{(i)} [-21] [\tfrac{35}{4}] + K_8^{(i)} [-30] [\tfrac{63}{4}] \} e^3 e'^4 \cos (3f \pm 4f' + i\phi)$$
$$+ \tfrac{1}{2} \{ K_4^{(i)} [5] [\tfrac{1}{16}] + K_5^{(i)} [\tfrac{15}{2}] [\tfrac{3}{8}] + K_6^{(i)} [\tfrac{21}{2}] [\tfrac{21}{16}] + K_7^{(i)} [14] [\tfrac{7}{2}]$$
$$+ K_8^{(i)} [18] [\tfrac{63}{8}] \} e^2 e'^5 \cos (2f \pm 5f' + i\phi)$$
$$+ \tfrac{1}{2} \{ K_5^{(i)} [-5] [\tfrac{1}{32}] + K_6^{(i)} [-6] [\tfrac{7}{32}] + K_7^{(i)} [-7] [\tfrac{7}{8}]$$
$$+ K_8^{(i)} [-8] [\tfrac{21}{8}] \} e e'^6 \cos (f \pm 6f' + i\phi)$$
$$+ \quad \{ K_6^{(i)} [\tfrac{1}{64}] [1] + K_7^{(i)} [\tfrac{1}{8}] [1] + K_8^{(i)} [\tfrac{9}{16}] [1] \} e'^7 \cos (7f' + i\phi)$$

$$
\begin{aligned}
+ \quad & \Big\{ K_1^{(i)} \left[\tfrac{1}{128}\right] [1] + K_2^{(i)} \left[\tfrac{9}{128}\right] [1] + K_3^{(i)} \left[\tfrac{45}{128}\right] [1] \\
& \qquad + K_4^{(i)} \left[\tfrac{165}{128}\right] [1] + K_5^{(i)} \left[\tfrac{495}{128}\right] [1] \\
& + K_6^{(i)} \left[\tfrac{1287}{128}\right] [1] + K_7^{(i)} \left[\tfrac{3003}{128}\right] [1] + K_8^{(i)} \left[\tfrac{6435}{128}\right] [1] \Big\} e^8 \cos (8f + i\phi) \\
+ \tfrac{1}{2} & \Big\{ K_1^{(i)} \left[-\tfrac{1}{64}\right] [2] + K_2^{(i)} \left[-\tfrac{1}{8}\right] [3] + K_3^{(i)} \left[-\tfrac{9}{16}\right] [4] + K_4^{(i)} \left[-\tfrac{15}{8}\right] [5] \\
& + K_5^{(i)} \left[-\tfrac{165}{32}\right] [6] + K_7^{(i)} \left[-\tfrac{99}{8}\right] [7] + K_7^{(i)} \left[-\tfrac{429}{16}\right] [8] \\
& \qquad + K_8^{(i)} \left[-\tfrac{429}{8}\right] [9] \Big\} e^7 e' \cos (7f \pm f' + i\phi) \\
+ \tfrac{1}{2} & \Big\{ K_1^{(i)} \left[\tfrac{1}{32}\right] \left[\tfrac{1}{2}\right] + K_2^{(i)} \left[\tfrac{7}{32}\right] \left[\tfrac{3}{2}\right] + K_3^{(i)} \left[\tfrac{7}{8}\right] [3] + K_4^{(i)} \left[\tfrac{21}{8}\right] [5] \\
& + K_5^{(i)} \left[\tfrac{105}{16}\right] \left[\tfrac{15}{2}\right] + K_6^{(i)} \left[\tfrac{231}{16}\right] \left[\tfrac{21}{2}\right] + K_7^{(i)} \left[\tfrac{231}{8}\right] [14] \\
& \qquad + K_8^{(i)} \left[\tfrac{429}{8}\right] [18] \Big\} e^6 e'^2 \cos (6f \pm 2f' + i\phi) \\
+ \tfrac{1}{2} & \Big\{ K_2^{(i)} \left[-\tfrac{3}{8}\right] \left[\tfrac{1}{4}\right] + K_3^{(i)} \left[-\tfrac{21}{16}\right] [1] + K_4^{(i)} \left[-\tfrac{7}{2}\right] \left[\tfrac{5}{2}\right] \\
& + K_6^{(i)} \left[-\tfrac{63}{8}\right] [5] + K_6^{(i)} \left[-\tfrac{63}{4}\right] \left[\tfrac{35}{4}\right] + K_7^{(i)} \left[-\tfrac{231}{8}\right] [14] \\
& \qquad + K_8^{(i)} \left[-\tfrac{99}{2}\right] [21] \Big\} e^5 e'^3 \cos (5f \pm 3f' + i\phi) \\
+ \tfrac{1}{2} & \Big\{ K_3^{(i)} \left[\tfrac{1}{8}\right] \left[\tfrac{1}{8}\right] + K_4^{(i)} \left[\tfrac{6}{8}\right] \left[\tfrac{5}{8}\right] + K_5^{(i)} \left[\tfrac{16}{8}\right] \left[\tfrac{16}{8}\right] + K_6^{(i)} \left[\tfrac{36}{8}\right] \left[\tfrac{35}{8}\right] \\
& + K_7^{(i)} \left[\tfrac{35}{4}\right] \left[\tfrac{35}{4}\right] + K_8^{(i)} \left[\tfrac{63}{4}\right] \left[\tfrac{63}{4}\right] \Big\} e^4 e'^4 \cos (4f \pm 4f' + i\phi) \\
+ \tfrac{1}{2} & \Big\{ K_4^{(i)} [-5] \left[\tfrac{1}{16}\right] + K_5^{(i)} \left[-\tfrac{35}{4}\right] \left[\tfrac{3}{8}\right] + K_6^{(i)} [-14] \left[\tfrac{21}{16}\right] \\
& + K_7^{(i)} [-21] \left[\tfrac{7}{2}\right] + K_8^{(i)} [-30] \left[\tfrac{63}{8}\right] \Big\} e^3 e'^5 \cos (3f \pm 5f' + i\phi) \\
+ \tfrac{1}{2} & \Big\{ K_5^{(i)} \left[\tfrac{16}{2}\right] \left[\tfrac{1}{32}\right] + K_6^{(i)} \left[\tfrac{21}{2}\right] \left[\tfrac{7}{32}\right] + K_7^{(i)} [14] \left[\tfrac{7}{8}\right] \\
& \qquad + K_8^{(i)} [18] \left[\tfrac{21}{8}\right] \Big\} e^2 e'^6 \cos (2f \pm 6f' + i\phi) \\
+ \tfrac{1}{2} & \Big\{ K_6^{(i)} [-6] \left[\tfrac{1}{64}\right] + K_7^{(i)} [-7] \left[\tfrac{1}{8}\right] - K_8^{(i)} [-8]]\tfrac{9}{16}] \Big\} e e'^7 \cos (f \pm 7f' + i\phi) \\
+ \quad & \Big\{ K_7^{(i)} \left[\tfrac{1}{128}\right] [1] + K_8^{(i)} \left[\tfrac{9}{128}\right] [1] \Big\} e'^8 \cos (8f' + i\phi \Big\}.
\end{aligned}
$$

MEMOIR No. 83.

On the Construction of Maps.

(This Memoir appears here for the first time.)

Maps being used for a great variety of purposes, many different methods of projecting them may be admitted; but when the chief end is to present to the eye a picture of what appears on the surface of the earth, we should limit ourselves to projections which are conformable. And, as the construction of the reseau of meridians and parallels is, except in maps of small regions, an important part of the labor involved, it should be composed of the most easily drawn curves. Accordingly, in a well known memoir* Lagrange recommended circles for this purpose, in which the straight line is included as being a circle whose centre is at infinity.

The circles which represent the meridians must then all pass through the projected points of the poles. The parallels must be represented by a system of circles intersecting the former orthogonally. It is plain the centres of the latter system of circles all lie in the straight line which is the projection of one of the meridians. If the projection of a parallel is to pass through a given point on a projected meridian the centre of the circle is found by the intersection of the tangent to the meridian at the point with the rectilinear meridian, and thus the parallel in question can be drawn. However, in practice, we should not depend on the graphical construction, employing rather the simple trigonometric formula for getting the length of the radius.

When we consider the utmost variety in this mode of projection, we see that it is readily divided into three species, characterized thus:

I. The projection of neither pole at infinity,

II. The projection of one pole only at infinity,

III. The projection of both poles at infinity.

In the first species both meridians and parallels are represented in general by circles, the exceptions being one projected meridian and one projected parallel having each an infinite radius. In the second species the

---

* *Oeuvres*, Tom. IV, p. 635. As far as possible the notation adopted here is identical with that of Lagrange.

meridians are projected into straight lines all passing through the projected pole and the latter point is the centre of all the circles forming the projections of the parallels. In the third species the projected meridians form a system of parallel straight lines and the projected parallels another system of parallel straight lines intersecting the former orthogonally. The three species are exemplified by the following three figures:

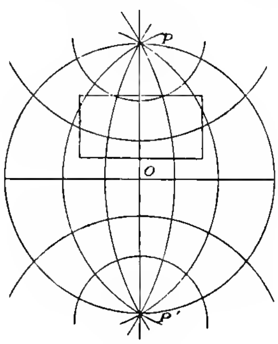


FIG. 1.

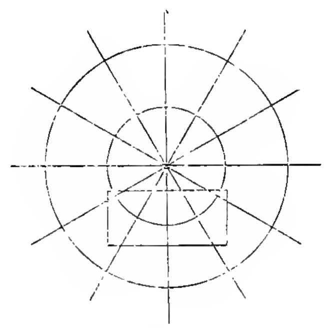


FIG. 2.

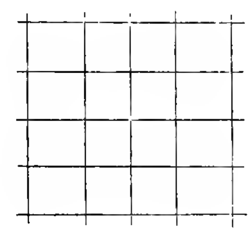


FIG. 3.

In Fig. 1, exhibiting the first species of our mode of projection, the projected meridians all pass through the projections of the poles P and P'. The straight line PP' is the projected meridian which has an infinite radius. If we bisect the distance PP' and at the point of bisection draw the perpendicular straight line we have the projected parallel which has an infinite radius. Along this line are situated the centres of all the circles forming the projected meridians. If δ denote half the distance PP', ω the angle which the projected meridian makes at either pole with the rectilinear meridian and d the distance of the centre of the representing circle from the mentioned point of bisection, we have

$$d = \frac{\delta}{\tan \omega}.$$

The graphical construction of this is apparent at once.

The centres of the circles representing parallels of latitude are situated on the rectilinear projected meridian. If we adopt a system of rectangular coordinates with P for origin and the rectilinear projected meridian as axis of x, the equation of the projected meridian which makes a right angle at P with the axis of x is

$$(x-\delta)^2 + y^2 = \delta^2.$$

Let it be proposed to draw the projected parallel which crosses the axis of x at a distance ξ from P, and let b denote the distance from P of the centre

of this projected parallel measured in the opposite direction. Then the equation of this parallel will be

$$(x+b)^2+y^2=(b+\xi)^2.$$

The intersection of these two circles must be orthogonal. Calling, therefore, the left members of the two equations U and U', this demands that

$$\frac{\partial U}{\partial x}\frac{\partial U'}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial U'}{\partial y}=0,$$

or that

$$(x-\delta)\,(x+b)+y^2=0.$$

We may take x, y, b as the unknowns in the three equations, and their solution gives

$$b=\frac{\xi^2}{2(\delta-\xi)},$$

or the proportion

$$2(\delta-\xi):\xi::\xi:b$$

has place, and from this b may be readily constructed. When ξ exceeds δ, b is negative and must therefore be measured from P in the opposite direction. When $\xi=\delta$ we have the rectilinear projected parallel.

Figures 2 and 3 do not require any explanation as to the mode of constructing them.

We now have it in our power to draw as many meridians and parallels as we choose, but we have said nothing as to how they correspond with the like curves on the surface of the earth. It will be seen that this question has not an absolute reply but depends on the interpretation of the formulæ involved which admits considerable latitude. Let us suppose that the whole surface of the map is divided into infinitesimal rectangles by drawing the meridians and parallels sufficiently near to each other. If we fix the correspondence in any way which permits the sides of these infinitesimal rectangles to have, all over the map, the proportions indicated by the equation of the earth's surface, the projection is a conformable one. It will be seen that, setting aside a constant multiplying all the formulæ and which indicates the general scale of the map, in Species I we have two arbitrary constants at our disposal, in Species II only one, while there are none in Species III.

The principal object of this memoir is to advocate the giving to these disposable constants such values as shall reduce as much as possible the variation of scale throughout the map.

Let ϕ denote the normal or geographical latitude, ρ the radius of the parallel and z the perpendicular distance of the point considered from the plane of the equator and a and $b = a\sqrt{1-\varepsilon^2}$ the equatorial and polar semi-diameters of the earth, the equation of the earth's surface, free from λ the longitude, is

$$\frac{\rho^2}{a^2} + \frac{z^2}{b^2} = 1.$$

This equation is satisfied by putting

$$\rho = \frac{a \cos\phi}{\sqrt{1-\varepsilon^2 \sin^2\phi}}, \quad z = \frac{a(1-\varepsilon^2)\sin\phi}{\sqrt{1-\varepsilon^2\sin^2\phi}},$$

where it is easy to be convinced that ϕ has the geometrical definition assigned to it. The length of a degree along the parallel belonging to latitude ϕ is (we adopt the degree as the unit of angular measure)

$$\frac{\pi}{180}\frac{a\cos\phi}{\sqrt{1-\varepsilon^2\sin^2\phi}}.$$

By differentiating we obtain

$$\frac{d\rho}{d\phi} = -\frac{a(1-\varepsilon^2)\sin\phi}{(1-\varepsilon^2\sin^2\phi)^{\frac{3}{2}}}, \quad \frac{dz}{d\phi} = \frac{a(1-\varepsilon^2)\cos\phi}{(1-\varepsilon^2\sin^2\phi)^3}.$$

The element along the meridian ds is then given by the equation

$$\frac{ds}{d\phi} = \sqrt{\left(\frac{d\rho}{d\phi}\right)^2 + \left(\frac{dz}{d\phi}\right)^2} = \frac{a(1-\varepsilon^2)}{(1-\varepsilon^2\sin^2\phi)^{\frac{3}{2}}}.$$

And the length of a degree along the meridian which belongs to the latitude ϕ is

$$\frac{\pi}{180}\frac{a(1-\varepsilon^2)}{(1-\varepsilon^2\sin^2\phi)^{\frac{3}{2}}}.$$

These two quantities are important in map construction. In computing them Clark's dimensions of the earth may be used which are

$a = 20\ 926\ 202$ feet $= 3963.295833$ miles.
$b = 20\ 854\ 895$ " $= 3949.790720$ " ,
$\varepsilon = 0.082483218.$

In establishing the formulæ of correspondence of the curves on the map with those on the surface of the earth, in case the projection is to be conformable, it is indispensable to have an auxiliary variable which with Lagrange we denote by θ. This satisfies the differential equation

$$\frac{d\theta}{\theta} = \frac{ds}{\rho}$$

In the following prosecution of the subject it is more suitable to employ the colatitude z rather than ϕ the latitude. We suppose that s is counted from the pole towards the equator. Then the differential equation being integrated on the condition that θ and z are to begin together, we have

$$\theta = \left(\frac{1+\varepsilon\cos z}{1-\varepsilon\cos z}\right)^{\frac{\varepsilon}{2}} \tan\frac{z}{2}.$$

Having now this auxiliary variable the distance southward from the projected north pole to the point where the parallel of z crosses the rectilinear meridian is given by the formula

$$\xi = \frac{2\delta\theta^c}{g+\theta^c},$$

δ, c and g are arbitrary constants. This equation shows the correspondence between parallels on the map and those on the earth's surface.

With regard to the meridians, if we denote longitudes on the earth's surface, counted from any meridian eastward as of one sign and westward as of the opposite, by λ, and suppose that the rectilinear meridian on the map corresponds to the principal meridian, and denote by ω the angle the projected meridian makes at either pole with the rectilinear meridian, the formula of correspondence is

$$\omega = c\lambda.$$

These two formulæ embody the whole theory of conformable projection in the case we treat. If they are substituted in the differential equations which belong to this subject the latter will be found to be satisfied; and a little additional consideration will make it evident that they are as general as possible. As Lagrange has shown c may be limited to positive numbers. Also, if the crossing of the parallel given by the quantity ξ is limited to the one which is between the two poles, g is also positive.

The projected coordinates measured southward from the north pole as origin are given by the equations

$$x = 2\delta\frac{g\theta^c\cos(c\lambda)+\theta^{2c}}{g^2+2g\theta^c\cos(c\lambda)+\theta^{2c}},$$

$$y = 2\delta\frac{g\theta^c\sin(c\lambda)}{g^2+2g\theta^c\cos(c\lambda)+\theta^{2c}}.$$

And the scale which belongs to any point of the map is given by the formula

$$m = \frac{2cg\delta}{a}\frac{\sqrt{1-\varepsilon^2\cos^2 z}}{\sin z}\frac{\theta^c}{g^2+2g\theta^c\cos(c\lambda)+\theta^{2c}}.$$

The right member takes the indeterminate form $\frac{0}{0}$ when $z = 0$, but this defect is remedied by dividing numerator and denominator by $\tan \frac{z}{2}$.

The formulæ just given are adapted to Species I. In Species II g and δ are both infinite, but their ratio is finite. Hence f being a new constant we have

$$\xi = f\theta^c.$$

The equation for correspondence of longitudes remains unchanged. The scale becomes

$$m = \frac{cf}{a} \frac{\sqrt{1 - \varepsilon^2 \cos^2 z}}{\sin z} \theta^c.$$

In Species III, since both poles are at infinity, we are under the necessity of measuring distances from another point. The equator serves this purpose in a symmetrical manner. This projection, generally known as Mercator's, is characterized by the condition $c = 0$. The formulæ for this Species are easily derived from those given for the first Species by making c infinitesimal. Those handy for use are

$$y = B\lambda, \quad x = -\frac{180}{\pi M} B \log \theta,$$

where λ is in degrees, B depends simply on the scale of the map, M the modulus of common logarithms, $\log \theta$ denoting the common logarithm of θ; common $\log\left(\frac{180}{\pi M}\right) = 2.1203383211$. The scale at any point of the map is the scale at the equator multiplied by the factor

$$\frac{\sqrt{1 - \varepsilon^2 \cos^2 z}}{\sin z}.$$

With regard to the scale m it must be noticed that δ and f are linear magnitudes. In the formulæ for m therefore δ and a or f and a must be expressed in the same linear unit. We thus obtain the fraction the scale of the map is to nature; it is customary to take unity for the numerator of this fraction. If one prefers to say the scale is so many miles to the inch the denominator of the fraction must be divided by 63360 the number of inches in a mile.

We come now to the question of advisable values for the disposable constants in the formulæ. In Species I, these are two, viz., g and c. In this connection the principle of Tchebicheff* that a map constructed on a

* *Sur la construction des cartes géographiques.* Acad. Sci. St. Pétersbourg. Bull. XIV, cols. 257-261, 1856.

conformable projection is the best possible when the scale is constant along the whole boundary of the map seems the most satisfactory that has been suggested. It differs, it is true, from the few hints thrown out by Lagrange, who proposed the notion that the variation of the scale should be zero in all directions about the point forming the centre of the map.

On account of the form of the pages of our books the vast number of published maps are bounded by rectangles. When the meridians and parallels are to be represented by circles, the principle of Tchebicheff cannot here be fulfilled, nevertheless we may have an approximation to it. I propose that the scale shall be the same at the middle points of the four sides of the rectangle bounding the map. This gives two independent equations for the determination of g and c.

Before we give the equations notice must be called to the circumstance that the poles are singular points in the projection, the meridians in the map in general do not meet at the poles under the same angles as their correspondents on the globe. They do this only when the exponent c of projection has unity for its value. If c exceeds unity there will be a lap over at the poles, and, if it falls short of it, a hiatus. This is no incongruity to the mathematician, but nevertheless offensive to the common eye. Therefore we must lay down the principle that if either or both poles are to be shown in a map we must put $c = 1$, that is, adopt the stereographic projection. However, the great number of maps do not show either pole; here we are at liberty to choose the value of c. In the following discussion we limit ourselves to this case; moreover we suppose that the rectilinear meridian divides the map into two symmetrical halves.

Let us adopt the subscripts (0), (1) and (2) to distinguish quantities belonging to the middle points of the upper, lower, and lateral lines bounding the map. We suppose that, before the map is constructed, we have chosen the values for z_0, z_1 and z_2 the colatitudes belonging to the mentioned points. It is true that in the case of z_2 it is generally difficult to imagine what should be its value in order that a definite region may just fill the contour of the map; but the difficulty of forming and solving the transcendental equations is greater.

Preceding equations give the following relations:

$$\xi_0 = \frac{2\delta\theta_0^c}{g+\theta_0}, \quad \xi_1 = \frac{2\delta\theta_1^c}{g+\theta_1}, \quad \xi_2 = \frac{\xi_0+\xi_1}{2} = \delta\left(\frac{\theta_0^c}{g+\theta_0} + \frac{\theta_1^c}{g+\theta_1}\right).$$

For brevity let us put

$$m_0^2 = \frac{\sin z_0}{\sqrt{1-\varepsilon^2\cos^2 z_0}}, \quad m_1^2 = \frac{\sin z_1}{\sqrt{1-\varepsilon^2\cos^2 z_1}}, \quad m_2^2 = \frac{\sin z_2}{\sqrt{1-\varepsilon^2\cos^2 z_2}},$$

which are all known quantities. Comparison of the values of m at the three points gives

$$m_0^2 \frac{(g + \theta_0^c)^2}{\theta_0^c} = m_1^2 \frac{(g + \theta_1^c)^2}{\theta_1^c} = m_2^2 \frac{g^2 + 2g\theta_2^c \cos(c\lambda_2) + \theta_2^{2c}}{\theta_2^c}.$$

But λ_2 can be eliminated from the last member by means of the equation

$$2\,\frac{g\theta_2^c \cos(c\lambda_2) + \theta_2^{2c}}{g^2 + 2g\theta_2^c \cos(c\lambda_2) + \theta_2^{2c}} = \frac{\theta_0^c}{g + \theta_0^c} + \frac{\theta_1^c}{g + \theta_1^c},$$

which affords

$$g^2 + 2g\theta_2^c \cos(c\lambda_2) + \theta_2^{2c} = \frac{(g + \theta_0^c)\,(g + \theta_1^c)\,(g^2 - \theta_2^{2c})}{g^2 - \theta_0^c\,\theta_1^c}$$

For convenience let us adopt

$$g = \eta\theta_0^c, \quad \frac{\theta_1}{\theta_0} = h_1, \quad \frac{\theta_2}{\theta_0} = h_2$$

Then the comparison of the values of m becomes

$$m_0^2(\eta + 1)^2 = m_1^2 \frac{(\eta + h_1^c)^2}{h_1^c} = m_2^2 \frac{(\eta + 1)\,(\eta + h_1^c)\,(\eta^2 - h_2^{2c})}{h_2^c(\eta^2 - h_1^c)}.$$

The first of these relations gives

$$\eta = \frac{m_0 h_1^{\frac{c}{2}} - m_1 h_1^c}{m_1 - m_0 h_1^{\frac{c}{2}}}, \quad \text{and } g = \theta_0^c\,\frac{m_0 h_1^{\frac{c}{2}} - m_1 h_1^c}{m_1 - m_0 h_1^{\frac{c}{2}}}.$$

The second relation will give

$$m_0^2(\eta + 1) = m_2^2 \frac{(\eta + h_1^c)\,(\eta^2 - h_2^{2c})}{h_2^c(\eta^2 - h_1^c)}.$$

Substituting in this the just obtained value of η we get

$$\eta + 1 = \frac{m_1(1 - h_1^c)}{m_1 - m_0 h_1^{\frac{c}{2}}}, \quad \eta + h_1^c = \frac{m_0 h_1^{\frac{c}{2}}(1 - h_1^c)}{m_1 - m_0 h_1^{\frac{c}{2}}},$$

consequently

$$m_0 m_1 = m_2^2 h_1^{\frac{c}{2}}\,\frac{\eta^2 - h_2^{2c}}{h_2^c(\eta^2 - h_1^c)}.$$

From this is derived

$$\eta^2 = \frac{m_2^2 h_2^{2c} - m_0\,m_1\,h_2^c\,h_1^{\frac{c}{2}}}{m_2^2 - m_0\,m_1\,h_2^c\,h_1^{-\frac{c}{2}}}.$$

Comparison of this with the former value of η gives us

$$\left(\frac{m_0 - m_1 h_1^{\frac{c}{2}}}{m_1 - m_0 h_1^{\frac{c}{2}}}\right)^2 = \frac{m_0 m_1 - m_2^2 h_2^c h_1^{-\frac{c}{2}}}{m_0 m_1 - m_2^2 h_2^{-c} h_1^{\frac{c}{2}}},$$

an equation from which c can be determined by the tentative process, and g follows from its value just given.

In Species II, as we have only one constant at our disposal, viz., c, we determine it from the condition that the scale should be the same at the ends of the central meridian of the map. That is, the equation

$$\frac{\sqrt{1 - \varepsilon^2 \cos^2 z_0}}{\sin z_0} \theta_0^c = \frac{\sqrt{1 - \varepsilon^2 \cos^2 z_1}}{\sin z_1} \theta_1^c$$

must be fulfilled; whence we have for the determination of c the exponential equation

$$h_1^c = \frac{\sin z_1}{\sin z_0} \sqrt{\frac{1 - \varepsilon^2 \cos^2 z_0}{1 - \varepsilon^2 \cos^2 z_1}}.$$

In all the trials I have made of the Tchebicheff principle, c has never turned out greater than unity. One example may be given for the sake of illustration. Desiring to construct a map of South America, and counting z from the South Pole, it was supposed that the limits given by the values $z_0 = 34°$, $z_1 = 102°$, $z_2 = 73°$ would include the whole of this continent in a rectangular map. These data conduct to

$$\log \theta_0 = 9.4866077, \quad \log h_1 = 0.6056371, \quad \log h_2 = 0.3834652$$

$$\log \frac{m_1}{m_0} = 0.1209441, \quad \log \frac{m_2^2}{m_0 m_1} = 0.1111990.$$

Thence there is concluded

$$c = 0.945576, \quad \log g = 0.2054407.$$

It is very evident that if the region to be included in a rectangular map is of nearly equal extension in the directions of longitude and latitude c will be found to be pretty close to unity. And c becomes smaller as the map deviates more from the square. It is of interest to determine what c becomes when the dimension in the direction of longitude is reduced to the last degree. In this case we do not know the value of z_2 and an extra

equation must be brought in to render the problem determinate. The comparison of scales gives the equations

$$m_0^2\frac{(g+\theta_0^c)^2}{\theta_0^c}=m_1^2\frac{(g+\theta_1^c)^2}{\theta_1^c}=m_2^2\frac{(g+\theta_2^c)^2}{\theta_2^c}.$$

The extra equation demanded is

$$\frac{\theta_0^c}{g+\theta_0^c}+\frac{\theta_1^c}{g+\theta_1^c}=2\frac{\theta_2^c}{g+\theta_2^c}.$$

We eliminate g from these equations by means of its value

$$g=\theta_0^c\frac{m_0h_1^{\frac{c}{2}}-m_1h_1^c}{m_1-m_0h_1^{\frac{c}{2}}}.$$

The two equations which remain are

$$\frac{m_0+m_1h_1^{\frac{c}{2}}}{h_1^{\frac{c}{2}}\theta_0^c}=2\frac{m_2}{\theta_2^c},$$

$$m_0m_1(1-h_1^c)\,\theta_2^c=m_2\left[\theta_2^c(m_1-m_0h_1^{\frac{c}{2}})+\theta_0^c(m_0h_1^{\frac{c}{2}}-m_1h_1^c)\right].$$

It would be troublesome to pursue the elimination further, but having a suspicion that the proper value of c in this extreme case is zero, we substitute this value in the second equation and it is rendered an identity. The first equation then gives for the determination of z_2 the relation

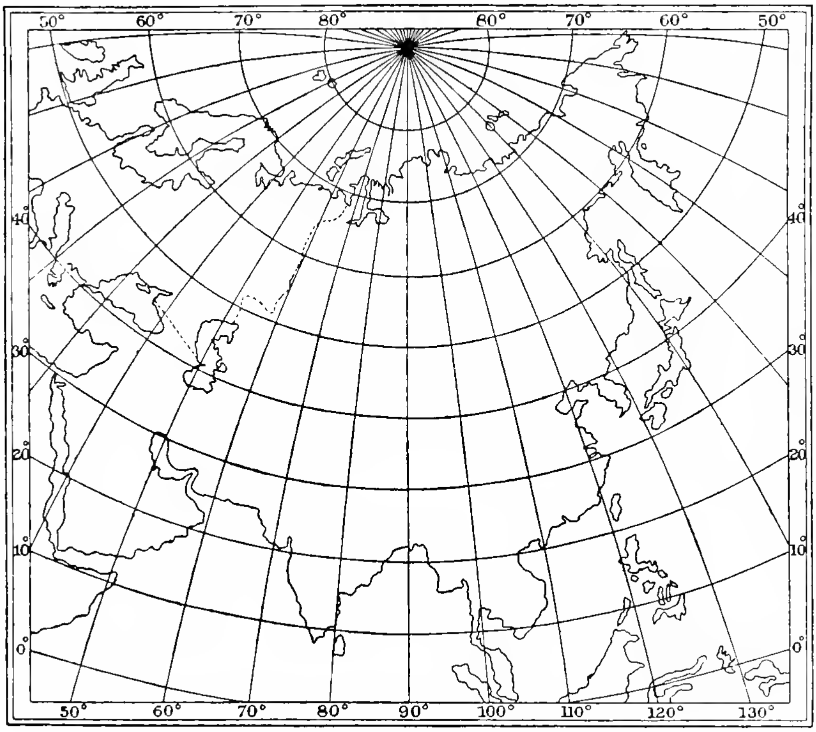
$$\frac{\sin z_2}{\sqrt{1-\varepsilon^2\cos^2 z_2}}=\tfrac{1}{4}(m_0+m_1)^2.$$

To inquire what the Tchebicheff principle gives when the dimension of the map in the direction of latitude is diminished to the last degree is quite unnecessary as it is obvious that such a map could only include a segment of the equator, when the projection that is proper is Mercator's. If, however, we are willing to depart from the rectangular boundary for a map, we can have one for an extremely narrow strip along a parallel by cutting it from the developed surface of the cone which envelops the earth along that parallel. This map would have the form of a segment of an annulus. In like manner the map of an extremely narrow strip along a meridian can be got from the surface of the cone to elliptical base which envelops the earth along that meridian. In the annular map along a parallel, the exponent of projection, neglecting ε, is $c=\cos z$.

All that has been said regarding terrestrial maps will apply as well to maps of the stars, provided we make $\varepsilon=0$, and, as the notion of a scale in

terrestrial maps is altogether out of place in celestial maps, we substitute the phrase "so many degrees to the inch."

As the reseaus of meridians and parallels in maps of large regions in all our atlases are unpleasant to the eye of a mathematician, I append a map of Asia constructed on the stereographic projection. On investigation it was found, in accordance with the Tchebicheff principle, that c in this instance differed from unity inappreciably.



MAP OF ASIA.

MEMOIR No. 84.

Dynamic Geodesy.

(This Memoir appears here for the first time.)

The formulæ which have been proposed for representing the intensity and direction of gravity over the earth's surface do not well satisfy the observations. It seems that more complex formulæ are needed. The notion that all observations should be reduced to sea level has long prevailed. It is not necessary to regard the surface of the ocean as forming a more natural boundary to the mass of the earth than the land-surface. The endeavor in this memoir is to elaborate a theory of gravity which dispenses with the idea that we must know the elevation of the place of observation before we can make comparisons. The boundary of the earth's mass may be looked upon as a surface oft-times discontinuous but nevertheless having an equation, so that to a given longitude and latitude always corresponds a unique point on that surface. Thus a table to double entry might be formed giving to the arguments longitude and latitude the three rectangular coordinates of the point with reference to the earth's centre of gravity. Of course the earth thus viewed must have no precipices or caves. But the actual precipices and caves of the earth have only the slightest influence on gravity, and the first at least can easily be avoided by pendulum observers.

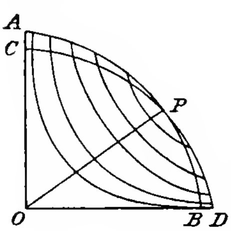
In order to have a starting point for the construction of a theory, an ideal earth is conceived; its mass is equivalent to that of the actual earth, the mass of the ocean and atmosphere being omitted, its boundary is a sphere and it is centrobaric. Its radius is equal to that of the actual earth in the latitude whose sine is $\frac{1}{\sqrt{3}}$, so that the volumes are nearly equal. The intensity of gravity is constant over the surface of this sphere. In all numerical work the radius of this sphere will be taken as the linear unit, and the temporal unit is so taken that its mass will be a unit; consequently the value of gravity over the surface is also unity. We proceed now to notice the deviations of the actual from the ideal earth. The most important of these are produced by the earth's rotation.

PART I.

The Effects of the Earth's Rotation on its Figure and on Gravity.

Were the matter of the earth in a state of perfect fluidity no asperities could exist on its surface nor any irregularities of density in its interior. The latter would be a function of the pressure and temperature. On the other hand, were the matter absolutely rigid, hydrostatics could not be appealed to for the decision of its figure. However, it is plain that the earth leans towards rigidity sufficiently for the preservation of the asperities and irregularities of density and is sufficiently viscous for hydrostatics to furnish a fair approximation to its figure.

If we suppose our ideal earth to receive a motion of rotation about a fixed axis the consequence will be that the matter will move away from the axis in the mode shown in the adjacent diagram. Let AO be half the axis



and $APBO$ a quadrant of a meridian section of the ideal earth. The rotation will cause the curve APB to fall into the position CPD. Five stream lines are drawn to exhibit the direction of motion of the matter. The motion ceases when the gravity becomes normal to the surface, or if by reason of momentum, it overshoots this mark, there is a tendency to return, and friction may be supposed to bring about a permanent state. The stream lines have been constructed from the cubic $x^2y = c^3$, (x being in the plane of the equator). This curve has the axes of coordinates for asymptotes and gives a node at P, the sine of whose latitude is $\frac{1}{\sqrt{3}}$. There is no motion of the matter at P. This is sufficiently accurate to give a rude notion of what occurs.

We now proceed to the determination of the permanent state of the earth. As the earth is evidently symmetric in regard to its axis of rotation, we are exempt from considering longitudes; thus calling the radius r, the sine of the latitude μ, and $\breve{V}$ the potential for points not interior to the mass, we have the partial differential equation

$$\frac{\partial \cdot r^2 \frac{\partial \breve{V}}{\partial r}}{\partial r} + \frac{\partial \cdot (1-\mu^2)\frac{\partial \breve{V}}{\partial \mu}}{\partial \mu} = 0.$$

For points interior to the mass we assume that the potential includes the centrifugal force, which adds to it the term $\frac{2\pi^2}{T^2} r^2 (1-\mu^2)$, if T is the time of

revolution. After this addition calling it $\hat{V}$, the equation of Poisson is transformed into

$$\frac{\partial \cdot r^2 \frac{\hat{V}}{\partial r}}{\partial r} + \frac{\partial \cdot (1-\mu^2)\frac{\partial \hat{V}}{\partial \mu}}{\partial \mu} + 4\pi\left(\rho - \frac{2\pi}{T^2}\right)r^2 = 0,$$

where ρ denotes the density. For the latter we adopt the value given by the differential equation $d\rho = \frac{m^2}{4\pi}\frac{dp}{\rho}$, p being the pressure, and m a constant. Substituting for dp its value ρdV and integrating, we have $\rho = \frac{m^2}{4\pi}V +$ a constant. Here we may modify the signification of $\hat{V}$ by including in it a constant such that

$$4\pi\left(\rho - \frac{2\pi}{T^2}\right) = m^2\hat{V}.$$

Thus the differential equation becomes

$$\frac{\partial \cdot r^2 \frac{\partial \hat{V}}{\partial r}}{\partial r} + \frac{\partial \cdot (1-\mu^2)\frac{\partial \hat{V}}{\partial \mu}}{\partial \mu} + m^2 r^2 \hat{V} = 0.$$

These two partial differential equations are in terms of polar coordinates, but they are briefer and more suitable to our purposes if we express them in terms of rectangular coordinates in the plane of the meridian. Thus making

$$r^2 = x^2 + y^2, \qquad \mu = \frac{y}{\sqrt{x^2+y^2}},$$

we have

$$\frac{\partial^2 \breve{V}}{\partial x^2} + \frac{\partial^2 \breve{V}}{\partial y^2} + \frac{1}{x}\frac{\partial \breve{V}}{\partial x} = 0, \tag{1}$$

$$\frac{\partial^2 \hat{V}}{\partial x^2} + \frac{\partial^2 \hat{V}}{\partial y^2} + \frac{1}{x}\frac{\partial \hat{V}}{\partial x} + m^2\hat{V} = 0. \tag{2}$$

Let q be used for $\frac{2\pi^2}{T^2}$ and $\bar{V} = \breve{V} + qx^2$. For determining gravity at points on the surface of the earth it is indifferent whether we differentiate partially $\hat{V}$ or $\bar{V}$. Hence the equations

$$\frac{\partial(\hat{V} - \bar{V})}{\partial x} = 0, \quad \frac{\partial(\hat{V} - \bar{V})}{\partial y} = 0,$$

are satisfied by these points. Hydrostatics furnishes the additional equations

$$\hat{V} = \text{a constant}, \qquad \bar{V} = \text{a constant},$$

which are also satisfied by the same points.

The first thing to be done in the treatment of the question before us is to integrate (1) and (2). Here recourse must be had to infinite series. In regard to (1) it will readily be found that it is satisfied by each of the following expressions:

$$\begin{aligned}
\breve{V}_0 &= (x^2 + y^2)^{-\frac{1}{2}}, \\
\breve{V}_1 &= (x^2 - 2y^2)(x^2 + y^2)^{-\frac{5}{2}}, \\
\breve{V}_2 &= (x^4 - 8x^2y^2 + \tfrac{8}{3}y^4)(x^2 + y^2)^{-\frac{9}{2}}, \\
\breve{V}_3 &= (x^6 - 18x^4y^2 + 24x^2y^4 - \tfrac{16}{5}y^6)(x^2 + y^2)^{-\frac{13}{2}}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

which may be continued as far as we please. As the differential equation is linear, taking a series of constants b_0, b_1, . . . , a more general integral will be

$$\breve{V} = b_0\breve{V}_0 + b_1\breve{V}_1 + b_2\breve{V}_2 + b_3\breve{V}_3 + \dots \tag{3}$$

Although this is not the most general integral of (1), it has sufficient generality for our purposes.

Next, treating (2), we find that it is satisfied (putting r for $\sqrt{x^2 + y^2}$) by each of the series of expressions

$$\begin{aligned}
\widehat{V}_0 &= r^{-1}\sin(mr), \\
\widehat{V}_1 &= r^{-1}\left(\frac{d}{dr}\frac{1}{r}\right)^2 \sin(mr)\,(x^2 - 2y^2), \\
\widehat{V}_2 &= r^{-1}\left(\frac{d}{dr}\frac{1}{r}\right)^4 \sin(mr)\,(x^4 - 8x^2y^2 + \tfrac{8}{3}y^4), \\
\widehat{V}_3 &= r^{-1}\left(\frac{d}{dr}\frac{1}{r}\right)^6 \sin(mr)\,(x^6 - 18x^4y^2 + 24x^2y^4 - \tfrac{16}{5}y^6), \\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

which may be continued as far as we please. Taking another series of constants a_0, a_1, . . . , a more general integral of (2) will be

$$\widehat{V} = a_0\widehat{V}_0 + a_1\widehat{V}_1 + a_2\widehat{V}_2 + a_3\widehat{V}_3 + \dots \tag{4}$$

As before, this expression has sufficient generality for our purposes. To equations (3) and (4) we add the equation

$$\frac{\partial(\widehat{V} - \breve{V})}{\partial r} = 0. \tag{5}$$

The following three equations

$$\left.\begin{aligned}
B + qx^2 + b_0\breve{V}_0 + b_1\breve{V}_1 + b_2\breve{V}_2 + b_3\breve{V}_3 + \dots &= 0, \\
B' + a_0\widehat{V}_0 + a_1\widehat{V}_1 + a_2\widehat{V}_2 + a_3\widehat{V}_3 + \dots &= 0, \\
\frac{\partial(\widehat{V} - \breve{V})}{\partial r} &= 0,
\end{aligned}\right\} \tag{6}$$

are satisfied by the surface of the earth, but it is obvious they cannot be unless certain relations are established between the a and b. The number of these relations is always two less than the number of the constants a and b taken into consideration. This is plain from the circumstance that the mass of the earth and its density when subjected to no pressure are data that must be afforded by observation. The treatment of the subject is greatly facilitated by the fact that a_i and b_i are of the ith order with respect to the centrifugal force. We propose to carry the approximation so as to include the third power of this force. As b_0 is the mass of the earth equivalent to unity with our units, this demands the consideration of the seven constants $a_0, a_1, a_2, a_3, b_1, b_2, b_3$. The number of arbitrary constants in equations (6) is ten; eight therefore are to be eliminated, and eight independent relations are needed. If, for $\frac{y^2}{x^2} = z$, we substitute a special value in equations (6), the left members will be reduced to functions of r. The solution of these equations, regarding r as the unknown, ought to show that they are satisfied by the same root. Thus the equating of the three roots affords two equations, and by assuming four values for z, eight equations result, serving to determine the superfluous constants of B, B', a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3.

The selection of the values of z is not indifferent; such values should be taken as will bring about the largest degree of independence in the resulting equations. Advantage may be taken of the vanishing of the expressions

$$x^2 - 2y^2, \qquad x^4 - 8x^2y^2 + \tfrac{8}{3}y^4, \qquad x^6 - 18x^4y^2 + 24x^2y^4 - \tfrac{16}{5}y^6,$$

for certain values of z. The equations in z are

$$z - \tfrac{1}{2} = 0, \qquad z^2 - 3z + \tfrac{3}{8} = 0, \qquad z^3 - \tfrac{15}{2}z^2 + \tfrac{45}{8}z - \tfrac{5}{16} = 0.$$

Their roots are all real and positive, and have the values

$$z = 0.5,$$
$$z = 0.130693605, \quad z = 2.8693064,$$
$$z = 0.0603769\!24, \quad z = 0.776823357, \quad z = 6.662799719.$$

In addition the values $z = 0$ and $z = \infty$ are worthy of consideration. The four values selected for z and the subscripts employed to distinguish quantities belonging to them are: $z = 0.5$, subscript $({}_0)$; $z = 0.0$, subscript $({}_1)$; $z = \infty$, subscript $({}_2)$; $z = 0.776823357$, subscript $({}_3)$.

When i is even, denote $r^{i-1}\left(\frac{d}{dr}\,\frac{1}{r}\right)^i \sin(mr)$ by f_i, and when i is odd, denote $r^i\left(\frac{d}{dr}\,\frac{1}{r}\right)^i \sin(mr)$ by f_i. In the prosecution of the investigation

we shall need the values of the f_i as far as f_7 for certain values of the argument r. Here we have the equations to recursion

$$f_2 = -\frac{3}{r^2} f_1 - m^2 f_0, \qquad f_3 = -5 f_2 - m^2 f_1,$$
$$f_4 = -\frac{7}{r^2} f_3 - m^2 f_2, \qquad f_5 = -9 f_4 - m^2 f_3,$$
$$f_6 = -\frac{11}{r^2} f_5 - m^2 f_4, \qquad f_7 = -13 f_6 - m^2 f_5,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

the law of progression is obvious. In spite of their great simplicity they are very unsuitable for accurate computation. The value of m we adopt is 2.5 or, in arc, 143° 14′ 22″.0156. Thus, commencing with f_0 and f_1 true to ten decimal places, when f_7 is arrived at, not more than three decimals can be depended on. It seems that the readiest method of obviating this difficulty is to employ the series in powers of r. We have

$$f_0 = m - \frac{1}{1..3} m^3 r^2 + \frac{1}{1..5} m^5 r^4 - \frac{1}{1..7} m^7 r^6 + \frac{1}{1..9} m^9 r^8 - ..,$$
$$f_1 = -\frac{2}{1..3} m^3 r^2 + \frac{4}{1..5} m^5 r^4 - \frac{6}{1..7} m^7 r^6 + \frac{8}{1..9} m^9 r^8 - ..,$$
$$f_2 = \frac{4.2}{1..5} m^5 r^2 - \frac{6.4}{1..7} m^7 r^4 + \frac{6.6}{1..9} m^9 r^6 - ..,$$
$$f_3 = -\frac{6.4.2}{1..7} m^7 r^4 + \frac{8.6.4}{1..9} m^9 r^6 - \frac{10.8.6}{1..11} m^{11} r^8 + ..,$$
$$f_4 = \frac{8.6.4.2}{1..9} m^9 r^4 - \frac{10.8.6.4}{1..11} m^{11} r^6 + \frac{12.10.8.6}{1..13} m^{13} r^8 - ..,$$
$$f_5 = -\frac{10.8.6.4.2}{1..11} m^{11} r^6 + \frac{12..4}{1..13} m^{13} r^8 - \frac{14..6}{1..15} m^{15} r^{10} + ..,$$
$$f_6 = \frac{12..2}{1..13} m^{13} r^6 - \frac{14..4}{1..15} m^{15} r^8 + ..,$$
$$f_7 = -\frac{14..2}{1..15} m^{15} r^8 + \ldots$$

About twelve terms of these series give fairly approximative values. The computation is facilitated by taking the terms two by two. For brevity we use the following notation

$$(0) = f_0, \qquad (\mathrm{I}) = f_1,$$
$$(\mathrm{II}) = f_2, \qquad (\mathrm{III}) = f_3 + 2\,(\mathrm{II}),$$
$$(\mathrm{IV}) = f_4, \qquad (\mathrm{V}) = f_5 + 4\,(\mathrm{IV}),$$
$$(\mathrm{VI}) = f_6, \qquad (\mathrm{VII}) = f_7 + 6\,(\mathrm{VI}).$$

For the value $m = 2.5$ and with the argument $\frac{1}{r}$ we have the following

Table of Values of (0), (I), &c.

$\frac{1}{r}$.	(0).	Δ.	Δ^2.	(I).	Δ.	Δ^2.	Δ^3.
0.998	0.59326 20026			−2.60359 17678			
		+260 69402			+112 58665		
0.999	59586 89428		−37389	60246 59013		+88520	
		260 32013			113 47185		−747
1.000	59847 21441		37404	60133 11828		87773	
		259 94609			114 34958		741
1.001	60107 16050		37419	60018 76870		87032	
		259 57190			115 21990		748
1.002	60366 73240		37431	59903 54880		86284	
		259 19759			116 08274		
1.003	60625 92999			59787 46606			

	(II).	Δ.	Δ².	(III).	Δ.	Δ².	Δ³.
0.998	4.07167 5929			4.05742 0762			
		—406 6105			+516 1649		
0.999	06760 9824		—1071	06258 2411		—52112	
		406 7176			510 9537		+302
1.000	06354 2648		1016	06769 1948		51810	
		406 8192			505 7727		299
1.001	05947 4456		961	07274 9675		51511	
		406 9153			500 6216		301
1.002	05540 5303		906	07775 5891		51210	
		407 0059			495 5006		
1.003	05133 5244			08271 0897			

	(IV).	Δ.	Δ².	(V).	Δ.	Δ².	Δ³.
0.998	3.03925 146			10.34081 205			
		—103 4062			—313 8352		
0.999	3.02891 084		+4166	30942 853		+10403	
		102 9896			312 7949		—45
1.000	3.01861 188		4150	27814 904		10358	
		102 5746			311 7591		43
1.001	3.00835 442		4136	24697 313		10315	
		102 1610			310 7276		42
1.002	2.99813 832		4120	21590 037		10273	
		101 7490			309 7003		
1.003	2.98796 342			18493 034			

	(VI).	Δ.	Δ².	(VII).	Δ.	Δ².
0.998	0.90297 7			5.03036 6		
		—502 3			—271 96	
0.999	89795 4		+28	5.00317 0		+180
		499 5			270 16	
1.000	89295 9		32	4.97615 4		165
		496 3			268 51	
1.001	88799 6		35	4.94930 3		147
		492 8			267 04	
1.002	88306 8		37	4.92259 9		134
		489 1			265 70	
1.003	87817 7			4.89602 9		

The equations we use for discovering the values of the constants are

$$\begin{cases} B + \frac{2}{3} q r_0^2 + \frac{b_0}{r_0} - \frac{28}{27}\frac{b_2}{r_0^5} - \frac{32}{45}\frac{b_3}{r_0^7} & = 0, \\ B' + (0)_0 a_0 + \frac{28}{27}(\mathrm{IV})_0 a_2 - \frac{32}{45}(\mathrm{VI})_0 a_3 & = 0, \\ (\mathrm{I})_0 a_0 - \frac{28}{27}(\mathrm{V})_0 a_2 - \frac{32}{45}(\mathrm{VII})_0 a_3 - \frac{4}{3} q r_0^2 + \frac{b_0}{r_0} - \frac{140}{27}\frac{b_2}{r_0^5} - \frac{224}{45}\frac{b_3}{r_0^7} & = 0, \end{cases}$$

$$\begin{cases} B + q r_1^2 + \frac{b_0}{r_1} + \frac{b_1}{r_1^3} + \frac{b_2}{r_1^5} + \frac{b_3}{r_1^7} & = 0, \\ B' + (0)_1 a_0 + (\mathrm{II})_1 a_1 + (\mathrm{IV})_1 a_2 + (\mathrm{VI})_1 a_3 & = 0, \\ (\mathrm{I})_1 a_0 + (\mathrm{III})_1 a_1 + (\mathrm{V})_1 a_2 + (\mathrm{VII})_1 a_3 - 2 q r_1^2 + \frac{b_0}{r_1} + 3\frac{b_1}{r_1^3} + 5\frac{b_2}{r_1^5} + 7\frac{b_3}{r_1^7} & = 0, \end{cases}$$

$$\begin{cases} B + \frac{b_0}{r_2} - 2\frac{b_1}{r_2^3} + \frac{8}{3}\frac{b_2}{r_2^5} - \frac{16}{5}\frac{b_3}{r_2^7} & = 0, \\ B' + (0)_2 - 2(\mathrm{II})_2 a_1 + \frac{8}{3}(\mathrm{IV})_2 a_2 - \frac{16}{5}(\mathrm{VI})_2 a_3 & = 0, \\ (\mathrm{I})_2 a_0 - 2(\mathrm{III})_2 a_1 + \frac{8}{3}(\mathrm{V})_2 a_2 - \frac{16}{5}(\mathrm{VII})_2 a_3 + \frac{b_0}{r_2} - 6\frac{b_1}{r_2^3} + \frac{40}{3}\frac{b_2}{r_2^5} - \frac{112}{5}\frac{b_3}{r_2^7} & = 0, \end{cases}$$

$$\begin{cases} B + 0.56280216\, q r_3^2 + \frac{b_0}{r_3} - 0.31159352\frac{b_1}{r_3^3} - 1.1419890\frac{b_2}{r_3^5} & = 0, \\ B' + (0)_3 a_0 - 0.31159352\,(\mathrm{II})_3 a_1 - 1.1419890\,(\mathrm{IV})_3 a_2 & = 0, \\ (\mathrm{I})_3 a_0 - 0.311\ldots(\mathrm{III})_3 a_1 - 1.141\ldots(\mathrm{V})_3 a_2 - 1.12560432\, q r_3^2 + \frac{b_0}{r_3} - 3(0.311\ldots)\frac{b_1}{r_3^3} & \\ \qquad - 5(1.141\ldots)\frac{b_2}{r_3^5} & = 0. \end{cases}$$

In the following work a value of q has been adopted such that

$$\log q = 7.23728\ 13257.$$

In order to have the equations to be solved linear, a preliminary investigation has been made leading to the following values of the constants:

$$B = -1.00115\ 01595, \quad B' = -0.22953\ 20069, \quad a_0 = 0.38353\ 06679,$$
$$a_1 = \ \ 0.00027\ 68007, \quad a_2 = \ \ 0.00000\ 01308, \quad b_1 = 0.00054\ 86250,$$
$$b_2 = \ \ 0.00000\ 10969.$$

In addition it has been assumed from induction that

$$a_3 = 0.00000\ 00001, \quad b_3 = 0.00000\ 00020.$$

With these values of the constants, the arguments for entering the preceding tables are

$$\frac{1}{r_1} = 0.99887\ 14383, \qquad \frac{1}{r_2} = 1.00225\ 18904, \qquad \frac{1}{r_3} = 1.00035\ 12904.$$

Whence result the following values

$(0)_1 =$	$0.59553\ 39995$	$(0)_2 =$	$0.60432\ 05706$	$(0)_3 =$	$0.59938\ 57360$
$(\mathrm{II})_1 =$	$4.06813\ 2630$	$(\mathrm{II})_2 =$	$4.05438\ 0175$	$(\mathrm{II})_3 =$	$4.06211\ 3643$
$(\mathrm{IV})_1 =$	$3.03023\ 791$	$(\mathrm{IV})_2 =$	$2.99557\ 149$	$(\mathrm{IV})_3 =$	$3.01500\ 381$
$(\mathrm{VI})_1 =$	$0.89859\ 7$	$(\mathrm{VI})_2 =$	$0.88183\ 2$	$(\mathrm{VI})_3 =$	$0.89121\ 2$
$(\mathrm{I})_1 =$	$-\ 2.60261\ 11420$	$(\mathrm{I})_2 =$	$-\ 2.59874\ 38968$	$(\mathrm{I})_3 =$	$-\ 2.60093\ 04800$
$(\mathrm{III})_1 =$	$4.06192\ 1746$	$(\mathrm{III})_2 =$	$4.07900\ 8822$	$(\mathrm{III})_3 =$	$4.06947\ 4566$
$(\mathrm{V})_1 =$	$10.31345\ 741$	$(\mathrm{V})_2 =$	$10.20808\ 966$	$(\mathrm{V})_3 =$	$10.26718\ 546$
$(\mathrm{VII})_1 =$	$5.00665\ 5$	$(\mathrm{VII})_2 =$	$4.91589\ 4$	$(\mathrm{VII})_3 =$	$4.96670\ 3$

The corrections δr_1, δr_2, δr_3 need be considered only in the terms $\frac{b_0}{r_1}$, $\frac{b_0}{r_2}$, $\frac{b_0}{r_3}$ and the quantities (0) and (1). From the first group of equations we derive

$$B = -1.00115\ 13041\ 4 + \tfrac{28}{77} b_2 + \tfrac{32}{45} b_8,$$
$$B' = -0.22953\ 40559 + 5.58262 a_2 + 1.449 a_3 + 1.192923 b_2 + 1.145 b_3,$$
$$a_0 = [9.58380\ 31899] - [0.612514] a_2 - [0.1336] a_3 - [0.299569] b_2 - [0.2818] b_3,$$

serving to eliminate B, B', a_0 from the remaining nine equations, which thus become

$$- 0.00054\ 90050\ 4 - [9.9975]\delta r_1 + [9.99852\ 879] b_1 + [0.30779\ 69] b_2 + [0.2313] b_3 = 0,$$
$$+ 0.00110\ 05862\ 6 - [0.0019]\delta r_2 - [0.30396\ 065] b_1 + [0.57215\ 85] b_2 - [0.4048] b_3 = 0,$$
$$+ 0.00017\ 12384\ 6 - [9.9995]\delta r_3 - [9.49404\ 603] b_1 - [9.02921\ 85] b_2 + [9.8519] b_3 = 0,$$
$$- 0.00112\ 68766 - [9.9987]\delta r_1 + 0.00585\ 6 b_2 + 0.0055 b_3 + 4.06813\ 26 a_1 + 6.17268\ 8 a_2 + 1.5376 a_3 = 0,$$
$$+ 0.00224\ 30670 - [9.9995]\delta r_2 - 0.01165\ 8 b_2 - 0.0114 b_3 - 8.10876\ 03 a_1 + 11.09463\ 9 a_2 - 2.195 a_3 = 0,$$
$$+ 0.00035\ 03930 - [9.9991]\delta r_3 - 0.00182\ 1 b_2 - 0.0018 b_3 - 1.26572\ 83 a_1 - 0.31643\ 3 a_2 + 0.634 a_3 = 0,$$
$$- 0.00277\ 85822 - [0.1574]\delta r_1 + 2.98985\ 442 b_1 + 10.15958\ 6 b_2 + 11.925 b_3 + 4.06192\ 17 a_1 + 20.97752\ 5 a_2 + 8.547 a_3 = 0,$$
$$+ 0.00554\ 68095 - [0.1616]\delta r_2 - 6.04062\ 546 b_1 + 18.66416\ 5 b_2 - 17.783 b_3 - 8.15801\ 76 a_1 + 37.86979\ 4 a_2 - 12.196 a_3 = 0,$$
$$+ 0.00086\ 50773 - [0.1593]\delta r_3 - 0.93576\ 606 b_1 - 0.53559\ 6 b_2 + 4.976 b_3 - 1.26802\ 19 a_1 - 1.06783\ 2 a_2 + 3.537 a_3 = 0.$$

The elimination gives the following values of the unknowns:

$B = -1.00115\,01641,$	$B' = -0.22953\,20133\,6,$
$a_0 = 0.38353\,06706,$	$a_1 = 0.00027\,68006\,8,$
$a_2 = 0.00000\,01307\,5,$	$a_3 = 0.00000\,00000\,5,$
$b_1 = 0.00054\,86250\,7,$	$b_2 = 0.00000\,10970\,1,$
$b_3 = 0.00000\,00033\,9,$	$\delta r_1 = -0.00000\,00011\,1,$
$\delta r_2 = -0.00000\,00057\,4,$	$\delta r_3 = -0.00000\,00047\,5.$

The addition of the corrections to the provisional values gives the following values of r, $\frac{1}{r}$, and com. log r:

	r.	$\frac{1}{r}$.	log r.
r_1	1.00112 98356 6	0.99887 14394 1	0.000 4904 044
r_2	0.99775 31635 1	1.00225 18961 7	9.999 0231 134
r_3	0.99964 88283 3	1.00035 12951 4	9.999 8474 613

From the values of r_0, r_1, r_2, r_3 are derived the following expressions for r and common log r in terms of μ the sine of the geocentric latitude:

$$\begin{aligned} r &= 1.00000\,17130\,1 + 0.00112\,64700\,2\,(1 - 3\,\mu^2) \\ &\quad + 0.00000\,16498\,10\,(1 - 10\,\mu^2 + \tfrac{35}{3}\mu^4) + 0.00000\,00029\,47\,(1 - 21\,\mu^2 + 63\,\mu^4 - \tfrac{231}{5}\mu^6) \\ &= 1.00112\,98358 - 0.00339\,59700\,\mu^2 + 0.00001\,94335\,\mu^4 - 0.00000\,01362\,\mu^6, \\ \log r &= 0.00000\,05235 + 0.00048\,93759\,(1 - 3\,\mu^2) \\ &\quad + 0.00000\,05042\,(1 - 10\,\mu^2 + \tfrac{35}{3}\mu^4) + 0.00000\,00008\,(1 - 21\,\mu^2 + 63\,\mu^4 - \tfrac{231}{3}\mu^6) \\ &= 0.00049\,04044 - 0.00147\,31873\,\mu^2 + 0.00000\,59355\,\mu^4 - 0.00000\,00392\,\mu^6. \end{aligned}$$

The values of the component $\frac{\partial \bar{V}}{\partial r}$ for the four values of μ, viz., μ_0, μ_1, μ_2, μ_3, were next computed and the results are:

for μ_0, $-0.99769\,16866,$	for μ_1, $-0.99593\,02672,$
for μ_2, $-1.00120\,21106,$	for μ_3, $-0.99823\,96845.$

For the other component we have the equation

$$\frac{1}{r}\frac{\partial \bar{V}}{\partial \mu}\sqrt{1-\mu^2} = -\mu\sqrt{1-\mu^2}\left[2qr + 6\frac{b_1}{r^4} + 20\frac{b_2}{r^6}(1 - \tfrac{7}{3}\mu^2) + 42\frac{b_3}{r^8}(1 - 6\,\mu^2 + \tfrac{33}{5}\mu^4)\right].$$

The values of the factor within the brackets for the special values of μ, viz., μ_0, μ_1, μ_2, μ_3 are

for μ_0, 0.00675 05005,	for μ_1, 0.00675 66636,
for μ_2, 0.00673 82341,	for μ_3, 0.00674 85836.

From these data it is easy to deduce the values of the actual gravity G for μ_0, μ_1, μ_2, μ_3. They are,

for μ_0, $G = 0.99769\,67615,$	for μ_1, $G = 0.99593\,02672,$
for μ_2, $G = 1.00120\,21106,$	for μ_3, $G = 0.99824\,52973.$

From these values we derive

$$G = 0.99768\,95187 - 0.00175\,92433\,(1 - 3\,\mu^2) - 0.00000\,35526\,6\,(1 - 10\,\mu^2 + \tfrac{35}{3}\,\mu^4)$$
$$- 0.00000\,00082\,4\,(1 - 21\,\mu^2 + 63\,\mu^4 - \tfrac{231}{5}\,\mu^6)$$
$$= 0.99593\,02672 + 0.00531\,34295\,\mu^2 - 0.00004\,19668\,\mu^4 + 0.00000\,03807\,\mu^6.$$

Calling the geocentric and geographical latitude severally θ and θ',

$$\theta' = \theta + 696''.8755 \sin 2\,\theta + 1''.3957 \sin 4\,\theta + 0''.0039 \sin 6\,\theta.$$

By means of this formula we derive

θ'.	θ.	$\log r$.
22° 30′	22° 21′ 48″.1916	0.000 2772 633
45 0	44 48 23 .1253	9.999 7602 471
67 30	67 21 46 .2741	9.999 2397 428

We can now deduce $\log r$ as a periodic function of θ':

$$\log r = 9.999\,7585\,033 + 0.000\,7336\,416 \cos 2\,\theta' - 0.000\,0017\,447 \cos 4\,\theta' + 0.000\,0000\,038 \cos 6\,\theta'.$$

From special values of $\dfrac{\partial \widehat{V}}{\partial r}$ we derive the values of G for evenly spaced values for θ' ($\widehat{V}$ is chosen in order to have a verification)

θ'.	G.
0°	0.99593 02672
22½	0.99669 85847
45	0.99855 87244
67½	1.00042 63297
90	1.00120 21106

Thence we get

$$G = 0.99856\,24569 - 0.00263\,59177 \cos 2\,\theta' + 0.00000\,37322 \cos 4\,\theta' - 0.00000\,00039 \cos 6\,\theta'.$$

From the same data is obtained

$$\theta = \theta' - 696''.8767 \sin 2\,\theta' + 0''.9578 \sin 4\,\theta' - 0''.0021 \sin 6\,\theta\,.$$

Part II.

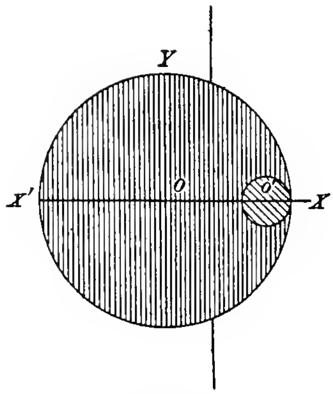
Effect of Departures from the Ideal Earth.

The departure of the actual from the ideal earth is so small that, at least in a first approximation, quantities of the order of the product of the departure by the centrifugal force may be neglected. Thus, in estimating the effect, we may always start from a centrobaric earth.

Of all volumes the centrobaric sphere has the simplest expression for the potential. The effect of the earth as an attractive body may be conceived as equivalent to that of a number of centrobaric spheres superposed on each other. With a finite number of spheres of course we obtain only an approximation; however, this augments as the number is increased; and, in the last analysis, when the spheres become material points, the approximation ends in exactitude. With a very moderate number results are reached having a practical value; and this may be considerably reduced by admitting the fiction of spheres of negative mass.

Let us apply this notion to the explanation of the mode in which sea and land are distributed over the earth's surface. This subject is rarely alluded to in our treatises on geography and geodesy. The prevailing notion is that the coast line is determined solely by what may be called the asperities of the surface, or that the theory which explains the shore line of a mountain turn may also be used for the Pacific Ocean. If a fluid is placed upon the ideal centrobaric earth, it will form a sea having everywhere the same depth, and there will be no land-surface. But let the earth's surface first undergo an irregular carving, the fluid will form a sea of irregular depth, and, if the amount of fluid is small enough, some land-surface will appear. These notions are correct; but, when it is added that the depth of the sea is in precise relation to the depth of the carving, we must demur. This theory takes no account of the shifting of the barycentre through the carving. The latter may be so performed that what appears as the bottom of the ocean is no nearer this point than the top of the highest mountain. For instance, in the case of a homogeneous sphere, the carving may be nothing at a point on one side and deep at the point directly opposite, and yet the resulting body be a homogeneous sphere to which there may belong an ocean of uniform depth. The present disposition of land and water on the earth is not to be accounted for on the asperity theory. To what then must it be attributed? It is evident that in an irregular distribution of matter in the interior of the globe we have a competent cause. We should expect that density would be superabundant on the side where the ocean predominates and defective on the side where the continents appear. The subject is well illustrated by a series of suppositions.

Supposition I.



The adjacent diagram shows two centrobaric spheres superposed, of which the larger represents the ideal earth, and the smaller is a sphere of negative mass. Let M denote the actual mass of the earth (the mass of the covering fluids not being included), — m the mass of the smaller sphere, d the distance oo' between the centres of the spheres, and a the radius of the earth. The origin of coordinates being at o the centre of figure of the earth, the positive direction of x towards the right, let h be the depth at X' of the ocean supposed without mass. Then the equation of sea-level will be

$$\frac{M+m}{\sqrt{x^2+y^2}}-\frac{m}{\sqrt{(x-d)^2+y^2}}=\frac{M+m}{a+h}-\frac{m}{a+h+d},$$

or, adopting polar coordinates such that

$$x = r\cos\phi', \qquad y = r\sin\phi',$$

$$\frac{M+m}{r}-\frac{m}{\sqrt{r^2-2dr\cos\phi'+d^2}}=\frac{M+m}{a+h}-\frac{m}{a+h-d}.$$

Calling the geographical colatitude ϕ and the downward force of gravity g, this potential furnishes the equations

$$g\cos(\phi-\phi') = \frac{M+m}{r^2}-\frac{m(r-d\cos\phi')}{[r^2-2dr\cos\phi'+d^2]^{\frac{3}{2}}},$$

$$g\sin(\phi-\phi') = \qquad -\frac{md\sin\phi'}{[r^2-2dr\cos\phi'+d^2]}.$$

The x coordinate of the centre of gravity of the earth is $-\frac{md}{M}$; and, if we wish to have the difference of the geographical colatitude over and above the colatitude measured from the latter centre instead of the centre of figure, it will be sufficiently approximate for points near the earth's surface to subtract the arc given by the formula

$$-\frac{md}{M}\sin\phi'.$$

For numerical illustration we assume $M = 1$, $m = 0.001$, $a = 1$, $h = 0.001$, and $d = 0.8$. If the smaller sphere touches the larger interiorly, and the mean density of the earth is put at 5.6 (that of water being unity) the diminution of the density of the earth by the superposition of the smaller sphere is 0.7. This is not an unreasonable supposition.

The following table shows the depth of the ocean at intervals of 30° in the colatitude, with the relative numbers of gravity and the comparison of the different colatitudes, all for points on the surface of the ocean or on

the surface of the land. There is therefore a discontinuity in the quantities tabulated at the point where $\phi' = 39° 56' 59''.4$.

ϕ'	Depth of ocean $r-1$.	g.	$\phi-\phi'$.	$-\frac{md}{M}\sin\phi'$.	$\phi-\phi'$ from cen. grav.
180°	0.0010000	0.9986926	0″	0″	0″
150	0.0009809	0.9987175	— 16	— 83	+ 67
120	0.0009153	0.9988029	— 37	—143	+106
90	0.0007748	0.9989749	— 79	—165	+ 86
60	0.0004643	0.9992920	—186	—143	— 43
39 56′ 59″.4	0.0000000	0.9995470	—399	—106	—293
30	—0.0004280	0.9986105	—643	— 82	—561
0	—0.0035183	0.9760000	0	0	0

We see from the table that the coast line is at $\phi' = 40°$ about, and the elevation of the continent above the geoid at the north pole is $3\frac{1}{2}$ times the depth of the ocean at the south pole. As to the gravity, it increases from the south pole to the coast line and thence diminishes to the summit of the continent at the north pole. As, at the latter point, there is a diminution of 2.4 per cent as compared with the value for an ideal earth, and, as a disturbance of latitude as great as $561''$ is noted, too influential elements have been attributed to the smaller sphere.

Supposition II.

As a second illustration let us introduce a second small sphere of positive mass directly opposite the first, so that with a notation similar to that just employed, the equation of the surface of the ocean (again without mass) will be

$$\frac{M}{\sqrt{x^2+y^2}} + m\left(\frac{1}{\sqrt{(x+d)^2+y^2}} - \frac{1}{\sqrt{(x-d)^2+y^2}}\right) = \text{a constant},$$

or, in polar coordinates,

$$\frac{M}{r} + m\left(\frac{1}{\sqrt{r^2+2dr\cos\phi'+d^2}} - \frac{1}{\sqrt{r^2-2dr\cos\phi'+d^2}}\right) = \text{a constant.}$$

If the limit of the continent is to be at the small circle of the earth's surface for which $x = \frac{1}{2}d$, the value of the constant forming the right member of the equation will be

$$\frac{M}{a} + m\left(\frac{1}{\sqrt{a^2+2d^2}} - \frac{1}{a}\right).$$

To determine g and ϕ we have

$$g\cos(\phi-\phi') = \frac{M}{r^2} - m\left(\frac{r-d\cos\phi'}{[r^2-2dr\cos\phi'+d^2]^{\frac{3}{2}}} - \frac{r+d\cos\phi'}{[r^2+2dr\cos\phi'+d^2]^{\frac{3}{2}}}\right),$$

$$g\sin(\phi-\phi') = -md\sin\phi'\left(\frac{1}{[r^2-2dr\cos\phi'+d^2]^{\frac{3}{2}}} + \frac{1}{[r^2+2dr\cos\phi'-d^2]^{\frac{3}{2}}}\right).$$

The x coordinate of the centre of gravity of the earth is $-\frac{2md}{M}$.

For numerical illustration we put $M=1$, $m=\frac{1}{8000}$, $a=1$, $d=0.9$. Thus the equation of the ocean surface is

$$\frac{1}{r}+\frac{1}{8000}\left[(r^2+1.8\,r\cos\phi'+0.81)^{-\frac{1}{2}}-(r^2-1.8\,r\cos\phi'+0.81)^{-\frac{1}{2}}\right]=0.9999522.$$

Then results the following table exactly analogous to that of the first supposition:

ϕ'.	Depth of ocean $r-1$.	g.	$\phi-\phi'$.	$-\frac{2md}{M}\sin\phi'$.	$\phi-\phi'$ from cen. grav.
180°	0.0012187	1.0097333	0″	0″	0″
150	0.0002291	0.9997246	—94	—23	—71
120	0.0001029	0.9998328	—28	—40	+12
90	0.0000478	0.9999044	—46	—46	0
60	—0.0000073	0.9999614	—28	—40	+12
30	—0.0001335	0.9998169	—94	—23	—71
0	—0.0011496	0.9875345	0	0	0

Here the greatest depth of the ocean and the greatest elevation of the continent are about the same, and the greatest deviation of gravity from that of the ideal earth is about one per cent.

Supposition III.

In the preceding illustrations the centres of the subsidiary spheres have been placed near the surface of the ideal earth. Let us now put them near the centre but still on the same diameter. If $-d$ and $-d'$ are the x coordinates of the centres of the subsidiary spheres the equation of the ocean's surface will be

$$\frac{M}{\sqrt{x^2+y^2}}+m\left[\frac{1}{\sqrt{(x+d)^2+y^2}}-\frac{1}{\sqrt{(x+d')^2+y^2}}\right]=C,$$

or, in terms of polar coordinates,

$$\frac{M}{r}+m\left[\frac{1}{\sqrt{r^2+2\,dr\cos\phi'+d^2}}-\frac{1}{\sqrt{r^2+d'r\cos\phi'+d'^2}}\right]=C.$$

Let $M=1$, $a=1$, h the depth of the ocean for $\phi'=180°$, and k the elevation of the continent for $\phi'=0°$, also f the value of $\cos\phi'$ at the coast, which, as we suppose the surface of the ocean to exceed half that of the earth, is necessarily positive and less than unity. We will assume that the continent is $\frac{5}{12}$ of the whole surface. Then $f=\cos\phi'$, where $\sin\frac{\phi'}{2}=\sqrt{\frac{5}{12}}$. Thus $\log 2f=9.9765190$. We put $h=0.00125$, and $k=0.00025$. For brevity let H stand for $1+h$, and K for $1-k$. These assumptions give

only three relations to determine four quantities. There is nothing to determine the value of m, only it must be reasonably small; we put it $= 0.1$. Subtract 1 from both members of the equation and still denote the right member by C. Then we have the three relations to determine d, d', C:

$$-\frac{h}{H}+m\left[\frac{1}{H-d}-\frac{1}{H-d'}\right]=C,$$

$$\frac{k}{K}+m\left[\frac{1}{K+d}-\frac{1}{K+d'}\right]=C,$$

$$m\left[\frac{1}{\sqrt{1+2fd+d^2}}-\frac{1}{\sqrt{1+2fd'+d'^2}}\right]=C.$$

For brevity putting

$$D^2=1+2fd+d^2, \qquad D'^2=1+2fd'+d'^2,$$

and eliminating C, we have

$$\left[\frac{1}{(H-d)(H-d')}+\frac{1}{(K+d)(K+d')}\right](d-d')=\frac{1}{m}\left(\frac{h}{H}+\frac{k}{K}\right),$$

$$\left[\frac{1}{(H-d)(H-d')}+\frac{d+d'+2f}{DD'(D+D')}\right](d-d')=\frac{1}{m}\frac{h}{H}.$$

These equations are solved most easily by the tentative process, and, with the assumed numbers, give

$$d=0.0977710, \qquad d'=0.0904648.$$

Using D and D' now to denote general distances from the two subsidiary centres, the equation to sea-level may be given the form

$$\frac{r-1}{r}+0.00073062\,\frac{2r\cos\phi'+0.1882358}{DD'(D+D')}=0.00036056,$$

where

$$D^2=r^2+2\,dr\cos\phi'+d^2, \qquad D'^2=r^2+2\,d'r\cos\phi'+d'^2.$$

To determine g and ϕ we have

$$g\cos(\phi-\phi')=\frac{M}{r^2}+m\left(\frac{r+d\cos\phi'}{D^3}-\frac{r+d'\cos\phi'}{D'^3}\right),$$

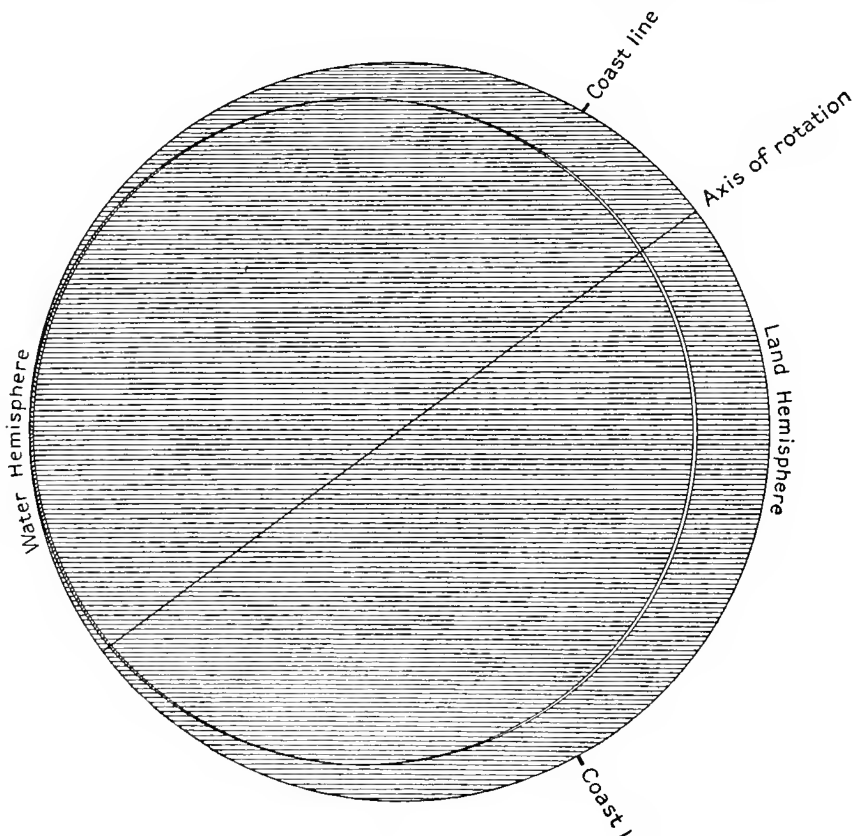
$$g\sin(\phi-\phi')=-m\sin\phi'\left(\frac{d}{D^3}-\frac{d'}{D'^3}\right).$$

The x coordinate of the centre of gravity of the earth is $-\dfrac{m(d-d')}{M}$.

Exactly as in the preceding illustrations we have the table:

ϕ'.	Depth of ocean $r-1$.	g.	$\phi-\phi'$.	$-m\frac{(d-d')}{M}\sin\phi'$.	$\varphi-\phi'$ from cen. grav.
180°	0.0012500	0.9994624	0″	0″	0″
150	0.0010850	0.9993768	—121	— 75	—46
120	0.0006995	0.9992432	—168	—131	—37
90	0.0002928	0.9992134	—145	—151	+ 6
61 43′ 34″.63	0.0000000	0.9992719	— 99	—133	+34
60	—0.0000142	0.9992483	— 96	—131	+35
30	—0.0001926	0.9989678	— 46	— 75	+29
0	—0.0002500	0.9988836	0	0	0

In this illustration gravity never deviates much more than $\frac{1}{10}$ per cent from that obtaining in the ideal earth, and the arcs in the last column of the table do not exceed the deviations of latitude which have been attributed to local attraction. The effect of the two subsidiary spheres is to add a meniscus of density 0.7 and having a maximum thickness of about 29 miles, situated immediately beneath the ocean; and to subtract a meniscus of the

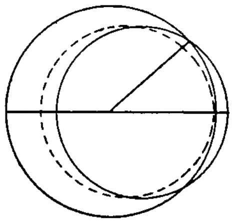


same volume and density situated in the opposite hemisphere. If the subsidiary spheres are homogeneous the region between the menisci is unaltered in density. If the change of 0.7 in the density is regarded as inadmissible. it can be diminished by assuming a smaller value for m. If m is assumed at 0.05, we have 0.35 in place of 0.7, and the maximum thickness of the menisci will be about 58 miles.

This illustration is of use in showing what variations may be expected in g, and what differences in the comparison of geodetic and astronomical latitudes. The diagram exhibits the general conditions assumed in this supposition.

Supposition IV.

Thus far the subsidiary spheres have been confined within the contour of the ideal earth; but, as asperities exist on the earth's surface, let us see whether our apparatus can be employed to evaluate their effects on gravity. It is natural to suppose that the continents may, in the lump, be regarded as menisci lying upon the surface of the ideal earth, their outside surfaces having a curvature slightly greater.



Let the adjacent figure represent a section through the centre of the ideal earth and the summit of the continent. The equation of the section of the ideal earth is

$$x^2 + y^2 = a^2.$$

If we take a subsidiary sphere of radius $a(1-\alpha)$, where α is a small quantity of the order of the altitude of the summit of the continent, and let it be tangent to the ideal earth immediately beneath the summit, the equation of its section (represented by the broken line in the figure) will be

$$(x - a\alpha)^2 + y^2 = a^2(1-\alpha)^2.$$

We suppose this sphere to contain within its contour matter of the density of the material of the continent. Then let it be moved outward from the centre of the ideal earth (taking its matter with it) a distance h the elevation of the summit of the continent (the smaller continuous circle in the figure represents its section after this transference). Its equation is

$$(x - a\alpha - h)^2 + y^2 = a^2(1-\alpha)^2.$$

If the angular radius of the continent is γ, we shall have

$$\left(\cos\gamma - \alpha - \frac{h}{a}\right)^2 + \sin^2\gamma = (1-\alpha)^2,$$

whence results

$$\alpha = \frac{\frac{h}{a}\left(\cos\gamma - \frac{1}{2}\frac{h}{a}\right)}{2\sin^2\frac{\gamma}{2} + \frac{h}{a}}.$$

The potential of the earth, in this condition, is

$$V = \frac{M}{r} + m\left(\frac{1}{\sqrt{(x - a\alpha - h)^2 + y^2}} - \frac{1}{\sqrt{(x - a\alpha)^2 + y^2}}\right)$$

$$= \frac{M}{r} + m\left(\frac{1}{\sqrt{r^2 - 2(a\alpha + h)\,r\cos\phi' + (a\alpha + h)^2}} - \frac{1}{\sqrt{r^2 - 2a\alpha\,r\cos\phi' + a^2\alpha^2}}\right),$$

where m denotes the mass contained in the subsidiary sphere. We have $\frac{m}{M} = \frac{\rho}{R}(1-\alpha)^3$, if ρ denotes the surface density and R the mean density of the earth; we have used $\frac{10}{21}$ for the value of $\frac{\rho}{R}$.

To determine gravity and its direction we have

$$g\cos(\phi-\phi') = \frac{M}{r^2} + m\left[\frac{r-(a\alpha+h)\cos\phi'}{[r^2-2(a\alpha+h)r\cos\phi'+(a\alpha+h)^2]^{\frac{3}{2}}} - \frac{r-a\alpha\cos\phi'}{[r^2-2a\alpha r\cos\phi'+a^2\alpha^2]^{\frac{3}{2}}}\right],$$

$$g\sin(\phi-\phi') = m\sin\phi'\left[\frac{a\alpha+h}{[r^2-2(a\alpha+h)r\cos\phi'+(a\alpha+h)^2]^{\frac{3}{2}}} - \frac{a\alpha}{[r^2-2a\alpha r\cos\phi'+a^2\alpha^2]^{\frac{3}{2}}}\right].$$

For points on the surface of the continent we take the value of r from

$$r = (a\alpha+h)\cos\phi' + \sqrt{a^2(1-\alpha)^2-(a\alpha+h)^2\sin^2\phi'},$$

but, for points on the surface of the ideal earth (ϕ' greater than γ) we put $r=a$. The x coordinate of the centre of gravity is $\frac{mh}{M}$.

These equations are rigorous, but, on account of the smallness of α and h, we may substitute for them the following

$$V = \frac{M}{r} + \frac{mh}{r^2}\cos\phi',$$

$$g\cos(\phi-\phi') = \frac{M}{r^2} + 2\frac{mh}{r^3}\cos\phi',$$

$$g\sin(\phi-\phi') = \frac{mh}{r^3}\sin\phi'.$$

It is worthy of remark that here the direction of gravity is towards the centre of gravity of the actual earth, precisely as it was before the introduction of the subsidiary sphere; hence, in the following table the two last columns are omitted.

We here make application of the preceding to the five continents of Eurasia, Africa, Australia, North America, and South America, as Prof. Helmert has done.* The elements of each continent may be derived from its area and the altitude of its centre counted from the surface of the ideal earth. In reference to the latter it must be noted that while the centres of Eurasia, Africa, and North America lie quite near the coast line of Supposition III, the centre of Australia is quite near the pole of the water surface of the earth, hence to the elevation of this centre above sea level ought to be added the greatest depth of the sea, and its area extended accordingly.

*Geodäsie, Vol. II, p. 313.

As to South America, its centre lies in the just mentioned water surface at a distance of 18° 16′ from the coast line; hence the depth of the ocean at this point ought to be added to the elevation of the centre above sea level. It is supposed that the elevation of the centre of a continent is a fourth greater than its mean elevation. The area of the earth being represented by 4π, the area of a continent is $4\pi \sin^2 \frac{\gamma}{2}$. With the explained modification of Prof. Helmert's numbers we have the following table:

Continent.	$\sin^2 \frac{\gamma}{2}$.	Altitude.	γ.	$\frac{h}{a}$.	α.
Eurasia	$\frac{990\,000}{9\,261\,238}$	625^m	38° 10′	0.0000981	0.0003606
Africa	$\frac{540\,000}{9\,261\,238}$	625	27 57	0.0000981	0.0007256
Australia	$\frac{800\,000}{9\,261\,238}$	7956	34 11	0.0012489	0.0059327
N. America	$\frac{3\,000\,000}{9\,261\,238 \times 7}$	512½	24 50⅔	0.0000804	0.0007881
S. America	$\frac{2\,250\,000}{9\,261\,238 \times 7}$	3008.4	33 54.2	0.0004723	0.0022983

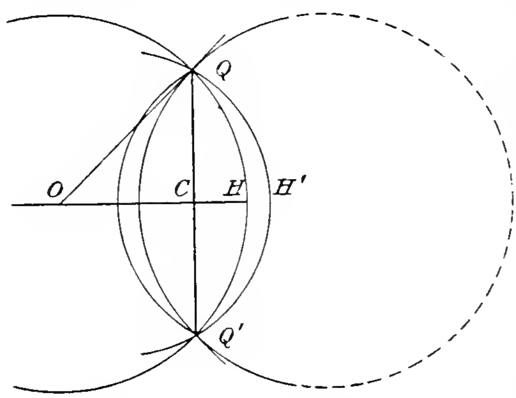
From these elements we obtain the following table of the values of gravity as modified by the action of each continent (the horizontal lines show the interval in which discontinuity occurs).

	Eurasia.		Africa.		Australia.		N. America.	
ϕ'.	g.	$\phi - \phi'$.	g.	$\phi - \phi'$.	g.	$\phi - \phi'$.	g.	$\phi - \phi'$.
180°	0.9999067	0″	0.9999068	0″	0.9988316	0″	0.9999236	0″
150	0.9999192	+ 5	0.9999193	+ 5	0.9989881	+ 60	0.9999338	+4
120	0.9999533	8	0.9999534	8	0.9994159	104	0.9999618	7
90	1.0000000	10	1.0000000	10	1.0000000	121	1.0000000	8
60	1.0000467	8	1.0000466	8	1.0005842	104	1.0000382	7
30	1.0000076	+ 5	1.0000807	+ 5	1.0004513	+ 60	1.0000662	+4
0	0.9998971	0	0.9998970	0	0.9986706	0	0.9999156	0

	S. America.	
ϕ'.	g.	$\phi - \phi'$.
180°	0.9995534	0″
150	0.9996132	+23
120	0.9997767	40
90	1.0000000	46
60	1.0002233	40
30	1.0001864	+23
0	0.9995018	0

Supposition V.

The foregoing distribution of density is extremely improbable, as the meniscus of negative density, intended to counterbalance the continent, is situated close to the surface of the opposite hemisphere. Therefore this hypothesis must be rejected. It is proposed to counterbalance the positive meniscus of the continent by a negative meniscus of the same magnitude, but reversed in position and so placed that the edges of the menisci are in contact. The adjacent figure exhibits a plane section of the two menisci through their common axis.



The combined action of the two menisci is equivalent to the differences of the action of four spherical segments standing on the same base QQ'. If the action of the segment $QH'Q'C$ is denoted by the symbol (1) and the action of $QHQ'C$ by (2), the action of this reversed by (3) and the action of the first reversed by (4), the action we seek for our two menisci is

$$(1) - (2) + (3) - (4),$$

or the segments of (2) and (4) should be regarded as having negative densities. Thus if we were in possession of the general formulæ for the attraction of the general spherical segment on an exterior point we should need nothing more. But no formulæ suitable for use are at hand and we must avail ourselves of the circumstance that the menisci may be regarded as infinitely thin.

Let x', 0, 0 be the coordinates of the attracted point, ρ the density of the menisci, the expressions for the forces are

$$X = \rho \iiint \frac{(x - x')\, dx\, dy\, dz}{[(x - x')^2 + y^2 + z^2]^{\frac{3}{2}}}, \qquad Y = \rho \iiint \frac{y\, dx\, dy\, dz}{[(x - x')^2 + y^2 + z^2]^{\frac{3}{2}}}.$$

Z evidently vanishes. The equation of the convex surface of the meniscus is

$$x^2 + y^2 + z^2 = a^2.$$

Let α be the maximum thickness of the meniscus, γ its angular radius, ϕ the angle its axis makes with the axis of x; ϕ ranges from 0 to π. Let us suppose that the equation of the concave surface is

$$(x + k \cos \phi)^2 + (y + k \sin \phi)^2 + z^2 = a'^2,$$

where k and a' are constants to be determined. This sphere must pass through the three points

$$\begin{cases} x = a \cos(\phi - \gamma) \\ y = a \sin(\phi - \gamma) \\ z = 0 \end{cases} \qquad \begin{cases} x = a \cos(\phi + \gamma) \\ y = a \sin(\phi + \gamma) \\ z = 0 \end{cases} \qquad \begin{cases} x = (a - \alpha) \cos\phi \\ y = (a - \alpha) \sin\phi \\ z = 0 \end{cases}$$

Thus result the two equations

$$2k \cos\gamma = a'^2 - a^2 - k^2, \qquad 2k(a - \alpha) = a'^2 - (a - \alpha)^2 - k^2,$$

whence

$$k = \frac{\alpha(2a - \alpha)}{2(a - \alpha - a\cos\gamma)}, \qquad a' = a - \alpha + k.$$

Rejecting quantities of the second order these values become

$$k = \frac{\alpha}{2\sin^2\frac{\gamma}{2}}, \qquad a' = a + \frac{\alpha\cos\gamma}{2\sin^2\frac{\gamma}{2}}.$$

The section of the meniscus by the plane $x = x$ is the whole or a portion of the circular ring in the plane yz having the radius

$$y^2 + z^2 = a^2 - x^2,$$

and, if the area of the section is A, we have

$$X = \rho\int \frac{(x - x')A}{[(x - x')^2 + a^2 - x^2]^{\frac{3}{2}}}dx = \rho\int \frac{(x - x')A}{[x'^2 + a^2 - 2x'x]^{\frac{3}{2}}}dx.$$

According to the position of the meniscus on the sphere in respect to the attracted point we have three different cases of limits in reference to x.

I. When $\phi + \gamma > 180°$,

$$X = \rho\int_{-a}^{a\cos(\phi+\gamma)} (\text{full ring})\,dx + \rho\int_{a\cos(\phi+\gamma)}^{a\cos(\phi-\gamma)} (\text{incomplete ring})\,dx.$$

II. When both $\phi - \gamma$ and $\phi + \gamma$ lie between $0°$ and $180°$.

$$X = \rho\int_{a\cos(\phi+\gamma)}^{a\cos(\phi-\gamma)} (\text{incomplete ring})\,dx.$$

III. When $\phi - \gamma < 0°$.

$$X = \rho\int_{a\cos(\phi+\gamma)}^{a\cos(\phi-\gamma)} (\text{incomplete ring})\,dx + \rho\int_{a\cos(\phi-\gamma)}^{a} (\text{full ring})\,dx.$$

In the first integral of Case I, A is the difference of the areas of the two circles whose radii are $\sqrt{a'^2 - (x + k\cos\phi)^2}$ and $\sqrt{a^2 - x^2}$; hence, for this term

$$X = \frac{\pi\rho\alpha}{\sin^2\frac{\gamma}{2}}\int_{-a}^{a\cos(\phi+\gamma)} \frac{(x - x')(a\cos\gamma - x\cos\phi)}{[x'^2 + a^2 - 2x'x]^{\frac{3}{2}}}dx.$$

In the second integral of the same case the circles bounding the section cross at two points whose coordinates are

$$y = \frac{a \cos \gamma - x \cos \phi}{\sin \phi}, \qquad z = \pm \sqrt{a^2 - x^2 - \left(\frac{a \cos \gamma - x \cos \phi}{\sin \phi}\right)^2}.$$

Taking θ from the equation

$$\cos \theta = \frac{x \cos \phi - a \cos \gamma}{\sin \phi \sqrt{a^2 - x^2}},$$

between the limits 0° and 180°, the area of the small circle diminished by the sector between $-z$ and $+z$ is $(\pi - \theta)\ (a^2 - x^2)$; and, similarly, the area of the large circle diminished by its corresponding sector is

$$(\pi - \theta')\,[a'^2 - (x + k \cos \phi)^2].$$

If, to the first, we add the area $(\theta - \theta')\ (a^2 - x^2)$, and subtract the area thus obtained from the latter area, we have the section of the meniscus

$$A = (\pi - \theta')\,[a'^2 - (x + k \cos \phi)^2] - (\pi - \theta')\,(a^2 - x^2) = \frac{a}{\sin^2 \frac{\gamma}{2}}(\pi - \theta)\,[a \cos \gamma - \cos \phi . x].$$

Consequently the second term in Case I has the value

$$X = \rho\, \frac{a}{\sin^2 \frac{\gamma}{2}} \int_{a \cos (\phi + \gamma)}^{a \cos (\phi - \gamma)} \frac{(x - x')\,(\pi - \theta)\,[a \cos \gamma - \cos \phi . x]}{[x'^2 + a^2 - 2x'x]^{\frac{3}{2}}}\, dx.$$

θ has the value 0 at the beginning of the integration and π at the end.

Integrating by parts, the second part of X is

$$X = \rho \frac{a}{\sin^2 \frac{\gamma}{2}} (\pi - \theta) \int_{a \cos (\phi + \gamma)}^{a \cos (\phi - \gamma)} \frac{(x - x')\,(a \cos \gamma - \cos \phi . x)}{[x'^2 + a^2 - 2x'x]^{\frac{3}{2}}}\, dx$$

$$+ \rho\, \frac{a}{\sin^2 \frac{\gamma}{2}} \int_{a \cos (\phi + \gamma)}^{a \cos (\phi - \gamma)} \frac{d\theta}{dx} \int \frac{(x - x')\,(a \cos \gamma - \cos \phi . x)}{[x'^2 + a^2 - 2x'x]^{\frac{3}{2}}}\, dx^2.$$

This joined to the first part, some of the terms cancel each other, and we have for the complete value of X in this case,

$$X = -\frac{\pi \rho a}{\sin^2 \frac{\gamma}{2}} [d_1 (x' + a)^{-1} + d_2 (x' + a) + d_3 (x' + a)^3]$$

$$+ \frac{\rho a}{\sin^2 \frac{\gamma}{2}} \int_{a \cos (\phi + \gamma)}^{a \cos (\phi - \gamma)} \frac{d\theta}{dx} [d_1 (x'^2 + a^2 - 2x'x)^{-\frac{1}{2}} + d_2 (x'^2 + a^2 - 2x'x)^{\frac{1}{2}} + d_3 (x'^2 + a^2 - 2x'x)^{\frac{3}{2}}]\, dx,$$

where

$$d_1 = -\frac{x'^2 - a^2}{4x'^3} [2ax' \cos \gamma - (x'^2 + a^2) \cos \phi], \qquad d_2 = \frac{1}{4x'^3} [2\,ax' \cos \gamma - 2\,a^2 \cos \phi],$$

$$d_3 = \frac{1}{12x'^3} \cos \phi.$$

From the preceding value of $\cos\theta$ we obtain

$$\frac{d\theta}{dx}=\frac{a\cos\gamma . x-a^2\cos\phi}{(a^2-x^2)\sqrt{a^2(\sin^2\phi-\cos^2\gamma)+2a\cos\phi\cos\gamma . x-x^2}}.$$

The radical of the denominator, as it vanishes for $x=a\cos(\phi+\gamma)$ and $x=a\cos(\phi-\gamma)$, must be equivalent to

$$\sqrt{[a\cos(\phi-\gamma)-x][x-a\cos(\phi+\gamma)]}.$$

And the factor a^2-x^2 maintains a finite value throughout the movement of x. Hence we put

$$x=a\cos\phi\cos\gamma-a\sin\phi\sin\gamma\cos u,$$

and thus have

$$\frac{dx}{\sqrt{[a\cos(\phi-\gamma)-x][x-a\cos(\phi+\gamma)]}}=du,$$

$$\frac{d\theta}{dx}dx=\frac{-\sin\gamma[\cos\phi\sin\gamma+\sin\phi\cos\gamma\cos u]}{1-\cos^2\phi\cos^2\gamma+\frac{1}{2}\sin2\phi\sin2\gamma\cos u-\sin^2\phi\sin^2\gamma\cos^2u}du.$$

The unintegrated part of X is then

$$X=\rho\frac{a\sin\gamma}{\sin^2\frac{\gamma}{2}}\int_0\frac{[\cos\phi\sin\gamma+\sin\phi\cos\gamma\cos u][d_6+d_7\cos u+d_8\cos^2u]}{[1-\cos^2\phi\cos^2\gamma+\frac{1}{2}\sin2\phi\sin2\gamma\cos u-\sin^2\phi\sin^2\gamma\cos^2u]\sqrt{d_4+d_5\cos u}}du,$$

where

$$d_4=x'^2+a^2-2ax'\cos\phi\cos\gamma,\qquad d_5=2ax'\sin\phi\sin\gamma,\qquad d_6=d_1+d_2d_4+d_3d_4^2,$$
$$d_7=d_2d_5+2d_3d_4d_5,\qquad d_8=d_3d_5^2.$$

This definite integral depends on elliptic integrals; a sufficiently accurate value of it can be obtained by computing the values of the quantity under the sign of integration for the five values of u, 0°, 45°, 90°, 135°, 180°; calling them U_0, U_1, U_2, U_3, U_4, the value of the integral is

$$X=\tfrac{1}{8}U_0+\tfrac{1}{4}U_1+\tfrac{1}{4}U_2+\tfrac{1}{4}U_3+\tfrac{1}{8}U_4.$$

In the second case we simply omit the part corresponding to the first term of Case I, and thus have

$$X=-\frac{\pi\rho a}{\sin^2\frac{\gamma}{2}}\Big[d_1[x'^2-2ax'\cos(\phi+\gamma)+a^2]^{-\frac{1}{2}}+d_2[x'^2-2ax'\cos(\phi+\gamma)+a^2]^{\frac{1}{2}}$$
$$+d_3[x'^2-2ax'\cos(\phi+\gamma)+a^2]^{\frac{3}{2}}\Big]$$
$$+\frac{\rho a\sin\gamma}{\sin^2\frac{\gamma}{2}}\int_0^\pi\frac{[\cos\phi\sin\gamma+\sin\phi\cos\gamma\cos u][d_6+d_7\cos u+d_8\cos^2u]}{[1-\cos^2\phi\cos^2\gamma+\frac{1}{2}\sin2\phi\sin2\gamma\cos u+\sin^2\phi\sin^2\gamma\cos^2u]\sqrt{d_4+d_5\cos u}}du.$$

In Case III the first term is the same as the second term of Case I, but the second term will need to be separated into two, in the latter of which we do not employ a development in powers of α. Thus this term will be

$$X=\frac{\pi\rho a}{\sin^2\frac{\gamma}{2}}\int_{a\cos(\phi-\gamma)}^{x_0}\frac{(x-x')(a\cos\gamma-\cos\phi . x)}{[x'^2+a^2-2x'x]^{\frac{3}{2}}}dx+\rho\iiint_{x_0}^{a}\frac{(x-x')\,dx\,dy\,dz}{[(x-x')^2+y^2+z^2]^{\frac{3}{2}}},$$

x_0 being an arbitrary quantity which will disappear. In the latter of the terms we can suppose

$$X = 2\pi\rho \iint \frac{x - x'}{[(x - x')^2 + r^2]^{\frac{3}{2}}} r\,dr\,dx .$$

The indefinite integral of the quantity under the signs of integration, with respect to r, is

$$-\frac{x - x'}{\sqrt{(x - x')^2 + r^2}} .$$

Taken between the limits $r = 0$, $r = r$, this gives

$$X = 2\pi\rho \int_{x_0}^{a} \left[\pm 1 - \frac{x - x'}{\sqrt{(x - x')^2 + r^2}} \right] dx ,$$

reading the upper member of the ambiguous sign when $x > x'$, and the lower when $x < x'$. But if the segment is cut off from the exterior sphere, $r^2 = a^2 - x^2$, and

$$X = 2\pi\rho \int_{x_0}^{a} \left[\pm 1 - \frac{x - x'}{\sqrt{x'^2 + a^2 - 2x'x}} \right] dx .$$

This gives for the exterior sphere

$$\begin{aligned} X = 2\pi\rho & \left[\pm a \mp \frac{x'^2 - a^2}{2x'^2}(a - x') \mp \frac{(a - x')^3}{6x'^2} \right] \\ & - 2\pi\rho \left[\mp x_0 - \frac{x'^2 - a^2}{2x'^2} \sqrt{x'^2 + a^2 - 2x'x_0} - \frac{1}{6x'^2}(x'^2 + a^2 - 2x'x_0)^{\frac{3}{2}} \right] . \end{aligned}$$

The limits for the interior sphere are $r^2 = 0$, $r^2 = a'^2 - x^2 - 2k \cos\phi \,.\, x$, and from $x = x_0$ to $x = a' - k\cos\phi$. Then

$$X = 2\pi\rho \int_{x_0}^{a' - k\cos\phi} \left[\pm 1 - \frac{x - x'}{\sqrt{x'^2 + a^2 - 2(x' + k\cos\phi)\,x}} \right] dx .$$

Consequently

$$\begin{aligned} X = 2\pi\rho & \left\{ \pm (a' - k\cos\phi) - \frac{(x' + k\cos\phi)^2 - a'^2}{2(x' + k\cos\phi)^2} \sqrt{x'^2 + a'^2 - 2(x' + k\cos\phi)(a' - k\cos\phi)} \right. \\ & \left. - \frac{1}{6(x' + k\cos\phi)^2}[x'^2 + a'^2 - 2(x' + k\cos\phi)(a' - k\cos\phi)]^{\frac{3}{2}} \right\} \\ - 2\pi\rho & \left\{ \mp x_0 - \frac{(x' + k\cos\phi)^2 - a'^2}{2(x' + k\cos\phi)^2} \sqrt{x'^2 + a^2 - 2(x' + k\cos\phi)\,x_0} \right. \\ & \left. - \frac{1}{6(x' + k\cos\phi)^2}[x'^2 + a^2 - 2(x' + k\cos\phi)\,x_0]^{\frac{3}{2}} \right\} . \end{aligned}$$

Proceeding to get the latter subtracted from the first, consider the terms which involve x_0. For brevity putting D for $x'^2 + a^2 - 2x'x_0$ the quantity we seek arises from

$$-2\pi\rho \left[\frac{1}{2}\left(1 - \frac{a^2}{x'^2}\right)\sqrt{D} + \frac{1}{6x'} D^{\frac{3}{2}} \right] ,$$

in which a^2 receives the augmentation $2a(a'-a)$, x' the augmentation $k\cos\phi$, and D the augmentation $2a(a'-a)-2ak\cos\phi$. Consequently we have

$$X=-2\pi\rho\left\{-\frac{\sqrt{D}}{x'}a(a'-a)+\left[\frac{a^2}{x'^3}\sqrt{D}-\frac{1}{3}\frac{D^{\frac{3}{2}}}{x'^3}\right]k\cos\phi\right.$$
$$\left.+\left[\frac{1}{2}\left(1-\frac{a^2}{x'^2}\right)\frac{1}{\sqrt{D}}+\frac{\sqrt{D}}{2x'^2}\right][a(a'-a)-2ak\cos\phi]\right\}.$$

But

$$a(a'-a)=\frac{a\alpha\cos\gamma}{2\sin^2\frac{\gamma}{2}},\quad ak\cos\phi=\frac{a\alpha\cos\phi}{2\sin^2\frac{\gamma}{2}},\quad a(a'-a)-ak\cos\phi=\frac{a\alpha}{2\sin^2\frac{\gamma}{2}}(\cos\gamma-\cos\phi).$$

Hence

$$X=-\pi\rho\frac{a\alpha}{\sin^2\frac{\gamma}{2}}\left\{-\cos\gamma\frac{\sqrt{D}}{x'^2}+\left[\frac{a}{x'^3}\sqrt{D}-\frac{1}{3}\frac{D^{\frac{3}{2}}}{ax'^3}\right]\cos\phi\right.$$
$$\left.+\frac{1}{2}\left[\left(1-\frac{a^2}{x'^2}\right)\frac{1}{\sqrt{D}}+\frac{\sqrt{D}}{x'^2}\right](\cos\gamma-\cos\phi)\right\}$$
$$=-\pi\rho\frac{a\alpha}{\sin^2\frac{\gamma}{2}}\left\{\frac{1}{2}\frac{x'^2-a^2}{x'^2}(\cos\gamma-\cos\phi)\frac{1}{\sqrt{D}}+\left(\frac{a\cos\phi}{x'}-\tfrac{1}{2}\cos\gamma-\tfrac{1}{2}\cos\phi\right)\frac{\sqrt{D}}{x'^2}\right.$$
$$\left.-\frac{1}{3}\frac{\cos\phi}{a}\frac{D^{\frac{3}{2}}}{x'^3}\right\}.$$

On putting $\sqrt{D}=a-x'$ this reduces to

$$X=-\frac{\pi\rho a}{x'^3\sin^2\frac{\gamma}{2}}[-a^2x'\cos\gamma+(\tfrac{1}{3}x'^3+a^2x'-\tfrac{1}{3}a^3)\cos\phi].$$

And the quantity resulting from the first integral at the upper limit is

$$X=\frac{\pi\rho a}{x'^3\sin^2\frac{\gamma}{2}}[a^2x'\cos\gamma+(-\tfrac{1}{3}x'^3+\tfrac{2}{3}a^3)\cos\phi].$$

The sum of these is

$$X=\frac{\pi\rho a}{x'^3\sin^2\frac{\gamma}{2}}[2a^2x'\cos\gamma+(-\tfrac{2}{3}x'^3-a^2x'+a^3)\cos\phi].$$

If $\sqrt{D}=x'-a$ this expression should be negatived. In fine, the difference of the two X's as far as it results from the terms independent of x_0, is

$$X=\frac{\pi\rho a}{x'^3\sin^2\frac{\gamma}{2}}[-a^2x'\cos\gamma+(\tfrac{1}{3}x'^3+a^2x'-\tfrac{1}{3}a^3)\cos\phi].$$

Hence the complete value of X in Case III is

$$X=\pm\frac{\pi\rho a}{x'^3\sin^2\frac{\gamma}{2}}[a^2x'\cos\gamma+(-\tfrac{1}{3}x'^3+\tfrac{2}{3}a^3)\cos\phi]$$
$$-\frac{\pi\rho a}{\sin^2\frac{\gamma}{2}}\left\{d_1[x'^2-2ax'\cos(\phi-\gamma)+a^2]^{-\frac{1}{2}}+d_2[x'^2-2ax'\cos(\phi-\gamma)+a^2]^{\frac{1}{2}}\right.$$
$$\left.+d_3[x'^2-2ax'\cos(\phi-\gamma)+a^2]^{\frac{3}{2}}\right\}$$
$$+\rho\int_{a\cos(\phi+\gamma)}^{a\cos(\phi-\gamma)}(\text{incomplete ring})\,dx,$$

where the upper member of the ambiguous sign must be taken when $a > x'$, and the lower when $a < x'$.

Taking up Y, the indefinite integral of the quantity under the sign of integration is

$$-\frac{1}{\sqrt{(x-x')^2+y^2+z^2}}.$$

The limits of integration are $y = \pm\sqrt{a^2-x^2-z^2}$ and

$$y = -k\sin\phi \pm \sqrt{a'^2-(x+k\cos\phi)^2-z^2}.$$

In the second portion of the course of integration with respect to x, we must integrate with respect to y from

$$y = -\sqrt{a^2-x^2-z^2} \text{ to } y = -k\sin\phi - \sqrt{a'^2-(x+k\cos\phi)^2-z^2},$$

and again from $y = -k\sin\phi + \sqrt{a'^2-(x+k\cos\phi)^2-z^2}$ to $y = \sqrt{a^2-x^2-z^2}$.

This gives

$$Y = \rho\iint\left[\frac{1}{\sqrt{(x-x')^2+a'^2-(x+k\cos\phi)^2+2k\sin\phi\sqrt{a^2-x^2-z^2}}}\right.$$
$$\left.+\frac{1}{\sqrt{(x-x')^2+a'^2-(x+k\cos\phi)^2-2k\sin\phi\sqrt{a^2-x^2-z^2}}}\right]dx\,dz.$$

In the first course of integration with respect to x, the limits of integration with respect to y are from

$$y = -k\sin\phi + \sqrt{a'^2-(x+k\cos\phi)^2-z^2} \text{ to } y = \sqrt{a^2-x^2-z^2},$$

which gives

$$Y = \rho\iint\left[\frac{1}{\sqrt{(x-x')^2+a'^2-(x+k\cos\phi)^2-2k\sin\phi\sqrt{a^2-x^2-z^2}}}\right.$$
$$\left.-\frac{1}{\sqrt{(x-x')^2+a^2-x^2}}\right]dx\,dz.$$

Rejecting quantities of the order of k^2, the two expressions become

$$Y = \rho\iint\frac{2k\sin\phi\sqrt{a^2-x^2-z^2}}{(x'^2+a^2-2x'x)^{\frac{3}{2}}}\,dx\,dz,$$

$$Y = \rho\iint\frac{a(a'-a)+k\cos\phi.x+k\sin\phi\sqrt{a^2-x^2-z^2}}{(x'^2+a^2-2x'x)^{\frac{3}{2}}}\,dx\,dz.$$

The limits of integration with respect to z are from $z = -\sqrt{a^2-x^2}$ to $z = \sqrt{a^2-x^2}$. In the terms involving the radical $\sqrt{a^2-x^2-z^2}$ we can put $z = \sqrt{a^2-x^2}\sin\eta$ with limits for η from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Thus

$$Y = \pi\rho k\sin\phi\int\frac{a^2-x^2}{(x'^2+a^2-2x'x)^{\frac{3}{2}}}\,dx,$$

$$Y = \rho\int\frac{2[a(a'-a)+k\cos\phi.x]\sqrt{a^2-x^2}+\frac{\pi}{2}k\sin\phi(a^2-x^2)}{(x'^2+a^2-2x'x)^{\frac{3}{2}}}\,dx.$$

The complete value of Y is the sum of these two definite integrals when the first is extended to all values of x for which the plane $x = x$ intersects the bounding surfaces of the meniscus four times, and when the second is extended to values for which the plane intersects once both the concave and convex surfaces. The values of x, for which the plane intersects only the convex surface, contribute nothing to the value of Y.

The indefinite integral

$$\int \frac{a^2 - x^2}{(x'^2 + a^2 - 2\,x'x)^{\frac{3}{2}}}\,dx = -\frac{(x'^2 - a^2)^2}{4\,x'^3}(x'^2 + a^2 - 2\,x'x)^{-\frac{1}{2}} - 2\,\frac{x'^2 + a^2}{4\,x'^3}(x'^2 + a^2 - 2\,x'x)^{\frac{1}{2}} + \frac{1}{12\,x'^3}(x'^2 + a^2 - 2\,x'x)^{\frac{3}{2}}.$$

Similarly as in X, we have the three cases as follows:

I. When $\phi + \gamma > 180°$.

$$Y = \frac{\rho a}{\sin^2 \frac{\gamma}{2}} \int_{a \cos(\phi + \gamma)}^{a \cos(\phi - \gamma)} \frac{[a \cos \gamma + \cos \phi . x]\sqrt{a^2 - x^2}}{(x'^2 + a^2 - 2\,x'x)^{\frac{3}{2}}}\,dx$$

$$+ \pi\rho\,\frac{a \sin\phi}{4 \sin^2 \frac{\gamma}{2}} \Big\{ -\frac{(x'^2 - a^2)^2}{4\,x'^3}\Big[(x'^2 - 2\,ax' \cos(\phi - \gamma) + a^2)^{-\frac{1}{2}} + (x'^2 - 2\,ax' \cos(\phi + \gamma) + a^2)^{-\frac{1}{2}} - \frac{2}{x' + a}\Big]$$

$$- 2\,\frac{x'^2 + a^2}{4\,x'^3}\Big[(x'^2 - 2\,ax' \cos(\phi - \gamma) + a^2)^{\frac{1}{2}} + (x'^2 - 2\,ax' \cos(\phi + \gamma) + a^2)^{\frac{1}{2}} - 2\,(x' + a)\Big]$$

$$+ \frac{1}{12\,x'^3}\Big[(x'^2 - 2\,ax' \cos(\phi - \gamma) + a^2)^{\frac{3}{2}} + (x'^2 - 2\,ax' \cos(\phi + \gamma) + a^2)^{\frac{3}{2}} - 2\,(x' + a)^3\Big]\Big\}.$$

II. When both $\phi - \gamma$ and $\phi + \gamma$ lie between 0° and 180°.

$$Y = \frac{\rho a}{\sin^2 \frac{\gamma}{2}} \int_{a \cos(\phi + \gamma)}^{a \cos(\phi - \gamma)} \frac{[a \cos \gamma + \cos \phi . x]\sqrt{a^2 - x^2}}{(x'^2 + a^2 - 2\,x'x)^{\frac{3}{2}}}\,dx$$

$$+ \pi\rho\,\frac{a \sin\phi}{4 \sin^2 \frac{\gamma}{2}} \Big\{ -\frac{(x'^2 - a^2)^2}{4\,x'^3}\Big[(x'^2 - 2\,ax' \cos(\phi - \gamma) + a^2)^{-\frac{1}{2}} - (x'^2 - 2\,ax' \cos(\phi + \gamma) + a^2)^{-\frac{1}{2}}\Big]$$

$$- 2\,\frac{x'^2 + a^2}{4\,x'^3}\Big[(x'^2 - 2\,ax' \cos(\phi - \gamma) + a^2)^{\frac{1}{2}} - (x'^2 - 2\,ax' \cos(\phi + \gamma) + a^2)^{\frac{1}{2}}\Big]$$

$$+ \frac{1}{12\,x'^3}\Big[(x'^2 - 2\,ax' \cos(\phi - \gamma) + a^2)^{\frac{3}{2}} - (x'^2 - 2\,ax' \cos(\phi + \gamma) + a^2)^{\frac{3}{2}}\Big]\Big\}.$$

III. When $\phi - \gamma < 0°$.

$$
\begin{aligned}
Y = {} & \frac{\rho a}{\sin^2 \frac{\gamma}{2}} \int_{a \cos(\phi+\gamma)}^{a \cos(\phi-\gamma)} \frac{[a \cos \gamma + \cos \phi . x] \sqrt{a^2 - x^2}}{(x'^2 + a^2 - 2x'x)^{\frac{3}{2}}} dx \\
& + \pi \rho \frac{a \sin \phi}{4 \sin^2 \frac{\gamma}{2}} \Big\{ - \frac{(x'^2 - a^2)^2}{4x'^3} \Big[-(x'^2 - 2ax' \cos(\phi - \gamma) + a^2)^{-\frac{1}{2}} \\
& \qquad\qquad - (x'^2 - 2ax' \cos(\phi + \gamma) + a^2)^{-\frac{1}{2}} \pm \frac{2}{x' - a} \Big] \\
& \qquad - 2 \frac{x'^2 + a^2}{4x'^3} \Big[-(x'^2 - 2ax' \cos(\phi - \gamma) + a^2)^{\frac{1}{2}} \\
& \qquad\qquad - (x'^2 - 2ax' \cos(\phi + \gamma) + a^2)^{\frac{1}{2}} \pm 2(x' - a) \Big] \\
& \qquad + \frac{1}{12x'^3} \Big[-(x'^2 - 2ax' \cos(\phi - \gamma) + a^2)^{\frac{3}{2}} \\
& \qquad\qquad - (x'^2 - 2ax' \cos(\phi + \gamma) + a^2)^{\frac{3}{2}} \pm 2(x' - a)^3 \Big] \Big\}.
\end{aligned}
$$

The upper member of the ambiguous sign must be taken when $x' > a$, and the lower when $x' < a$. The definite integral appearing in all the cases is elliptic, and may be reduced to the simplest form. In order to do this we assume the change of variables

$$\frac{a - x}{a + x} = \pm \frac{a - x'}{a + x'} \frac{1 - y}{1 + y},$$

the ambiguous sign to be so read that $\pm (a - x')$ shall be positive (the case $a - x' = 0$ will be treated shortly). Thus

$$\frac{dx}{\sqrt{(a^2 - x^2)(x'^2 + a^2 - 2x'x)}} = m \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}}.$$

When $x' > a$ the values of m and k are

$$m = \sqrt{\frac{2}{x'}}, \qquad k^2 = \frac{a^2}{x'^2},$$

and when $x' < a$,

$$m = \sqrt{\frac{2x'}{a^2}}, \qquad k^2 = \frac{x'^2}{a^2}.$$

When $x' > a$ the limits of integration are from

$$y = \frac{a \cos(\phi + \gamma) + x'}{a + x' \cos(\phi + \gamma)} \text{ to } y = \frac{a \cos(\phi - \gamma) + x'}{a + x' \cos(\phi - \gamma)},$$

and when $x' < a$ the reciprocal of these.

In the first case

$$\frac{[a \cos \gamma + \cos \phi . x](a^2 - x^2)}{x'^2 + a^2 - 2x'x} = -a^3 \frac{[(a + x'y) \cos \gamma + (x' + ay) \cos \phi](1 - y^2)}{(a + x'y)^2 (a - x'y)},$$

and, in the second case,

$$\frac{[a \cos \gamma + \cos \phi . x](a^2 - x^2)}{x'^2 + a^2 - 2x'x} = -a^3 \frac{[(x' + ay) \cos \gamma + (a + x'y) \cos \phi](1 - y^2)}{(x' + ay)^2 (x' - ay)}.$$

We can put $y = \sin\theta$, and then

$$\frac{[a\cos\gamma + \cos\phi . x]\sqrt{a^2 - x^2}}{[x'^2 + a^2 - 2x'x]^{\frac{3}{2}}}dx = -a^3\sqrt{\frac{2}{x'}}\frac{[(a + x'\sin\theta)\cos\gamma + (x' + a\sin\theta)\cos\phi]\cos^2\theta}{(a + x'\sin\theta)^2(a - x'\sin\theta)\sqrt{1 - \frac{a^2}{x'^2}\sin^2\theta}}d\theta,$$

and, in the second case,

$$\frac{[a\cos\gamma + \cos\phi . x]\sqrt{a^2 - x^2}}{[x'^2 + a^2 - 2x'x]^{\frac{3}{2}}}dx = -a^3\sqrt{\frac{2x'}{a^2}}\frac{[(x' + a\sin\theta)\cos\gamma + (a + x'\sin\theta)\cos\phi]\cos^2\theta}{(x' + a\sin\theta)^2(x' - a\sin\theta)\sqrt{1 - \frac{x'^2}{a^2}\sin^2\theta}}d\theta.$$

It seems that farther prosecution of the reduction would lead to more complication. Thus it will be less laborious if mechanical quadratures are used. For this purpose we transform by making $x = a\cos\eta$. Then the definite integral becomes

$$\frac{\rho a a^3}{\sin^2\frac{\gamma}{2}}\int_{\phi-\gamma}^{\phi+\gamma}\frac{(\cos\gamma + \cos\phi\cos\eta)\sin^2\eta}{[(x' - a)^2 - 2ax'\cos\eta]^{\frac{3}{2}}}d\eta.$$

After X and Y have been determined for the anti-meniscus, as these forces are referred to the centre of its sphere, they must be changed to the centre of the meniscus. Let the forces here determined with a positive density for the anti-meniscus be called X' and Y'. Then the forces which ought to be added to X and Y are severally

$$\frac{-(x' - 2a\cos\gamma\cos\phi)X' + 2a\cos\gamma\sin\phi . Y'}{\sqrt{(x' - 2a\cos\gamma\cos\phi)^2 + 4a^2\cos^2\gamma\sin^2\phi}},$$

$$\frac{2a\cos\gamma\sin\phi . X' - (x' - 2a\cos\gamma\cos\phi)Y'}{\sqrt{(x' - 2a\cos\gamma\cos\phi)^2 + 4a^2\cos^2\gamma\sin^2\phi}}.$$

The preceding equations have purposely been made general. However, a great reduction results when we suppose that $x' = a$. In the case of the attraction of the continent on any point situated on the earth's surface, this supposition can be adopted as the meniscus is assumed to be infinitely thin. But in the case of the anti-meniscus the relation holds only when the attracted point is at the edge of the continent. The principal modifications in the formulæ may be noted.

For X we have

$$X = -\frac{\rho}{(\sqrt{2a})^3}\int\frac{A\,dx}{\sqrt{a - x}}.$$

In the first integral of Case I,

$$X = -\frac{\pi\rho a}{(\sqrt{2a})^3\sin^2\frac{\gamma}{2}}\int_{-a}^{a\cos(\phi+\gamma)}\frac{a\cos\gamma - \cos\phi . x}{\sqrt{a - x}}dx.$$

And the second term

$$X = -\frac{\rho a}{(\sqrt{2a})^3\sin^2\frac{\gamma}{2}}\int_{a\cos(\phi+\gamma)}^{a\cos(\phi-\gamma)}\frac{(\pi - \theta)(a\cos\gamma - \cos\phi . x)}{\sqrt{a - x}}dx.$$

Integrating by parts

$$X = -\frac{\rho a}{(\sqrt{2a})^3 \sin^2\frac{\gamma}{2}}(\pi-\theta)\int_{a\cos(\phi+\gamma)}^{a\cos(\phi-\gamma)} \frac{a\cos\gamma - \cos\phi . x}{\sqrt{a-x}}\,dx$$

$$-\frac{\rho a}{(\sqrt{2a})^3 \sin^2\frac{\gamma}{2}}\int_{a\cos(\phi+\gamma)}^{a\cos(\phi-\gamma)} \frac{d\theta}{dx}\int \frac{a\cos\gamma - \cos\phi . x}{\sqrt{a-x}}\,dx^2.$$

We have here

$$d_1 = 0, \qquad d_2 = \frac{\cos\gamma - \cos\phi}{2a}, \qquad d_3 = \frac{\cos\phi}{12a^3},$$

and the complete value of X in this case is

$$X = -\frac{\pi\rho a}{\sin^2\frac{\gamma}{2}}(\cos\gamma - \tfrac{1}{3}\cos\phi)$$

$$+\frac{\rho a}{\sqrt{2a}\sin^2\frac{\gamma}{2}}\int_{a\cos(\phi+\gamma)}^{a\cos(\phi-\gamma)} \frac{d\theta}{dx}\left[(\cos\gamma-\cos\phi)+\tfrac{1}{3}\cos\phi\frac{a-x}{a}\right]\sqrt{a-x}\,dx.$$

Also here we have

$$d_4 = 2a^2(1-\cos\phi\cos\gamma), \qquad d_5 = 2a^2\sin\phi\sin\gamma,$$
$$d_6 = a(1-\cos\phi\cos\gamma)\left[\tfrac{2}{3}(\cos\gamma-\cos\phi)+\tfrac{1}{3}\sin^2\phi\cos\gamma\right],$$
$$d_7 = a\sin\phi\sin\gamma\left[\tfrac{1}{3}(\cos\gamma-\cos\phi)+\tfrac{2}{3}\sin^2\phi\cos\gamma\right], \qquad d_8 = \tfrac{1}{3}\cos\phi\sin^2\phi\sin^2\gamma.$$

The definite integral in X is

$$X = \frac{\rho a\sin\gamma\sin\phi}{\sin^2\frac{\gamma}{2}}\int_0^\pi \frac{[\cos\phi\sin\gamma+\sin\phi\cos\gamma\cos u][\cos\gamma\sin\phi+\sin\gamma\cos\phi\cos u]}{[1+\cos\phi\cos\gamma-\sin\phi\sin\gamma\cos u]\sqrt{1-\cos\phi\cos\gamma+\sin\phi\sin\gamma\cos u}}\,du.$$

In the second case

$$X = -\frac{\pi\rho a}{\sin^2\frac{\gamma}{2}}\left[(\cos\gamma-\cos\phi)\sin\frac{\phi+\gamma}{2}+\tfrac{2}{3}\cos\phi\sin^3\frac{\phi+\gamma}{2}\right] + \text{same definite integral.}$$

The complete value of X in Case III is

$$X = -\frac{\pi\rho a}{\sin^2\frac{\gamma}{2}}\left[\cos\gamma+\tfrac{1}{3}\cos\phi+(\cos\gamma-\cos\phi)\sin\frac{\phi-\gamma}{2}+\tfrac{2}{3}\cos\phi\sin^3\frac{\phi-\gamma}{2}\right]$$
$$+ \text{same definite integral.}$$

The two expressions for Y become

$$Y = \frac{\pi\rho k\sin\phi}{(\sqrt{2a})^3}\int\frac{a+x}{\sqrt{a-x}}\,dx,$$

$$Y = \frac{\rho}{(\sqrt{2a})^3}\int\left\{2\left[a(a'-a)+k\cos\phi . x\right]\frac{\sqrt{a+x}}{a-x}+\frac{\pi}{2}k\sin\phi\frac{a+x}{\sqrt{a-x}}\right\}dx.$$

The indefinite integral

$$\frac{1}{(\sqrt{2a})^3}\int\frac{a+x}{\sqrt{a-x}}\,dx = -\frac{1}{\sqrt{2a}}\left(\frac{5}{3}+\frac{1}{3}\frac{x}{a}\right)\sqrt{a-x}.$$

The indefinite integral

$$\int \frac{[a\,\gamma \cos - \cos\phi . x]\,\sqrt{a+x}}{(\sqrt{2a})^3\,(a-x)}\,dx$$

$$= \tfrac{2}{3}\cos\phi\left(\frac{a+x}{2a}\right)^{\frac{3}{2}} - (\cos\gamma - \cos\phi)\left[\frac{\sqrt{a+x}}{2a} - \tfrac{1}{2}\log\frac{\sqrt{2a}+\sqrt{a+x}}{\sqrt{2a}-\sqrt{a+x}}\right].$$

In consequence the definite integral which appears in all three cases of Y, is integrable and has the value

$$Y = \frac{\rho a}{\sin^2\frac{\gamma}{2}}\left\{\tfrac{2}{3}\cos\phi\left[\cos^3\frac{\phi-\gamma}{2} - \cos^3\frac{\phi+\gamma}{2}\right]\right.$$

$$\left. - (\cos\gamma - \cos\phi)\left[\cos\frac{\phi-\gamma}{2} - \cos\frac{\phi+\gamma}{2} - \log\frac{\cot\frac{\phi-\gamma}{4}}{\cot\frac{\phi+\gamma}{4}}\right]\right\}.$$

The subsidiary terms which must be added to this in each case are

Case I. $\dfrac{\pi\rho a \sin\phi}{2\sin^2\frac{\gamma}{2}}\left\{-\left[\sin\frac{\phi-\gamma}{2} + \sin\frac{\phi+\gamma}{2} - 2\right] + \tfrac{1}{3}\left[\sin^3\frac{\phi-\gamma}{2} + \sin^3\frac{\phi+\gamma}{2} - 2\right]\right\},$

Case II. $\dfrac{\pi\rho a \sin\phi}{2\sin^2\frac{\gamma}{2}}\left\{-\left[\sin\frac{\phi-\gamma}{2} - \sin\frac{\phi+\gamma}{2}\right] + \tfrac{1}{3}\left[\sin^3\frac{\phi-\gamma}{2} - \sin^3\frac{\phi+\gamma}{2}\right]\right\},$

Case III. $\dfrac{\pi\rho a \sin\phi}{2\sin^2\frac{\gamma}{2}}\left\{\left[\sin\frac{\phi-\gamma}{2} + \sin\frac{\phi+\gamma}{2}\right] - \tfrac{1}{3}\left[\sin^3\frac{\phi-\gamma}{2} + \sin^3\frac{\phi+\gamma}{2}\right]\right\}.$

When the attracted point is on the axis of the menisci, Y vanishes and for X the action of the meniscus we have

$$X = -\frac{\pi\rho a}{3\sin^2\frac{\gamma}{2}}\left[1 - 3\frac{a^2}{x'^2}\cos\gamma + 2\frac{a^3}{x'^3} - \left(3\frac{D}{x'^2} - 2\frac{\Delta^2}{x'^3}\right)\Delta\right],$$

where

$$D = x' - a\cos\gamma, \qquad \Delta^2 = D^2 + a^2\sin^2\gamma.$$

The action of the anti-meniscus is

$$X = -\frac{\pi\rho a}{3\sin^2\frac{\gamma}{2}}\left[1 - 3\frac{a^2}{u'^2}\cos\gamma - 2\frac{a^3}{u'^3} - \left(3\frac{D}{u'^2} - 2\frac{\Delta^2}{u'^3}\right)\Delta\right],$$

where D and Δ have the same signification as before, but $u' = x' - 2a\cos\gamma$. This must be subtracted from the action of the meniscus.

Some remarks should be added respecting the degree of approximation obtained in the preceding investigation. While it is true that when the attracted point is at a finite distance from the surface of the meniscus, the attraction can be developed in a series arranged according to ascending integral powers of α, this is no longer true when the point is on the surface, when the proper form of the series is

$$A_1\,\alpha + A_2\,\alpha^{\frac{3}{2}} + A_3\,\alpha^2 + A_4\,\alpha^{\frac{5}{2}} + \ldots$$

If α is about $\frac{1}{10000}$, in taking $A_1\,\alpha$ as the value, we must expect that the error committed is about 1 per cent, instead of a $\frac{1}{10000}$ part. Also the expansion of the potential in a series of integral powers of $\frac{1}{x'}$ is, in this case, divergent. Thus the method proposed by Tisserand* is quite nugatory. The trouble appears to be that the surface of the meniscus is discontinuous at the edge.

I have computed the values of g from the preceding data for the five continents at their summits and their antipodes; the results are in the following table:

	Eurasia.	Africa.	Australia.	N. America.	S. America.
ϕ'.	g.	g.	g.	g.	g.
180°	0.9999993	0.9999998	0.9999946	0.9999999	0.9999980
0	0.9998669	0.9998557	0.9982539	0.9998796	0.9993395

As in the former suppositions we must ascertain the displacement in the centre of gravity of the earth produced by the new arrangement of the matter. Let V_1, V_2, V_3, V_4 be the volumes of the spherical segments in the order of their previous consideration. We have $V_3 = V_1$, and $V_4 = V_2$. Next let x_1, x_2, x_3, x_4 be the x coordinates of their several centres of gravity; we have

$$x_3 = 2\,a\cos\gamma - x_1, \qquad x_4 = 2\,a\cos\gamma - x_2.$$

The shift of the centre of gravity of the earth along the axis of, and towards, the summit of the continent will be

$$\mathrm{X} = \frac{\rho}{M}(V_2 x_2 - V_1 x_1 + V_3 x_3 - V_4 x_4).\dagger$$

On making the just mentioned substitutions this becomes

$$\mathrm{X} = \frac{\rho}{M}[2\,(V_2 x_2 - V_1 x_1) - 2\,a\cos\gamma\,(V_2 - V_1)].$$

---

* Mécanique Céleste, Tom. II, p. 319.

† Poisson, Mécaniqe, Tom. I, pp. 155–156.

But if H denote the altitude of the first segment and J the radius of its base, we have

$$V_1 = \tfrac{2}{3}\pi a^2 H - \tfrac{1}{3}\pi J^2 (a - H),$$
$$V_1 x_1 = \tfrac{1}{2}\pi a^2 H (a - \tfrac{1}{2} H) - \tfrac{1}{4}\pi J^2 (a - H)^2.$$

In order to obtain the values of $V_2 - V_1$ and $V_2 x_2 - V_1 x_1$, we must suppose that a receives an increment $= -\dfrac{\alpha \cos\gamma}{2\sin^2\frac{\gamma}{2}}$, and H an increment α. Thus

$$V_2 - V_1 = [\tfrac{2}{3}\pi a^2 + \tfrac{1}{3}\pi J^2]\,\alpha - [\tfrac{4}{3}\pi a H - \tfrac{1}{3}\pi J^2]\,\frac{\alpha\cos\gamma}{2\sin^2\frac{\gamma}{2}},$$

$$V_2 x_2 - V_1 x_1 = [\pi a^2 (a - \tfrac{3}{4} H) + \tfrac{1}{2}\pi J^2 (a - H)]\,\frac{\alpha}{2\sin^2\frac{\gamma}{2}} - [\pi a (H + a)(a - \tfrac{1}{2} H)]\,\frac{\alpha\cos\gamma}{2\sin^2\frac{\gamma}{2}}.$$

Recalling that $H = a(1 - \cos\gamma)$ and $J = a\sin\gamma$, $M = \frac{4\pi}{3} R a^3$, on making the substitutions we get

$$x = \frac{3}{8}\,\frac{\alpha}{2\sin^2\frac{\gamma}{2}}\,\frac{\rho}{R}\,(1 - \cos\gamma)(1 - 2\cos\gamma)(\tfrac{1}{4} - \cos\gamma).$$

The corresponding changes in the colatitude are tabulated below :

	Eurasia.	Africa.	Australia.	N. America.	S. America.
ϕ'.	$\phi - \phi'$.	$\phi - \phi'$.	$\phi - \phi'$.	$\phi - \phi'$.	$\phi - \phi'$.
180°	0″	0″	0″	0″	0″
150	+1	+2	+18	+2	+ 7
120	2	3	30	3	12
90	2	3	35	3	13
60	2	3	30	3	12
30	+1	+2	+18	+2	+ 7
0	0	0	0	0	0

After having considered the irregularities of distribution of matter in the solid portion of the earth, it remains to consider the actions of the fluids upon the earth's surface on the intensity and direction of gravity. It suffices to suppose that the ocean has the form of a meniscus invaded however by the continents. Of the latter there may be several sorts. A continent may be untouched by the ocean, or a certain portion of its rim be under water or again the whole of it while the summit is still above, and, lastly, the continent may be wholly submerged. In all these cases, except the first, the proper allowance must be made for the displacement of the water.

As to the action of the atmosphere on gravity, it is plain that it is too minute to need consideration. For this fluid may be supposed to form a spherical shell whose action on interior points is nothing. Nevertheless the mass of the earth for use in pendulum experiments ought to be less than that used for the action on distant bodies, as the moon, by the amount of the mass of the atmosphere. A sufficiently approximate value of the latter is obtained by supposing it equivalent to that of a sea of mercury $0\overset{m}{.}75$ deep and covering the surface of the earth. The protrusion of the continents tends to diminish this estimate, but the diminution of gravity, in going upward, tends to increase it. These two causes may nearly offset each other.

INDEX

The Roman numerals indicate the volumes, the Arabic the pages, except for the Introduction where Roman are used for both.

ERRATA

(Lines counted from the bottom of the page are noted as negative.)

VOL. I.

Page	Line	
10	14	for T read T_i
13	2	for $\frac{8\pi^2}{3T}$ read $\frac{8\pi^2}{3T^2}$
13	12	for $\frac{1}{9}q$ read $\frac{1}{9}q^2$
13	15	for $2V_2''$ read $2V_1''$
14	9	for $-(\frac{1}{2}e_2 - \frac{1}{4}e_1^2)$ read $+(\frac{1}{2}e_2 - \frac{1}{4}e_1^2)$
18	2	for $m_2 V_0$ read $m^2 V_0$
18	−5	for $\frac{V_0'}{m^2 V_0}$ read $-\frac{V_0'}{m^2 V_0}$
19	21	for $-1.''3679$ read $+1.''3679$
60	−3	for $n\frac{dm}{dt}$ read $n\frac{dm}{dt})$
62	−9	add the term $6\mu'^2$ to the factor of $\tan\delta$
65	−9	for $+\frac{1}{k^2}$ read $+\frac{1}{k^2})$
69	14	for $\varDelta a$ read $\varDelta_{\odot} a$
78	20	for λ read λ_1
79	21	for an read and
124	15	for $\cos\theta'$ read $\cos\theta$
133	11	for $\cos\varphi$ read $\cos\varphi_1$
159	17	add dv to second member of equation
185	5	for $D_{\frac{1}{2}}^{-1}$ read $D_{\frac{1}{2}}^{-2}$
193	−3	for $\frac{\partial T}{d q_i}$ read $\frac{\partial T}{d q_i'}$
201	4	add $\sqrt{}$ before $(a^2 - 2al\cos\theta + l^2)$
204	16	for $\int_0^{\pi}$ read $\int_0^{\frac{\pi}{2}}$
210	19	for established read establishes
210	20	for $\int^{\pi}$ read $\int_0^{\pi}$
219	−3	for $3\,\mathrm{E}\,\mathrm{E}_{-6}'$ read $3\,\mathrm{E}_5\,\mathrm{E}_{-6}'$
224	12	for $\frac{I}{1}$ read $\frac{I}{2}$ and transpose an arm of parenthesis to one line lower
225	10	for $\frac{r}{a}\varepsilon^{v\sqrt{-}}$ read $\frac{r}{a}\varepsilon^{v\sqrt{-1}}$
226	4	for $J_{\frac{ie}{2}}^{(i-1}$ read $J_{\frac{ie}{2}}^{(i-1)}$
230	2	for $=$ read $-$
230	3	for $=$ read $-$
238	19	for $\frac{\partial\Omega}{y}$ read $\frac{\partial\Omega}{\partial y}$
242	−7	for $\mathrm{w}\nu + \mathrm{w}\nu' + \mathrm{w}\nu''$ read $\mathrm{w}\nu + \mathrm{w}'\nu' + \mathrm{w}''\nu''$
251	9	for af read of
252	6	for a_{i-} read a_{i-j}
252	18	for $\frac{x}{r^3} + \mathrm{m}$ read $\frac{x}{r^3} + \mathrm{m}^2$
252	−2	for U_- read U_-
259	−7	for m^5 read m^6
264	4	for $\pi\theta_1$ read $\pi\theta_1^2$
265	−1	for 4θ read $4\theta^2$
266	11	for Σ read Σ
274	−1	for aP read aP^{β}
278	13	for aP^{β} read aP_0^{β}
293	13	add the exponent $\frac{1}{2}$ to the right member
293	17	add the exponent $\frac{1}{2}$ to the denominator
296	17	for $(\mu n)^2$ read $(\mu n)^{\frac{2}{3}}$
304	22	for $(2C)$ read $(2C)^{\frac{3}{2}}$
308	2	for (us) read $(us)^{\frac{1}{2}}$
308	−2	for $\mathrm{a}_i\mathrm{a}_{-i--1}$ read $\mathrm{a}_i\mathrm{a}_{-i-j-1}$
309	10	for $\mathrm{a}_i\mathrm{a}_{-i--1}$ read $\mathrm{a}_i\mathrm{a}_{-i-j-1}$
310	−1	for $+[2, 2]$ read $+[2, -2]$
314	20	for N_3 read N_2
320	3	for $\left[\frac{\mu}{n^2}\right]$ read $\left[\frac{\mu}{n^2}\right]^{\frac{1}{3}}$ and add the exponent $\frac{1}{3}$ to the last factor
324	−6	for a_0^2 read a_0^2
324	−1	for $\left[\frac{\mu}{n^3}\right]$ read $\left[\frac{\mu}{n^2}\right]^{\frac{1}{3}}$

Page	Line	
326	23	for r sin read $r \sin v$
326	27	for $r\cos, r\sin\upsilon$ read $r\cos\upsilon, r\sin\upsilon$
326	31	" " " " " "
326	36	" " " " " "
328	6	for $\frac{y}{r^2}$ read $\frac{y}{r^3}$
331	-9	for $\frac{47}{9072}$ read $\frac{47}{9072}a^4$
340	-9	for $-\frac{3}{4}n$ read $-\frac{3}{4}n'$
346	8	for $d.e'$ read $d.e'^2$
346	-5	for n' read n'^2
349	-9	for $J^{(2)}_{\frac{}{2}}$ read $J^{(2)}_{\frac{e}{2}}$
353	-7	add exponent $\frac{1}{2}$ to last factor in denominator
359	-9	for λ' read λ_0
360	26	for $\frac{1}{4}\eta^4$ read $\frac{1}{2}\eta^4$
361	15	for $\frac{1}{8}\eta e^3$ read $\frac{1}{6}\eta e^3$
362	22	for Γ'_2 read Γ_2
362	4	for l_2 read l'_2

Vol. II.

Page	Line	
10	15	for $G'\alpha''\gamma''$ read $G''\alpha''\gamma''$
13	16	for G in denom. read G'
14	4	for $G-G'$ in denom. read $G-G''$
16	9	for $\cos T$ read $\cos^2 T$
19	4	for NG'^2 read $N'G'^2$
24	15	for $\log k$ read $\log k'$
43	6	for $\nu -$ read $\nu =$
56	17	for $[n-2]$ read $[n-2]_1$
59	9	for s_{+1} read s_{p+1}
70	-5	for -0τ read -2τ
71	16	for 3^2 read $3m^2$
71	16	for $\mathrm{m}\frac{3}{2}^3$ read $\frac{3}{2}\mathrm{m}^3$
75	9	for $b_5^{(0)}$ read $b_{\frac{5}{2}}^{(0)}$
108	7	for $m\rho^2 u$ read $m'\rho^2 u$
108	19	for $h_1\frac{d}{dt}$ read $h_1\frac{ds}{dt}$
110	13	for $a_2 - a$ read $a_2 - a_1$
122	11	for $2\psi + 2\mathrm{a}_i$ read $2\psi + a_i$
150	8	for $s - \omega'$ read $s' - \omega'$
152	16	for $E^{(j)}_{-}$ read $E^{(j)}_{-i}$
164	11	for $\sin\varphi$ read $\sin\varphi'$
175	-2	for $e'\begin{matrix}\sin\\ \cos\end{matrix}\tilde{\omega}$ read $e'\begin{matrix}\sin\\ \cos\end{matrix}\tilde{\omega}'$

Vol. IV.

Page	Line	
4	7	for $+3\cos l$ read $+3\cos l'$
5	5	for e'^2 read e'^2_0
15	12	for $(1-4\gamma^2)U^2$ read $(1-4\gamma^2)U_1^2$
29	8	for $e'\cos\phi'$ read $(e'\cos\phi'$
37	-6	for $\cos 2n't + c$ read $\cos 2(n't + c)$
81	17	for $\frac{dr}{dt^2}$ read $\frac{d^2r}{dt^2}$
122	3	for $\sin i\,\Omega$ read $\sin i\delta\,\Omega$
142	17	symbol θ in too small type
148	9	for $3\gamma(1-c^2)^2$ read $3\gamma(1-e^2)^2$
174	8	for $>$ read $<$
182	14	for θ read θ_1
182	17	for $[\theta_0]$ read $[\theta_0$
182	-1	for 2θ read 2θ
204	6	for $e\,a_0$ read $e = a_0$
210	12	for $(1+x^2)^2$ read $(1+x^2)^{\frac{1}{2}}$
225	2	for $\alpha'\beta''$ read $\alpha''\beta''$
225	-1	for $G + b^2$ read $G_z + b^2$
233	19	for $-cx$ read $c_z x$
256	2	for $\int^{\pi}$ read $\int^{\pi}$
271	26	for $+j'\frac{A}{\nu}$ read $-j\frac{A}{\nu}$
271	26	for $\frac{1}{\nu}\frac{\partial A}{\partial F'}$ read $\frac{1}{\nu}\frac{\partial A}{\partial G'}$
277	17	for $-C^2 - 0$ read $-C^2 = 0$
278	$-2,3$	for $+\sqrt{4-c^2}\ \&c$ read $-\sqrt{4-c^2}\ \&c$
288	6	for a_γ read a_ν
291	-3	for g read g_i
310	16	for F_{xxx} read F_{xxx0}
311	-1	for $x^{2\ +1}x^{2i+}x^{2\ +1}x^0$ read $x^{2i+1}x^{2i+1}x^{2i+1}x^0$
314	30	add an additional "
322	-10	for $\frac{1}{2}(\psi+\phi)$ read $\frac{1}{2}(\psi-\phi)$
332	23	for μ^3 read μ_3
332	24	for m and m' read m and m'
342	29	supply the sign $=$
349	6	for $\frac{1}{2}h\eta$ read $\frac{1}{2}h_0\eta$
349	-4	for $h_4\eta^2\eta$ read $h_4\eta^2\eta'$
349	7	for $-h$ read $-h_1$
350	12	for $a_{14}\eta'_4$ read $a_{14}\eta'^4$
352	4	for $-a$ read a_5
352	10	for $c_5\eta'_3$ read $c_5\eta'^3$

Page	Line	
352	11	for $b\,\eta'$ read $b_2\,\eta'$
353	19	for $-2\,h_0\,h_2$ read $-2\,h_0\,h_2\,\eta\eta'^2$
354	28	for $a\,\eta^2\,\eta'^2$ read $a_{34}\,\eta^2\,\eta'^2$
356	9	for $c_{22}\,K^3$ read $c_{22}\,K^3\,x$
356	22	for $e_{12} = e_6 + \ldots$ read $e_{12} = c_6 + \ldots$
356	−6	for $c_{23}\,e_1^2$ read $c_{23}\,e_1^3$
357	7	for $2\,c_{17}\,e_1^2$ read $2\,c_{17}\,e_1$
357	−9	for $KA_3\,x^2$ read $KA_3\,x^3$
359	−2	for $n_{11}\,K$ read $n_{11}\,K +$
362	13	for p_4 read $(p_4$
363	−8	for $s_{56}\,C^4\,K^3$ read $s_{56}\,C^4)\,K^3$
369	10	for B_1 read D
369	−1	for C read C^2
377	−4	insert arm of parenthesis before $\frac{t_1}{s_{16}}$
382	16	for a_{12} read a_{22}
382	17	for a_{53} read a_{35}
382	18	for a_{12} read a_{22} twice
384	25	for $2\,s_1\,\delta_8$ read $2\,s_1^3\,\delta_8$
384	−10	for $-4\,s_1^3\,\beta_{37}$ read $+4\,s_1^3\,\beta_{37}$
386	14	for β_{33} read β_{38}
388	−12	for β read β_{18}
389	23	for t_{86} read t_{81}
389	25	for $\zeta_2\,\lambda_6$ read $-\zeta_2\,\lambda_6$
390	−7	for $+\eta_6$ read $-\eta_6$
395	4	add the factor u to the first term
396	9	for μ read μ'
402	11	for $-\frac{}{16}$ read $-\frac{5}{16}$
403	−10	for $\frac{7}{2}\,c^2$ read $7\,e^2$
405	3	for $+K_4^{(i)}\,[\frac{35}{4}$ read $+K_4^{(i)}\,[\frac{35}{8}$
406	16	for $\frac{5}{16}$ read $\frac{15}{16}$
406	18	for 5 read $\frac{5}{2}$
406	−9	insert $+K_6^{(i)}\,[\frac{35}{4}]\,[5]$
407	5	for the first $K_7^{(i)}$ read $K_6^{(i)}$
411	14	for exponent 3 read $\frac{3}{2}$
411	16	for exponent 2 read $\frac{1}{2}$
429	21	for tnrn read tarn
430	21	add exp. $\frac{3}{2}$ to the denominator
431	−2	for $-d_2$ read $+d^2$
441	13	for $\int_0$ read $\int_0^\pi$
441	−5	for $\sin^2\phi\,\sin^2\gamma\,\cos^2 u$ read $-\sin^2\phi\,\sin^2\gamma\,\cos^2 u$
449	2	for $a\,\gamma\cos$ read $a\cos\gamma$